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ANALYSIS OF A FINITE VOLUME ELEMENT SCHEME FOR SOLVING THE MODEL TWO-PHASE NONEQUILIBRIUM FLOW PROBLEM

The paper proposes a hybrid numerical method for solving a model problem of two-phase nonequilibrium flow of an incompressible fluid in a porous medium. This problem is relevant in the modern theory of the motion of multiphase fluids in porous media and has many applications. The studied model is based on the assumption that the relative phase permeabilities and capillary pressure depend not only on saturation, but also on its time derivative. The saturation equation in this problem refers to the type of convection-diffusion with a predominance of convection, which also includes a third-order term to account for the nonequilibrium effects. Due to the hyperbolic nature of the equation, its solution is accompanied by a number of difficulties that lead to the need for an appropriate choice of the solution method. In contrast to previous works, this paper uses a finite volume element method for solving the problem, the construction of which is based on integral balance equations, and an approximate solution is chosen from the finite element space. To discretize the problem, two different dual grids are used based on the main triangulation. In this paper, a number of a priori estimates are obtained which yields the unconditional stability of the scheme as well as its convergence with the second order. The advantages of the approach used include the local conservatism of the scheme, as well as the comparative simplicity of the software implementation of the method. These results are confirmed by a numerical test carried out on the example of a model problem.
Key words: Finite volume element method, nonequilibrium fluid flow, dynamic capillary pressure, dual mesh, a priori estimate, convergence, stability, computational experiment.

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Екі фазалы теңөлшемсіз фильтрацияның модельді есебін шешудің ақырлы көлемді элементті сұлбасын талдау

Бұл жұмыста екі фазалы сығылмайтын сұйықтықтың теңөлшемсіз фильтрациясының модельдік есебін шешудің гибридті сандық әдісі ұсынылған. Бұл есеп көпфазалы сұйықтықтардың кеуекті ортадағы қозғалысының заманауи теориясында өзекті болып табылады және көптеген қолданбаларға ие. Зерттелетін модель салыстырмалы фазалық өткізгіштіктер мен капиллярлық қысымның қанықтықтан ғана емес, сонымен қатар оның уақыт бойынша туындыларынан да тәуелді деген болжамға негізделген. Бұл есептегі қанықтық теңдеуі конвекциясы басым болатын конвекция-диффузия түріне жатады, сонымен қатар оның құрамына теңөлшемсіздік әсерлерін ескеретін үшінші ретті қосылғыш кіреді. Теңдеудің гиперболалық сипатына байланысты оның шешімі бірқатар қиындықтарға ие болады, сондықтан оны шешу әдісін лайықты таңдау қажет етіледі. Алдыңғы жұмыстарға қарағанда бұл жұмыста есепті шешудің ақырлы көлемді-элементтік әдісі қолданылады. Бұл әдіс интегралдық баланс теңдеулері негізінде құрастырылған, ал жуық шешім ақырлы элементтер кеңістігінен таңдалады. Бұл жағдайда есепті дискретизациялау үшін негізгі триангуляция негізінде екі түрлі қосарланған тор қолданылады. Бұл жұмыста бірқатар априорлық бағалаулар алынған, олардан сұлбаның шартсыз орнықтылығы, сондай-ақ екінші ретпен жинақталуы шығады. Қолданылатын тәсілдің артықшылығына сұлбаның локальды консервативтілігі, сонымен қатар әдісті бағдарламалық жүзеге асырудың салыстырмалы қарапайымдылығы жатады. Бұл нәтижелер модельдік есеп мысалында жүргізілген сандық тәжірибемен расталады.

Түйін сөздер: Ақырлы көлемді-элементті әдіс, теңөлшемсіз фильтрация, динамикалық капиллярлық қысым, қосарланған тор, априорлық бағалау, жинақтылық, орнықтылық, есептеу тәжірибесі.

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Анализ конечно-объемно-элементной схемы решения модельной задачи двухфазной неравновесной фильтрации

В работе предлагается гибридный численный метод решения модельной задачи двухфазной неравновесной фильтрации несжимаемой жидкости. Данная задача является актуальной в современной теории движения многофазных жидкостей в пористых средах и имеет множество приложений. Изучаемая модель основана на предположении, что относительные фазовые проницаемости и капиллярное давление зависят не только от насыщенности, но также от ее временной производной. Уравнение для насыщенности в данной задаче относится к типу конвекции-диффузии с преобладанием конвекции, которое также содержит слагаемое третьего порядка для учета эффектов неравновесности. В силу гиперболического характера уравнения его решение сопровождается рядом трудностей, которые приводят к необходимости надлежащего выбора метода решения. В отличие от предыдущих работ, в данной работе применяется конечно-объемно-элементный метод решения задачи, построение которого основывается на интегральных уравнениях баланса, а приближенное решение выбирается из конечно-элементного пространства. При этом для дискретизации задачи используются две различные двойственные сетки на базе основной триангуляции. В работе получен ряд априорных оценок, из которых следует безусловная устойчивость схемы, а также ее сходимость со вторым порядком. К числу преимуществ используемого подхода можно отнести локальную консервативность схемы, а также сравнительную простоту программной реализации метода. Данные результаты подтверждаются численным тестом, проведенным на примере модельной задачи.

Ключевые слова: Конечно-объемно-элементный метод, неравновесная фильтрация, динамическое капиллярное давление, двойственная сетка, априорная оценка, сходимость, устойчивость, вычислительный эксперимент.

1 Introduction

Modeling the flow of a multiphase fluid in porous media is of great economic importance in the petroleum engineering, hydrology, carbon sequestration, and nuclear waste management [1–3]. These models form the basis of fluid dynamics simulators used in the development of oil fields, allowing predictive calculations of development indicators. Most simulators contain descriptions of the so-called classical fluid flow models in porous media that do not take into account a number of important factors. One of these factors is the phenomenon of a delay in the establishment of saturations, which is observed in microheterogeneous fractured rocks.

There are several approaches to modeling nonequilibrium effects. The first approach [4] is based on thermodynamic arguments and volume averaging of the microscopic equations of conservation of mass and momentum, as a result of which the authors of [4] came to the conclusion that it is necessary to add additional terms to the macroscopic equations. [4] introduced the concept of dynamic capillary pressure, i.e. instantaneous local difference between phase pressures. Dynamic capillary pressure has been the subject of many experimental [5] and theoretical [6, 7] studies.

The second approach [8] is based on the assumption that relative phase permeabilities and the capillary pressure are considered as functions not only of saturation, but also of the derivative of the saturation with respect to time $\frac{\partial s_w}{\partial t}$. Thus, a characteristic feature of nonequilibrium flows, i.e. the dependence on the rate of the process is taken into account.

Many works [9–12] are devoted to the numerical implementation of the two-phase fluid flow model with the nonequilibrium law from [4]. For example, in [9], a second-order numerical scheme for both spatial and temporal variables is proposed using a mixed finite element method with the lowest order Thomas-Raviar elements and an implicit Euler scheme. To show the convergence of the scheme, the error estimates for saturation, fluxes and phase pressures are obtained in $L^{\infty}(0,T;L^2(\Omega))$ norm for temporal and spatial triangulation. The authors of [10] present an a posteriori error estimate for (piecewise linear) approximation of finite elements, which corresponds to some linear Sobolev equations using the implicit Euler scheme.

Also, a class of quasiparabolic equations is considered in [11]. Such equations simulate the two-phase flow in porous media, where dynamic effects are included in capillary pressure. The existence and uniqueness of the weak solution were proved, and the error estimates for the implicit Euler time discretization were obtained.

The paper [12] analyzes the convergence of a "two-point flow" finite volume scheme to approximate the flow of two incompressible phases with dynamic capillary pressure in porous media. In that work, a fully implicit scheme is based on a non-standard approximation of mobility and capillary pressure on a double grid is proposed. A discrete variational formulation was derived and a new result of convergence in a two-dimensional and three-dimensional porous medium was presented. Compared to static capillary pressure, the nonequilibrium capillary model requires more powerful methods, especially not standard discrete energy estimates.

This work is devoted to the construction of a numerical method for solving the problem of two-phase fluid flow with the inequality law proposed in [4]. In contrast to the above works, we use the finite volume element method (FVEM). The essence of this method consists in constructing dual grids based on the main grid and generating control volumes on the dual grid. Compared to finite difference and finite element methods, the finite volume element method is simple to implement and provides flexibility in handling complex geometric domains, as well as automatically provides local mass conservation. The last property is most important in problems of fluid flow in porous media.

The FVEM has been successfully applied to problems of non-stationary equations of an incompressible fluid in the Boussinesq approximation [13], for problems of fluid flow of an incompressible fluid [14], for a non-stationary equation of convection-diffusion [15] and many others. In the cited works, the accuracy theoretical estimates of the proposed schemes are obtained, a comparative analysis of the estimates obtained with the results of numerical tests is carried out, and the implementation advantages of the method are shown.

In the present paper, two dual grids are constructed on the base of the main triangulation. The first dual grid is used for the velocity and pressure equations, and the second one is used for the saturation equation. Theoretical estimates are obtained which show the stability of the scheme, as well as the convergence of the scheme with the second order.

2 Materials and Methods

2.1 Statement of the Problem

In $Q_T = \Omega \times [0,T]$, where $\Omega \subset \mathbb{R}^2$ is a convex bounded domain with a Lipschitz-continuous boundary Γ, $T > 0$, the following model problem of two-phase nonequilibrium flow is considered under the assumption of incompressibility of phases and the absence of gravitational forces:

$$\nabla \cdot \vec{u} = 0, \quad (x,t) \in Q_T, \tag{1}$$

$$(k\lambda(s))^{-1} \vec{u} + \nabla p = 0, \quad (x,t) \in Q_T, \tag{2}$$

$$\phi s_t + f_w \vec{u} \cdot \nabla s - \nabla \cdot (\gamma \nabla s) - \nabla \cdot (\gamma_1 \nabla (L s_t)) = 0, \quad (x,t) \in Q_T, \tag{3}$$

$$s(x,0) = s_0(x), \quad x \in \Omega, \tag{4}$$

$$\vec{u} \cdot \vec{n} = 0, \quad \nabla s \cdot \vec{n} = 0, \quad (x,t) \in \Gamma \times (0,T], \tag{5}$$

where $\vec{u}$ is the total velocity vector, p is pressure, $s = s(x,t)$ is the water saturation, ϕ is porosity, k is the absolute permeability, L is the replacement time; f_w, γ, γ_1 are some positive constants; $\lambda(s) = \lambda_w(s) + \lambda_o(s)$, $\lambda_\alpha(s) = k_\alpha(s)\mu_\alpha^{-1}$, $k_\alpha(s)$ and μ_α are relative permeability and viscosity of the phase α; $\vec{n}$ is the outer unit normal to the boundary Γ.

Introduce the following functional spaces:

$$U = \{\vec{v} \in H(\mathrm{div};\Omega) \ : \ \vec{v} \cdot \vec{n} = 0 \text{ on } \Gamma\}, \quad M = L^2(\Omega)/\mathbb{R}, \quad W = H_0^1(\Omega).$$

The mixed variational formulation of Problem (1)-(5) is as follows: find $(\vec{u}, p, s) \in U \times M \times H^1(0,T;W)$ such that the following identities hold for all $\vec{v} \in U$, $w \in M$, $\varphi \in W$, $t \in (0,T)$

$$(\nabla \cdot \vec{u}, w) = 0, \tag{6}$$

$$\left((K\lambda(s))^{-1} \vec{u}, \vec{v}\right) - (p, \nabla \cdot \vec{v}) = 0, \tag{7}$$

$$(s_t, \varphi) + a(\vec{u}, s, \varphi) + d(s, \varphi) + d_1(s_t, \varphi) = 0, \tag{8}$$

$$s(x,0) = s_0(x), \quad x \in \Omega, \tag{9}$$

where

$$a(\vec{u}, \eta, \varphi) = \frac{f_w}{2} \int_\Omega \vec{u} \cdot (\varphi \nabla \eta - \eta \nabla \varphi)\, dx,$$

$$d(s, \varphi) = \int_\Omega \gamma \nabla s \cdot \nabla \varphi \, dx, \quad d_1(s, \varphi) = \int_\Omega \gamma_1 \nabla (Ls) \cdot \nabla \varphi \, dx$$

for all $(\vec{u}, \eta, \varphi) \in U \times W \times W$. It is known that [16]

$$a(\vec{u}, \eta, \varphi) = -a(\vec{u}, \varphi, \eta), \quad a(\vec{u}, \varphi, \varphi) = 0. \tag{10}$$

2.2 The Finite Volume Element Method

Let us first discretize Problem (1)-(5) with respect to time. Let $\{t_n\}_{n=0}^{N}$ be a uniform partitioning introduced in the time interval $[0,T]$. Further, let $(\vec{u}^n, p^n, s^n)$ denote the semi-discrete approximation of $(\vec{u}, p, s)$ at $t = t_n$.

Introduce the notations $\Delta_t s^{n-1/2} = \dfrac{s^n - s^{n-1}}{\tau}$, $s^{n-1/2} = \dfrac{s^n + s^{n-1}}{2}$.

The semi-discrete formulation of Problem (6)-(9) reads: find $(\vec{u}^n, p^n, s^n) \in U \times M \times W$, $n = 1, 2, ..., N$ such that for all $\vec{v} \in U$, $w \in M$, $\varphi \in W$:

$$(\nabla \cdot \vec{u}^n,\, w) = 0, \tag{11}$$

$$\left((k\lambda)^{-1}\, \vec{u}^n, \vec{v}\right) - (p^n, \nabla \cdot \vec{v}) = 0, \tag{12}$$

$$(s^n, \varphi) + \tau a\left(\vec{u}^{n-1/2}, s^{n-1/2}, \varphi\right) + \tau d\left(s^{n-1/2}, \varphi\right) + d_1\left(s^n - s^{n-1}, \varphi\right) = \left(s^{n-1}, \varphi\right), \tag{13}$$

$$s^0 = s_0(x), \quad x \in \Omega. \tag{14}$$

To solve Problem (1)-(5), we use the finite volume element method. In $\overline{\Omega}$ introduce a quasi-uniform triangulation $\mathfrak{T}_h$ and let h be its diameter. Let us construct two dual partitions on the basis of the basic partition $\mathfrak{T}_h$.

Let $\{V_i\}_{i=1}^{N_h}$, $\{E_i\}_{i=1}^{N_e}$ and $\{M_i\}_{i=1}^{N_m}$ denote the sets of vertices, edges and midpoints of the triangles in $\mathfrak{T}_h$, respectively. Consider two adjacent triangles $T_i \in \mathfrak{T}_h$ and $T_j \in \mathfrak{T}_h$ and let E_k be their common edge, and M_k be the midpoint of E_k. We form a quadriterial Q_k^* by connecting the barycenters of T_i and T_j with the ends of E_k. In the case when T_i is a boundary element and E_k is its edge lying on the domain boundary, Γ, we form a triangle T_l^* by connecting the barycenter T_i with the ends of the edge E_k. The set of internal elements Q_k^* and boundary triangles T_l^* is called the dual partition for the pressure and velocity equation and is denoted by $\mathfrak{T}_h^*$.

To construct the second dual partition, $\mathfrak{V}_h^*$, we connect the barycenter C_i of the triangle $T_i \in \mathfrak{T}_h$ with the midpoints of its edges by straight lines. This leads to the partitioning of T_i into three quadrilaterals. By combining them, we obtain a control volume $T_{V_i}^*$ which surround the vertex V_i. A set of control volumes cover Ω, and is called the dual partition of Ω of the barycentric type corresponding to the triangulation $\mathfrak{T}_h$.

Let us define the functional spaces U_h, W_h and M_h of trial functions as

$$U_h = \left\{\vec{v}_h \in U \cap \left(C\left(\overline{\Omega}\right)\right)^2 : \quad \vec{v}_h|_K \in (P_1(K))^2 \ \ \forall K \in \mathfrak{T}_h\right\},$$

$$M_h = \left\{w_h \in W \cap C\left(\overline{\Omega}\right) : \quad w_h|_K \in P_0(K) \ \ \forall K \in \mathfrak{T}_h\right\},$$

$$W_h = \left\{q_h \in M : \quad q_h|_K \in P_1(K) \ \ \forall K \in \mathfrak{T}_h\right\},$$

where $P_l(K)$ is the space of polynomial functions of degree not greater than l on K.

Define the spaces of test functions $\tilde{U}_h$ and $\tilde{W}_h$ in the following form:

$$\tilde{U}_h = \left\{\vec{v}_h \in \left(L^2(\Omega)\right)^2 : \quad \vec{v}_h|_V \in (P_0(V))^2, \quad \vec{v}_h \cdot \vec{n}|_V = 0 \ \ \forall V \in \mathfrak{T}_h^*\right\},$$

$$\tilde{W}_h = \left\{w_h \in L^2(\Omega) : \quad w_h|_V \in P_0(V), \quad w_h|_V = 0 \ \ \forall V \in \mathfrak{V}_h^*\right\}. \tag{15}$$

Let $\Pi_h^* \vec{u}$ and $\rho_h^* w$ be the interpolation projections of $\vec{u} \in U$ and $w \in W$ into the spaces of trial functions $\tilde{U}_h$ and $\tilde{W}_h$ defined as

$$\Pi_h^* \vec{u}_h(x) = \sum_{i=1}^{N_m} \vec{u}_h(M_i)\,\zeta_i(x), \quad \rho_h^* w_h(x) = \sum_{i=1}^{N_h} w_h(P_i)\,\chi_i(x)$$

for all $x \in \Omega$, where ζ_i is the characteristic function of Q_i^* and $\chi_i(x)$ is the characteristic function of $T_{V_i}^*$.

Now we define a fully discrete scheme corresponding to Problem (1)-(5): find $(\vec{u}_h^n, p_h^n, s_h^n) \in U_h \times M_h \times W_h$ $(1 \le n \le N)$, such that the following identities hold for all $\vec{v}_h \in \tilde{U}_h$, $w_h \in \tilde{M}_h$:

$$\left((k\lambda)^{-1}\vec{u}_h^n, \Pi_h^* \vec{v}_h\right) + l_h\left(\Pi_h^* \vec{v}_h, p_h^n\right) = 0, \tag{16}$$

$$\left(\nabla \cdot \vec{u}_h^n,\, w_h\right) = 0, \tag{17}$$

$$\left(\Delta_t s_h^{n-1/2}, \rho_h^* \varphi_h\right) + a_h\left(\vec{u}_h^{n-1/2}, s_h^{n-1/2}, \rho_h^* \varphi_h\right) + d_h\left(s_h^{n-1/2}, \rho_h^* \varphi_h\right) +$$
$$+ d_{1h}\left(\Delta_t s_h^{n-1/2}, \rho_h^* \varphi_h\right) = 0, \tag{18}$$

$$s_h^0 = \rho_h \varphi(x), \tag{19}$$

where

$$a_h\left(\vec{u}_h^n, s_h^n, \rho_h^* \varphi_h\right) = \sum_{V_z \in \mathfrak{V}_h^*} f_w \varphi_h(z) \int_{\partial V_z} s_h^n \vec{u}_h^n \cdot \vec{n}\, ds,$$

$$d_h\left(s_h^n, \rho_h^* \varphi_h\right) = \sum_{V_z \in \mathfrak{V}_h^*} \varphi_h(z) \int_{\partial V_z} \gamma \nabla s_h^n \cdot \vec{n}\, ds,$$

$$d_{1h}\left(s_h^n, \rho_h^* \varphi_h\right) = L \sum_{V_z \in \mathfrak{V}_h^*} \varphi_h(z) \int_{\partial V_z} \gamma_1 \nabla s_h^n \cdot \vec{n}\, ds,$$

$$l_h\left(\Pi_h^* \vec{v}_h, p_h^n\right) = -\sum_{V_z \in \mathfrak{T}_h^*} \vec{v}_h(z) \int_{\partial V_z} p_h^n \cdot \vec{n}\, ds.$$

Let us formulate the following lemmas from [13, 14] without proof.

Lemma 1 *The following results hold [13]:*

$$d_h\left(s_h, \rho_h^* \varphi_h\right) = d\left(s_h, \varphi_h\right), \quad a_h\left(\vec{u}_h, s_h, \rho_h^* \varphi_h\right) = a\left(\vec{u}_h, s_h, \varphi_h\right),$$

$$d_{1h}\left(s_h, \rho_h^* \varphi_h\right) = d_1\left(s_h, \varphi_h\right),$$

$$a_h\left(\vec{u}_h, s_h, \rho_h^* s_h\right) = 0, \quad \forall s_h, \varphi_h \in W_h, \quad \forall \vec{u}_h \in U_h.$$

Moreover, $d_h\left(s_h, \rho_h^* w_h\right)$ *is a symmetric, bounded and positive definite form, i.e.*

$$d_h\left(s_h, \rho_h^* w_h\right) = d_h\left(w_h, \rho_h^* s_h\right), \quad \forall s_h, w_h \in W_h$$

and there are constants h_0, C_0, *such that for* $0 < h \le h_0$,

$$d_h\left(s_h, \rho_h^* s_h\right) \ge \gamma_0 \left|s_h\right|_1^2, \quad \left|d_h\left(s_h, \rho_h^* w_h\right)\right| \le C_0 \left\|s_h\right\|_1 \left\|w_h\right\|_1, \quad \forall s_h, w_h \in W_h.$$

Lemma 2 *The following result is valid [13]:*

$$\left(\vec{u}_h, \Pi_h^* \vec{v}_h\right) = \left(\vec{v}_h, \Pi_h^* \vec{u}_h\right), \quad \forall \vec{u}_h, v_h \in U_h.$$

For every $\vec{u} \in H^m(\Omega)^2$, $m = 0, 1$ *and* $\vec{v}_h \in U_h$,

$$\left|(\vec{u}, \vec{v}_h) - (\vec{u}, \Pi_h^* \vec{v}_h)\right| \le Ch^{m+n} \|\vec{u}\|_m \|\vec{v}_h\|_n, \quad n = 0, 1. \tag{20}$$

Lemma 3 *The operator* Π_h^* *satisfies the following inequalities for all* $\vec{v}_h \in U_h$ *and* $w_h \in M_h$ *provided* $\nabla \cdot \vec{v}_h = 0$ *[14]:*

1) $\|\Pi_h^* \vec{v}_h\|_{(L^2(\Omega))^2} \le \|\vec{v}_h\|_{(L^2(\Omega))^2}$,
2) $\|\vec{v}_h - \Pi_h^* \vec{v}_h\|_{(L^2(\Omega))^2} \le Ch \|\vec{v}_h\|_{H(\mathrm{div};\Omega)}$,
3) $l_h\left(\Pi_h^* \vec{v}_h, w_h\right) = -\left(\nabla \cdot \vec{v}_h, w_h\right)$,
4) $\left((k\lambda(\eta_h))^{-1} \vec{v}_h, \Pi_h^* \vec{v}_h\right) \ge C \|\vec{v}_h\|^2_{H(\mathrm{div};\Omega)}$.

Let us introduce the norm $|||\vec{u}_h|||_0 = \left(\vec{u}_h, \Pi_h^* \vec{u}_h\right)^{1/2}$. It is shown in [13] that $|||\cdot|||_0$ equivalent to $\|\cdot\|_0$ on U_h.

Let us formulate the following result obtained in [14] without proof.

Theorem 1 *([14]) Let* $(\vec{u}, p)$ *and* $(\vec{u}_h, p_h)$ *be the solutions of (6)-(7) and (16)-(17), respectively. Then there exists a positive constant* C *independent of* h *such that*

$$\|\vec{u} - \vec{u}_h\|_{(L^2(\Omega))^2} + \|p - p_h\|_0 \le Ch^2 \left(\|\vec{u}\|_{(H^1(\Omega))^2} + \|p\|_1\right), \tag{21}$$

$$\|\nabla \cdot (\vec{u} - \vec{u}_h)\|_0 \le Ch \|\nabla \cdot \vec{u}\|_1, \tag{22}$$

provided $\vec{u}(t) \in \left(H^1(\Omega)\right)^2$, $\nabla \cdot \vec{u}(t) \in H^1(\Omega)$ *and* $p(t) \in H^1(\Omega)$.

Now we prove the main results of the paper.

Theorem 2 *The sequence of solutions* $(u_h^n, p_h^n, s_h^n) \in U_h \times M_h \times W_h$, $n = 1, 2, ..., N$ *of Problem (16)-(19) satisfies the inequality*

$$\|\vec{u}_h^n\|_{H(\mathrm{div};\Omega)} + \|s_h^n\|_0 + \tau \sqrt{\frac{\gamma_0}{2T}} \|\nabla s_h^n\|_0 \le C \|s_0\|_1. \tag{23}$$

Proof. Taking $\varphi_h = \overline{s}_h^n$ in (18), we get:

$$\left(\Delta_t s_h^{n-1/2}, \rho_h^* s_h^{n-1/2}\right) + b_h\left(u_h^{n-1/2}, s_h^{n-1/2}, \rho_h^* s_h^{n-1/2}\right) +$$

$$+ d_h\left(s_h^{n-1/2}, \rho_h^* s_h^{n-1/2}\right) + d_{1h}\left(\Delta_t s_h^{n-1/2}, \rho_h^* s_h^{n-1/2}\right) = 0. \tag{24}$$

Estimate the terms in (24):

$$\left(\Delta_t s_h^{n-1/2}, \rho_h^* s_h^{n-1/2}\right) = \frac{1}{2\tau} |||s_h^n|||_0^2 - \frac{1}{2\tau} \left|\left|\left|s_h^{n-1}\right|\right|\right|_0^2,$$

$$d_h\left(s_h^{n-1/2}, \rho_h^* s_h^{n-1/2}\right) = \frac{\gamma}{2} \left\|\nabla\left(s_h^n + s_h^{n-1}\right)\right\|_0^2,$$

$$d_{1h}\left(\Delta_t s_h^{n-1/2}, \rho_h^* s_h^{n-1/2}\right) \geq \frac{L\gamma_1}{2\tau}\left(\|\nabla s_h^n\|_0^2 - \left\|\nabla s_h^{n-1}\right\|_0^2\right).$$

Then it follows from (24) that

$$|||s_h^n|||_0^2 - \left|\left|\left|s_h^{n-1}\right|\right|\right|_0^2 + \frac{\tau\gamma}{2}\left\|\nabla\left(s_h^n + s_h^{n-1}\right)\right\|_0^2 + L\gamma_1\left(\|\nabla s_h^n\|_0^2 - \left\|\nabla s_h^{n-1}\right\|_0^2\right) \leq 0. \quad (25)$$

Sum (25) by n from 1 to n:

$$|||s_h^n|||_0^2 + \frac{\tau\gamma}{2}\sum_{i=1}^{n}\left\|\nabla\left(s_h^i + s_h^{i-1}\right)\right\|_0^2 + L\gamma_1\|\nabla s_h^n\|_0^2 \leq |||s_0|||_0^2 + L\gamma_1\|\nabla s_0\|_0^2.$$

By extracting the square root from the last inequality, and using Lemma 3, we arrive at the statement of the theorem.

Lemma 4 *Let $\Theta_h : W \to W_h$ be the projection operator such that there exists $\Theta_h s^n \in W_h$ for $s^{n-1}, s^n \in W$, $s_h^{n-1} \in W_h$, and $\vec{u}_h^n \in U_h$ satisfying [13]*

$$\left(\Theta_h \Delta_t s^{n-1/2} - \Delta_t s^{n-1/2}, w_h\right) + d_0\left(\Theta_h s^n - s^n, w_h\right) + d_0\left(\Theta_h s^{n-1} - s^{n-1}, w_h\right) +$$

$$+a\left(\vec{u}_h^{n-1/2}, \Theta_h s^{n-1/2}, w_h\right) - a\left(\vec{u}^{n-1/2}, s^{n-1/2}, w_h\right) = 0$$

for any $w_h \in W_h$, where $d_0(s, w_h) = \dfrac{1}{2}\left(d(s, w_h) + d_1(s, w_h)\right)$. Moreover,

$$\|\Theta_h s^n - s^n\|_0 + \tau\|\nabla(\Theta_h s^n - s^n)\|_0 \leq Ch^2\|s_0\|_2, \quad n = 0, 1, ..., N, \quad (26)$$

provided $s^n \in H^2(\Omega) \cap W$.

Theorem 3 *Let $(\vec{u}, p, s)$ and $(\vec{u}_h^n, p_h^n, s_h^n)$ be the solutions of Problem (1)-(5) and Problem (16)-(19), respectively. Then*

$$\|\vec{u}(t_n) - \vec{u}_h^n\|_{(L^2(\Omega))^2} + \|p(t_n) - p_h^n\|_0 + \|s(t_n) - s_h^n\|_0 +$$

$$+c\tau\|\nabla(s(t_n) - s_h^n)\|_0 \leq Ch^2 \quad (27)$$

provided $\tau = O(h)$.

Proof. First, consider the difference of Problems (11)-(14) and (16)-(19) to obtain

$$(s^n, \varphi_h) - (s_h^n, \rho_h^*\varphi_h) + \tau a\left(\vec{u}^{n-1/2}, s^{n-1/2}, \varphi_h\right) - \tau a_h\left(\vec{u}_h^{n-1/2}, s_h^{n-1/2}, \rho_h^*\varphi_h\right) +$$

$$+\tau d\left(s^{n-1/2}, \varphi_h\right) - \tau d_h\left(s_h^{n-1/2}, \rho_h^*\varphi_h\right) + d_1\left(s^n - s^{n-1}, \varphi_h\right) - d_{1h}\left(s_h^n - s_h^{n-1}, \rho_h^*\varphi_h\right) =$$

$$= \left(s^{n-1}, \varphi_h\right) - \left(s_h^{n-1}, \rho_h^*\varphi_h\right). \quad (28)$$

By applying obvious transformations and the projection defined in Lemma 4, we obtain

$$(\Theta_h s^n - s_h^n, \varphi_h) - (s_h^n, \rho_h^*\varphi_h - \varphi_h) + \tau a\left(\vec{u}_h^{n-1/2}, \Theta_h s^{n-1/2}, \varphi_h\right) -$$

$$-\tau a_h\left(\vec{u}_h^{n-1/2}, s_h^{n-1/2}, \rho_h^*\varphi_h\right) + \tau d\left(\Theta_h s^{n-1/2} - s_h^{n-1/2}, \varphi_h\right) +$$

$$+d_1\left(\Theta_h\left(s^n - s^{n-1}\right) - \left(s_h^n - s_h^{n-1}\right), \varphi_h\right) = \left(\Theta_h s^{n-1} - s_h^{n-1}, \varphi_h\right) - \left(s_h^{n-1}, \rho_h^*\varphi_h - \varphi_h\right).$$

Let $\psi^n = \Theta_h s^n - s_h^n$ and choose $\varphi_h = \psi^{n-1/2}$:

$$\frac{1}{2}\|\psi^n\|_0^2 + \tau a\left(\vec{u}_h^{n-1/2}, \Theta_h s^{n-1/2}, \psi^{n-1}\right) - \tau a_h\left(\vec{u}_h^{n-1/2}, s_h^{n-1/2}, \rho_h^*\psi^{n-1}\right) +$$

$$+\tau\gamma\left\|\nabla\psi^{n-1}\right\|_0^2 + \frac{\gamma_1}{2}\|\nabla\psi^n\|_0^2 - \frac{\gamma_1}{2}\left\|\nabla\psi^{n-1}\right\|_0^2 \leq$$

$$= \frac{1}{2}\left\|\psi^{n-1}\right\|_0^2 + \left(s_h^n - s_h^{n-1}, \rho_h^*\psi^{n-1/2} - \psi^{n-1/2}\right). \tag{29}$$

Let us estimate the scalar products in (29):

$$\left|a\left(\vec{u}_h^{n-1/2}, \Theta_h s^{n-1/2}, \psi^{n-1/2}\right) - a_h\left(\vec{u}_h^{n-1/2}, s_h^{n-1/2}, \rho_h^*\psi^{n-1/2}\right)\right| \leq$$

$$\leq \frac{1}{4\gamma}\left\|\nabla\psi^{n-1/2}\right\|_0^2 + Ch^4, \tag{30}$$

$$\left|\left(s_h^n - s_h^{n-1}, \rho_h^*\psi^{n-1/2} - \psi^{n-1/2}\right)\right| \leq$$
$$\leq Ch\left(\|\psi^n\|_0^2 + \left\|\psi^{n-1}\right\| + Ch^4\right) + \frac{\tau}{8\gamma}\left\|\nabla\psi^{n-1/2}\right\|_0^2 + Ch^2\left\|s^n - s^{n-1}\right\|_1\left\|\nabla\psi^{n-1/2}\right\|_0 \leq$$
$$\leq Ch\left(\|\psi^n\|_0^2 + \left\|\psi^{n-1}\right\| + Ch^4\right) + C\tau^2h^3 + \frac{\tau}{4\gamma}\left\|\nabla\psi^{n-1/2}\right\|_0^2. \tag{31}$$

Taking into account the inequalities (30), (31) and the assumption $\tau = O(h)$, it follows from (29) that

$$\frac{1}{2}\|\psi^n\|_0^2 + c\tau\left\|\nabla\psi^{n-1/2}\right\|_0^2 + \frac{\gamma_1}{2}\left(\|\nabla\psi^n\|_0^2 - \left\|\nabla\psi^{n-1}\right\|_0^2\right) \leq$$

$$\leq \frac{1}{2}\left\|\psi^{n-1}\right\|_0^2 + C\tau\left(\left\|\psi^{n-1}\right\|_0^2 + h^4 + \tau^2h^2\right) + C\tau h^4. \tag{32}$$

Summing (32) with respect to n from 1 to n, we obtain

$$\frac{1}{2}\|\psi^n\|_0^2 + c\tau\sum_{i=1}^{n}\left\|\nabla\psi^{i-1/2}\right\|_0^2 \leq \frac{1}{2}\left\|\psi^0\right\|_0^2 + \frac{\gamma_1}{2}\left\|\nabla\psi^0\right\|_0^2 + C\tau\sum_{i=1}^{n}\left\|\psi^{i-1}\right\|_0^2 + Ch^4.$$

Applying the discrete Gronwall's lemma yields

$$\|\psi^n\|_0^2 + c\tau\sum_{i=1}^{n}\left\|\nabla\psi^{i-1/2}\right\|_0^2 \leq C\left(\left\|\psi^0\right\|_0^2 + \left\|\nabla\psi^0\right\|_0^2\right) + Ch^4.$$

Finally, by taking into account Theorem 1, we arrive at the statement of the theorem.

3 Results

To validate the finite volume element scheme (16)-(19), the following five-spot test problem was solved. The problem (1)-(5) in the square $\Omega = [-1, 1] \times [-1, 1]$ was considered in which an injection and a production well were placed in the lower left and upper top corner of Ω, respectively. The following dimensionless values were taken as initial data: $k = 1$, $\mu_w = \mu_o = 1$, $\tau = 10^{-3}$, $\gamma = \gamma_1 = 1$, $L = 1$, $s_0(x) \equiv 0$, and the functions $k_\alpha(s)$ were defined as $k_w(s) = s^2$, $k_o(s) = (1 - s)^2$.

Since the exact solution to the problem cannot be found analytically, more attention is paid to the qualitative characteristics of the solutions obtained using the scheme (16)-(19). To programmatically construct the dual grids used in the finite volume element method, we first introduced a quasiuniform triangulation in Ω. In the numerical test, the triangular decomposition of Ω containing 357 nodes, 648 triangles, and 1004 edges was used. A coarser grid was chosen intentionally to assess the stability of the scheme to the appearance of non-physical oscillations. Then, in construction of the dual grid used for the velocity and pressure equations, the algorithm given in [14] was utilized. The dual grid for the saturation equation was built based on an algorithm presented in [21].

Firstly, the total velocity and water saturation obtained by the scheme (16)-(19) at $t = 5600\tau$ are shown in Figure 1.

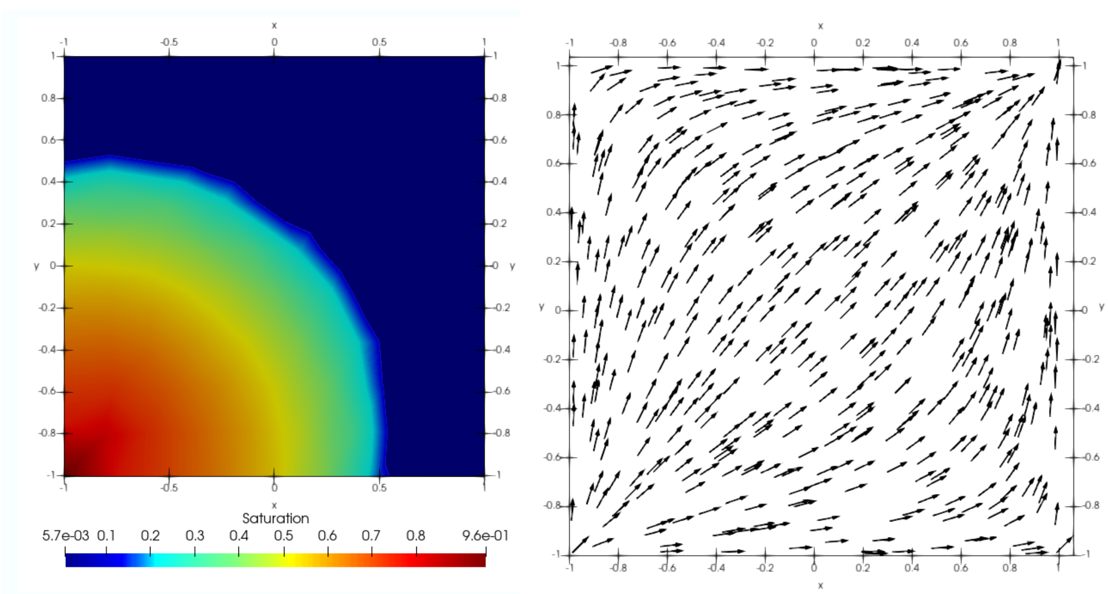

Figure 1: Numerical solution of the model problem at $t = 5600\tau$, saturation (left), total velocity (right)

In our previous work [22], a stabilized finite element method was applied to solving the problem of two-phase nonequilibrium flow. Using the model convection-diffusion equation as an example, it was shown that the use of the standard Galerkin method can lead to the appearance of nonphysical oscillations near the phase separation line. It is shown that their suppression without stabilization can be partially implemented by thickening the grid near the discontinuity line. Stabilization of the saturation equation significantly reduces nonphysical oscillations; however, the choice of the stabilization parameter itself is a separate problem. Using the stabilization approach, in [22] the model problem of two-phase nonequilibrium flow with the parameters indicated above was solved. It was noted that the stabilization of the equation imposes an additional computational complexity associated with the need to recalculate the stabilization parameter for each finite element at each time layer.

Due to its local conservativeness, the finite volume element scheme (16)-(19) did not lead to the appearance of non-physical oscillations which in turn did not require the addition of stabilizing terms in the scheme. As can be seen from Figure 1, the scheme allows obtaining non-oscillating solutions even on a coarse mesh. In addition, the scheme turned out to be simpler in software implementation.

4 Conclusion

Thus, in this paper, a fully discrete mixed finite volume element method was studied for the problem of two-phase non-equilibrium flow in porous media. It was shown on a synthetic example that the constructed method can be considered as an alternative method that allows obtaining non-oscillating solutions to the problem without the stabilization of the equation, and also requires less computational complexity in comparison, for example, with discontinuous Galerkin methods. The constructed method can be generalized to solve filtration problems with more real input data. A separate work will be devoted to this problem.

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References

[1] Xiong Q., Baychev T.G., and Jivkov A.P., "Review of pore network modelling of porous media: Experimental characterisations, network constructions and applications to reactive transport", *Journal of Contaminant Hydrology,* 192 (2016): 101–117.

[2] Al-Bayati D., Saeedi A., Ktao I., Myers M., White C., and Xie Q., "Insight into Influence of Crossflow in layered Sandstone porous media during Miscible and Immiscible CO2 WAG Flooding", InterPore 2019 Valencia: Book of Abstracts (2019): 66.

[3] Allen G.H. and Sahimi M., "Flow, Transport, and Reaction in Porous Media: Percolation Scaling, Critical-Path Analysis, and Effective Medium Approximation" (AGU Publications, 2017).

[4] Hassanizadeh S.M. and Graw W.G., "Toward an improved description of the physics of twophase flow", *Advances in Water Resources,* 16 (1993): 53–67.

[5] O'Carroll D.M., Phelan T.J., and Abriola L.M., "Exploring dynamic effects in capillary pressure in multistep outflow experiments", *Water Resour Res.,* 41 (2005): W11419.

[6] Cuesta C., van Duijn C.J., and Hulshof J., "Infiltration in porous media with dynamic capillary pressure: travelling waves", *Eur. J. Appl. Math.,* 11 (2000): 381–397.

[7] DiCarlo D.A., Juanes R., LaForce T., and Witelski T.P., "Nonmonotonic travelling wave solutions of infiltration in porous media", *Water Resour. Res.,* 44 (2008): W02406.

[8] Barenblatt G.I., ”Filtratsiya dvuh nesmeshivayuschihsya zhidkostei v odnorodnoi poristoi srede [Filtration of two nonmixing fluids in a homogeneous porous medium]“ , *Mehanika gazov i zhidkostei,* 5 (1971): 57–64 (in Russian).

[9] Cao X. and Mitra K., ”Error estimates for a mixed finite element discretization of a two-phase porous media flow model with dynamic capillarity“ , *Journal of Computational and Applied Mathematics,* 353 (2019): 164–178.

[10] Nicaise S. and Bbekkouche F., ”A posteriori error estimates for a fully discrete approximation of sobolev equations“ , *Confluentes Math.,* 11 (2019): 3–28.

[11] Fan Y., and Pop I.S., ”A class of pseudo-parabolic equations: existence, uniqueness of weak solutions, and error estimates for the Euler-implicit discretization“ , *Math. Meth. Appl. Sci.,* 34 (2011): 2329–2339.

[12] Bouadjila K., Mokrane A., Saad A.S., and Saad M., ”Numerical analysis of a finite volume scheme for two incompressible phase flow with dynamic capillary pressure“ , *Computers and Mathematics with Applications,* 75:10 (2018): 3614–3631.

[13] Luo Z.D., ”A Stabilized Crank-Nicolson Mixed Finite Volume Element Formulation for the Non-stationary Incompressible Boussinesq Equations“ , *Journal of Scientific Computing,* 66 (2016): 555–576.

[14] Kumar S., ”Finite volume element methods for incompressible miscible displacement problems in porous media (Ph.D. thesis)“ (Department of mathematics indian institute of technology, Bombay, 2008).

[15] Vasilevsky Yu.V. and Kapyrin I.V., ”Dve shemy rasshheplenija dlja nestacionarnoj zadachi konvekcii-diffuzii na tetrajedralnyh setkah [Two splitting schemes for the nonstationary convection-diffusion problem on tetrahedral meshes]“ , *Zhurnal vychislitel'noj matematiki i matematicheskoj fiziki,* 48 (2008): 1429–1447 (in Russian).

[16] Luo Z., Li H., and Sun P., ”A fully discrete stabilized mixed finite volume element formulation for the non-stationary conduction–convection problem“ , *Journal of Mathematical Analysis and Applications,* 404 (2013): 71–85.

[17] Ferraris S., Bevilacqua I., Canone D., Pognant D., and Previati M., ”The Finite Volume Formulation for 2D Second-Order Elliptic Problems with Discontinuous Diffusion/Dispersion Coefficients“ , *Mathematical Problems in Engineering,* 187634 (2012): 1–24.

[18] Nikitin K.D., ”Nelinejnyj metod konechnyh obyomov dlja zadach dvuhfaznoj filtracii [Nonlinear finite volume method for two-phase filtration problems]“ , *Matematicheskoe modelirovanie,* 22 (2010): 131–147 (in Russian).

[19] Lin Y., Liu J., and Yang M., ”Finite volume element methods: an overview on recent developments“ , *International Journal of Numerical analysis and modeling, Series B.,* 4 (2013): 14–34.

[20] Sayevand K. and Arjang F., ”Finite volume element method and its stability analysis for analyzing the behavior of sub-diffusion problems“ , *Applied Mathematics and Computation,* 290 (2016): 224–239.

[21] Voller V. R., ”Basic control volume finite element methods for fluids and solids“ (World Scientific Publishing, 2009).

[22] Omariyeva D., Temirbekov N., Madiyarov M., ”Stabilized finite element method for solving the saturation equation in the two-phase non-equilibrium flow problem“ , *Joint Issue of Bulletin of the National Engineering Academy of the Republic of Kazakhstan and Computational Technologies,* 3:1 (2020): 209–216.