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The Cauchy problems for q-difference equations with the Caputo fractional derivatives

The fractional differential equations play important roles due to their numerous applications and also for the important role they play not only in mathematics but also in other sciences. In the present research work, we build up the explicit solutions to linear fractional q-differential equations with the q-Caputo fractional derivative of real order $\alpha > 0$. To speak more precisely, we will achieve our main results we use that this Cauchy type q-fractional problem is equivalent to a corresponding Volterra q-integral equation. After that, by using the method of successive approximations is applied to solve the Volterra q-integral equation we construct the the explicit solutions to linear fractional q-differential equations. In the same way we have the more general homogeneous fractional q-differential equation with the Caputo fractional q-derivative of real order $\alpha > 0$ and we give other The (Mittag-Leffler) q-function. Finally, some examples are presented to illustrate our main results in cases where we can even give concrete formulas for these explicit solutions.

Key words: Cauchy type q-fractional problem, existence, uniqueness, q-derivative, q-calculus, fractional calculus, fractional derivative, Caputo fractional derivatives.

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Капуто бөлшек туындысы бар q-айрымдық теңдеулер үшін Коши есептері

Бөлшек туындылары бар теңдеулер өздерінің көп салаларда қолданылуына байланысты маңызды рөл атқарады, сонымен қатар олар тек математикада ғана емес, сонымен қатар басқа ғылымдарда да маңызды рөл атқарады. Бұл зерттеу жұмысында біз $\alpha > 0$ нақты ретті Капутоның q-бөлшек туындысы бар бөлшек-сызықтық q- дифференциалдық теңдеулердің нақты шешімдерін құрамыз. Нақтырақ айтсақ, біз осы Коши типтес q-бөлшек есептің сәйкес Вольтердің q-интегралдық теңдеуіне эквивалентті болатындығын қолдана отырып, негізгі нәтижелерге қол жеткіземіз. Осыдан кейін Вольтердің q-интегралдық теңдеуінің шешіміне тізбектеп жуықтау әдісін қолдана отырып, бөлшек-сызықтық q-дифференциалдық теңдеулердің нақты шешімдерін құрамыз. Сол сияқты бізде $\alpha > 0$ нақты ретті Капутоның бөлшек q-туындысы бар жалпы біртекті бөлшек q-дифференциалдық теңдеу бар және біз басқа q-функциясын береміз (Миттаг-Леффлер). Соңында, біз осы нақты шешімдерге нақты формулалар бере алатын жағдайларда негізгі нәтижелерімізді көрсететін бірнеше мысалдар келтірілген.
Түйін сөздер: Коши типтес q-бөлшек есеп, бар болу, жалғыз болу, q-туынды, q-есептеу, бөлшек есептеу, бөлшек туынды, Капутоның бөлшек туындысы.

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Задачи Коши для q-разностных уравнений с дробными производными Капуто

Уравнения с дробными производными играют важную роль из-за их многочисленных применений, а также из-за той важной роли, которую они играют не только в математике, но и в других науках. В данной исследовательской работе мы строим явные решения дробно-линейных q-дифференциальных уравнений с q-дробной производной Капуто действительного порядка $\alpha > 0$. Точнее говоря, мы достигнем наших основных результатов, используя то, что эта q-дробная задача типа Коши эквивалентна соответствующему q-интегральному уравнению Вольтерра. После этого, применяя метод последовательных приближений к решению q-интегрального уравнения Вольтерра, строим явные решения дробно-линейных q-дифференциальных уравнений. Таким же образом у нас есть более общее однородное дробное q-дифференциальное уравнение с дробной q-производной Капуто действительного порядка $\alpha > 0$, и мы даем другую q-функцию (Миттаг-Леффлера). Наконец, представлены некоторые примеры, иллюстрирующие наши основные результаты в тех случаях, когда мы даже можем дать конкретные формулы для этих явных решений.

Ключевые слова: q-дробная задача типа Коши, существование, единственность, q-производная, q-исчисление, дробное исчисление, дробная производная, дробные производные Капуто.

1 Introduction

The fractional calculus is the field of mathematics that investigates the integration and differentiation of real or complex orders. The fractional differential equations based on the Caputo fractional derivative require initial conditions for integer order derivatives. Consequently, fractional differential equations have grasped the interest of many researchers working in diverse applications [1]- [10]. Recently, there has been a significant development in ordinary and partial differential equations involving fractional derivatives and a huge amount of papers and also some books devoted to this subject in various spaces have appeared, see e.g. the monographs of T. Sandev and Z. Tomovski [7], A.A. Kilbas et al. [8], R. Hilfer [9], K.S. Miller and the B. Ross [11], the papers [12], [13], [14], [15], [16], [17], [18], [19] and [20] and the references therein.

The origin of the q-difference calculus can be traced back to the works in [21, 22] by F. Jackson and R.D. Carmichael [23] from the beginning of the twentieth century. For more interesting theory results and scientific applications of the q-difference calculus, we cite the monographs [24–26] and the references therein. In the last decades, the fractional q-difference calculus has been proposed by W. Al-salam [27] and R.P. Agarwal [28] and P.M. Rajkovic', S.D. Marinkovic', and M.S. Stankovic [29]. Recently, many researchers got much interested in looking at fractional q-differential equations (FDEs) as new model equations for many physical problems. For example, some researchers obtained q-analogues of the integral and differential fractional operators properties such as the q-Laplace transform and q-Taylor's formula [30], q-Mittage Leffler function [28] and so on.

However, the theory of the q-difference equations with constant and variable coefficients is still in the initial stages and many aspects of this theory need to be explored. For some recent developments on the subject, see e.g. [31], [32], [33], [34] and the references therein. To the best of our knowledge, the theory of the Cauchy problem for linear, homogeneous and nonhomogeneous q-difference equations based the basic Caputo fractional derivative is yet to be developed.

Motivated by this, we discuss to construct the explicit solution to linear fractional q-differential equation with the Caputo fractional q-derivative ${}^cD^{\alpha}_{q,0+}$ order of $\alpha > 0$ in the following form (see Definition 2.2):

$$\left({}^cD^{\alpha}_{q,0+}y\right)(x) - \lambda y(x) \;=\; f\left(x\right), \quad 0 \leq a < x \leq b, \alpha > 0; \lambda \in \mathbb{R}, \tag{1}$$

with the initial conditions

$$D^k_q(y(0+)) = b_k, b_k \in \mathbb{R}, k = 0, 1, 2, \ldots, n = -[-\alpha], \tag{2}$$

where $f \in C_{q,\lambda}[a, b]$ (see 2.8) with $0 \leq \gamma \leq 1$, $\gamma \leq \alpha$ and where $[\alpha]$ denotes the smallest integer greater or equal to α. Moreover, we consider the Cauchy problem for the following more general homogeneous fractional q-differential equation than

$$\left({}^cD^{\alpha}_{q,0+}y\right)(x) - \lambda x^{\beta} y(x) \;=\; 0, \quad 0 \leq a < x \leq b, \alpha > 0; \lambda \in \mathbb{R}, \tag{3}$$

with the initial conditions

$$D^k_q(y(0+)) = b_k, \quad b_k \in \mathbb{R}, k = 0, 1, 2, \ldots, n = -[-\alpha], \tag{4}$$

with $\beta > -\alpha$.

In Section 3 of this paper we construct explicit solutions to linear fractional q-differential equations with the Caputo fractional q-derivative ${}^cD^{\alpha}_{q,a+}f$ of order $\alpha > 0$ given by Definition 2.2. in the space $C^{\alpha,n-\alpha}_{q,\gamma}[0, a]$, denned in (13). The main result in this Section is Theorems 2.1 but in order to prove this result we need to prove two results (Theorem 3.1 and 3.3) of independent interest.

The paper is organized as follows: The main results are presented and proved in subsection 3 and the announced examples are given in subsection 4. In order to not disturb these presentations we include in Section 1 some necessary Preliminaries.

2 Preliminaries

First we recall some elements of q-calculus, for more information see e.g. the books [24], [26] and [34]. Throughout this paper, we assume that $0 < q < 1$ and $0 \leq a < b < \infty$.

Let $\alpha \in \mathbb{R}$. Then a q-real number $[\alpha]_q$ is defined by

$$[\alpha]_q = \frac{1 - q^{\alpha}}{1 - q},$$

where $\lim\limits_{q \to 1} \frac{1-q^{\alpha}}{1-q} = \alpha$.

We introduce for $k \in \mathbb{N}$:

$$(a; q)_0 = 1,\ (a; q)_n = \prod_{k=0}^{n-1} \left(1 - q^k a\right),\ (a; q)_{\infty} = \lim_{n \to \infty} (a, q)_n,\ (a; q)_{\alpha} = \frac{(a; q)_{\infty}}{(q^{\alpha} a; q)_{\infty}}.$$

For any two real numbers α and β, we have

$$(a - b)^{\alpha}_q \left(a - q^{\alpha} b\right)^{\beta}_q = (a - b)^{\alpha+\beta}_q. \tag{5}$$

The q-analogue of the binomial coefficients $[n]_q!$ are defined by

$$[n]_q! = \begin{cases} 1, & \text{if n} = 0, \\ [1]_q \times [2]_q \times \cdots \times [n]_q, & \text{if n} \in \text{N}, \end{cases}$$

The gamma function $\Gamma_q(x)$ is defined by

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x},$$

for any $x > 0$. Moreover, it yields that

$$\Gamma_q(x)[x]_q = \Gamma_q(x+1).$$

The q-analogue differential operator $D_q f(x)$ is

$$D_q f(x) = \frac{f(x) - f(qx)}{x(1-q)},$$

and the q-derivatives $D_q^n(f(x))$ of higher order are defined inductively as follows:

$$D_q^0(f(x)) = f(x), \quad D_q^n(f(x)) = D_q\left(D_q^{n-1} f(x)\right), (n = 1, 2, 3, \dots)$$

The q-integral (or Jackson integral) $\int_0^a f(x) d_q x$ is defined by

$$\int_0^a f(x) d_q x := (1-q)a \sum_{m=0}^{\infty} q^m f(aq^m)$$

and

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x,$$

for $0 < a < b$. Notice that

$$\int_a^b D_q f(x) d_q x = f(b) - f(a).$$

For any $t, s > 0$ the definition of q-Beta function is that:

$$B_q(t,s) := \frac{\Gamma_q(t)\Gamma_q(s)}{\Gamma_q(t+s)} := \int_0^1 x^{t-1}(qx;q)_{s-1} d_q x \tag{6}$$

The (Mittag-Leffler) q-function $E_{q,\alpha,\beta}(z)$ is defined by

$$E_{\alpha,\beta,a}\left[zx^\alpha(a/x;q)_\alpha;q\right] := \sum_{k=0}^{\infty} \frac{z^k x^{k\alpha}(a/x;q)_{k\alpha}}{\Gamma_q(\alpha k+\beta)} \tag{7}$$

and

$$E_{\alpha,m,l}\left[z;q\right] := \sum_{k=0}^{\infty} c_k z^k \tag{8}$$

where c_0 and $c_k = \prod_{j=0}^{k-1} \frac{\Gamma_q[\alpha(jm+l)+1]}{\Gamma_q[\alpha(jm+l+1)+1]}(k \in \mathbb{N})$.

A q-analogue of the classical exponential function e^x is

$$e_q^x := \sum_{j=0}^{\infty} \frac{x^j}{[j]!}. \tag{9}$$

Moreover, the multiple q-integral $\left(I^n_{q,a+}f\right)(x)$ is

$$\begin{aligned}\left(I^n_{q,a+}f\right)(x) &= \int_a^x \int_a^t \int_a^{t_{n-1}} \ldots \int_a^{t_2} d_qt_1 d_qt_2 \ldots d_qt_{n-1} d_qt \\ &= \frac{x^{n-1}}{\Gamma_q(n)} \int_a^x (qt/x;q)_{n-1} f(t) d_qt.\end{aligned}$$

Definition 1 *The Riemann-Liouville q-fractional integrals $I^\alpha_{q,a+}f$ of order $\alpha > 0$ are defined by*

$$\left(I^\alpha_{q,0+}f\right)(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x;q)_{\alpha-1} f(t) d_qt.$$

Definition 2 *The Caputo fractional q-derivative ${}^cD^\alpha_{q,a+}f$ of order $\alpha > 0$ is defined as*

$$\left({}^cD^\alpha_{q,a+}f\right)(x) := \left(I^{[\alpha]-\alpha}_{q,a+} D^{[\alpha]}_{q,a+} f\right)(x).$$

Notice that

$$\left(I^\alpha_{q,a+} x^\lambda (a/x;q)_\lambda\right)(x) = \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\alpha+\lambda+1)} x^{a+\lambda} (a/x;q)_{\alpha+\lambda}, \tag{10}$$

for $\lambda \in (-1, \infty)$.

For $1 \le p < \infty$ we define the space $L^p_q = L^p_q[a,b]$ by

$$L^p_q[a,b] := \left\{ f : \left(\int_a^b |f(x)|^p d_qx \right)^{\frac{1}{p}} < \infty \right\}.$$

Let $\alpha > 0$, $\beta > 0$ and $1 \le p < \infty$. Then the q–fractional integration has the following semigroup property

$$\left(I^\alpha_{q,a+} I^\beta_{q,a+} f\right)(x) = \left(I^{\alpha+\beta}_{q,a+} f\right)(x), \tag{11}$$

for all $x \in [a,b]$ and $f(x) \in L_q^p[a,b]$.

Let $0 < a < b < \infty$, $0 \le \lambda \le 1$ and $n \in \mathbb{N}$. Then we introduce the spaces $C_{q,\lambda}[a,b]$ and $C_q^n[a,b]$ of functions f given on $[a,b]$, such that the functions with the norms,respectively

$$\|f\|_{C_{q,\lambda}[a,b]} \;:=\; \max_{x\in[a,b]} \left|x^{\lambda}\left(qa/x;q\right)_{\lambda} f(x)\right| < \infty. \tag{12}$$

$$\|f\|_{C_q^n[a,b]} \;:=\; \sum_{k=0}^{n} \max_{x\in[a,b]} \left|D_q^k f(x)\right| < \infty.$$

and the space $C_{q,\lambda}^{\alpha,n}[0,a]$ defined for $n-q < \alpha \le n$, $n \in \mathbb{N}$ by

$$C_{q,\lambda}^{\alpha,n}[0,a] := \{f(x) : f(x) \in C_q^n[a,b], \left({}^{c}D_{q,a+}^{\alpha} f\right)(x) \in C_{q,\lambda}[a,b]\}. \tag{13}$$

In the classical case several authors have considered such problems even in linear cases, see e.g. [8, subection 4.1.3] and the references therein.

Theorem 2.1 *(See [35, Theorem 8.1]) Let $n-1 < \alpha \le n; n \in \mathbb{N}$, G be an open set in $\mathbb{R}$ and $f(.,.) : (0,a] \times G \to \mathbb{R}$ be a function such that $F(x,y(x)) = f(x) + \lambda y(x) \in L_q^1[0,a]$ for any $y \in G$. If $y(x) \in L_q^1[0,a]$, then $y(t)$ satisfies a.e. the relations (1)-(2) if and only if $y(x)$ satisfies a.e. the integral equation*

$$y(x) := \sum_{k=0}^{n-1} \frac{b_k}{[k]_q!} x^{\alpha-k} + \left(I_{q,0}^{\alpha} f(t,y(t))\right)(x), \forall x \in (0,a]. \tag{14}$$

3 The Main results

3.1 The Cauchy Problems for q-difference equation with the Caputo fractional q-derivative

In this section we construct the explicit solution to linear fractional q-difference equations (1) and with the initial conditions (2). From here we obtain the following result.

Theorem 3.1 *Let $n-1 < \alpha < n (n \in \mathbb{N})$ and $0 \le \gamma < 1$ be such that $\gamma \geqq \alpha$. Also let $\lambda \in \mathbb{R}$. If $f(x) \in C_{q,\gamma}[0,a]$, the Cauchy problem (1)-(2) has a unique solution $y(x) \in C_{q,\gamma}^{\alpha,n-\alpha}[0,a]$ and this solution is given by*

$$\begin{aligned} y(x) \;&:=\; \sum_{k=0}^{n-1} b_k x^k E_{\alpha,k+1,0}\left(\lambda x^{\alpha};q\right) \\ &+ \int_0^x x^{\alpha-1}\left(qt/x;q\right)_{\alpha-1} E_{\alpha,\alpha,t}\left(\lambda x^{\alpha}(q^{\alpha}t/x;q)_{\alpha};q\right) f(t) d_q t. \end{aligned} \tag{15}$$

Proof. First, we solve the Volterra q-integral equation (14), we apply the method of successive approximations by setting

$$y_0(x) = \sum_{k=0}^{n-1} \frac{b_k}{[k]_q!} x^k$$

and

$$\begin{aligned} y_i(x) &= y_0(x) + \frac{\lambda x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x;q)_{\alpha-1} y_{i-1}(t) d_q t \\ &+ \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x;q)_{\alpha-1} f(t) d_q t \end{aligned} \tag{16}$$

Using Definition 2.1 and (10) and (16) we find $y_1(x)$:

$$y_1(x) = y_0(x) + \lambda \left(I^{\alpha}_{q,0+} y_0\right)(x) + \left(I^{\alpha}_{q,0+} f\right)(x)$$

that is,

$$\begin{aligned} y_1(x) &= \sum_{k=0}^{n-1} \frac{b_k}{[k]_q!} x^k + \lambda \sum_{k=0}^{n-1} \frac{b_k}{[k]_q!} \left(I^{\alpha}_{q,0+} t^k\right)(x) + \left(I^{\alpha}_{q,0+} f\right)(x) \\ &= \sum_{k=0}^{n-1} \frac{b_k}{[k]_q!} x^k + \lambda \sum_{k=1}^{n-1} \frac{b_k x^{\alpha+k}}{\Gamma_q(\alpha+k+1)} + \left(I^{\alpha}_{q,0+} f\right)(x) \\ &= \sum_{k=0}^{n-1} b_k \sum_{m=0}^{1} \frac{\lambda^m x^{m\alpha+k}}{\Gamma_q(\alpha m+k+1)} + \left(I^{\alpha}_{q,0+} f\right)(x). \end{aligned} \tag{17}$$

Similarly, using Definition 2.1 and (10), (11) and (17) we have for $y_2(x)$ that

$$\begin{aligned} y_2(x) &= y_0(x) + \lambda \left(I^{\alpha}_{q,0+} y_1\right)(x) + \left(I^{\alpha}_{q,0+} f\right)(x) \\ &= \sum_{k=0}^{n-1} \frac{b_k}{[k]_q!} x^k + \sum_{k=0}^{n-1} b_k \sum_{m=0}^{1} \frac{\lambda^{m+1}}{\Gamma_q(\alpha m+k+1)} \left(I^{\alpha}_{q,0+} t^{m\alpha+k}\right)(x) \\ &+ \lambda \left(I^{\alpha}_{q,0+} I^{\alpha}_{q,0+} f(t)\right)(x) + \left(I^{\alpha}_{q,0+} f\right)(x) \\ &= \sum_{k=0}^{n-1} \frac{b_k}{[k]_q!} x^k + \sum_{k=0}^{n-1} b_k \sum_{m=0}^{1} \frac{\lambda^{m+1}}{\Gamma_q(\alpha(m+1)+k+1)} x^{\alpha(m+1)+k} \\ &+ \lambda \left(I^{2\alpha}_{q,0+} f(t)\right)(x) + \left(I^{\alpha}_{q,0+} f\right)(x) \\ &= \sum_{k=0}^{n-1} \frac{b_k}{[k]_q!} x^k + \sum_{k=0}^{n-1} b_k \sum_{m=0}^{1} \frac{\lambda^{m+1}}{\Gamma_q(\alpha(m+1)+k+1)} x^{\alpha(m+1)+k} \\ &+ \frac{\lambda x^{2\alpha-1}}{\Gamma(2\alpha)} \int_0^x f(t) (qt/x;q)_{2\alpha-1} d_q t + \left(I^{\alpha}_{q,0+} f\right)(x). \end{aligned} \tag{18}$$

Now for $\alpha m-1=\alpha(m-1)+\alpha-1$, using (5) we get

$$x^{\alpha m-1}(qt/x;q)_{\alpha m-1}=x^{\alpha-1}(qt/x;q)_{\alpha-1}x^{\alpha(m-1)}(q^{\alpha}t/x;q)_{\alpha(m-1)}. \tag{19}$$

Thus combined with (18) and (19). gives

$$\begin{aligned}
y_2(x) &= \sum_{k=0}^{n-1} b_k \sum_{m=0}^{2} \frac{\lambda^m x^{\alpha m+k}}{\Gamma_q(\alpha m+k+1)} \\
&+ \int_0^x \left[\sum_{m=1}^{2} \frac{\lambda^{m-1} x^{\alpha m-1}(qt/x;q)_{\alpha m-1}}{\Gamma_q(\alpha m)}\right] f(t)d_qt \\
&= \sum_{k=0}^{n-1} b_k \sum_{m=0}^{2} \frac{\lambda^m x^{\alpha m+k}}{\Gamma_q(\alpha m+k+1)} \\
&+ \int_0^x \left[\sum_{m=0}^{1} \frac{\lambda^{m} x^{\alpha(m+1)-1}(qt/x;q)_{\alpha(m+1)-1}}{\Gamma_q(\alpha(m+1))}\right] f(t)d_qt \\
&= \sum_{k=0}^{n-1} b_k \sum_{m=0}^{2} \frac{\lambda^m x^{\alpha m+k}}{\Gamma_q(\alpha m+k+1)} \\
&+ \int_0^x \left[\sum_{m=0}^{1} \frac{\lambda^{m} x^{\alpha-1}(qt/x;q)_{\alpha-1}x^{\alpha m}(q^{\alpha}t/x;q)_{\alpha m}}{\Gamma_q(\alpha(m+1))}\right] f(t)d_qt.
\end{aligned}$$

Continuing this process, we derive the following relation for $y_i(x)$:

$$\begin{aligned}
y_i(x) &= \sum_{k=0}^{n-1} b_k \sum_{m=0}^{i} \frac{\lambda^m x^{\alpha m+k}}{\Gamma_q(\alpha m+k+1)} \\
&+ \int_0^x \left[\sum_{m=1}^{i} \frac{\lambda^{m-1} x^{\alpha m-1}(qt/x;q)_{\alpha m-1}}{\Gamma_q(\alpha m)}\right] f(t)d_qt \\
&= \sum_{k=0}^{n-1} b_k \sum_{m=0}^{i} \frac{\lambda^m x^{\alpha m+k}}{\Gamma_q(\alpha m+k+1)} \\
&+ \int_0^x \left[\sum_{m=0}^{i-1} \frac{\lambda^{m} x^{\alpha(m+1)-1}(qt/x;q)_{\alpha(m+1)-1}}{\Gamma_q(\alpha(m+1))}\right] f(t)d_qt \\
&= \sum_{k=0}^{n-1} b_k \sum_{m=0}^{i} \frac{\lambda^m x^{\alpha m+k}}{\Gamma_q(\alpha m+k+1)} \\
&+ \int_0^x \left[\sum_{m=0}^{i-1} \frac{\lambda^{m} x^{\alpha-1}(qt/x;q)_{\alpha-1}x^{\alpha m}(q^{\alpha}t/x;q)_{\alpha m}}{\Gamma_q(\alpha(m+1))}\right] f(t)d_qt
\end{aligned}$$

.

Taking the limit as $i\to\infty$, we obtain the following explicit solution $y(x)$ to the q-integral equation (16):

$$y(x) = \sum_{k=0}^{n-1} b_k \sum_{m=0}^{\infty} \frac{\lambda^m x^{\alpha m+k}}{\Gamma_q(\alpha m+k+1)} + \int_0^x \left[\sum_{m=0}^{\infty} \frac{\lambda^{m} x^{\alpha-1}(qt/x;q)_{\alpha-1}x^{\alpha m}(q^{\alpha}t/x;q)_{\alpha m}}{\Gamma_q(\alpha m+\alpha)}\right] f(t)d_qt.$$

On the basis of Theorem 2.1 an explicit solution to the Volterra q-integral equation (14) and hence to the Cauchy type problem (1)-(2).

Theorem 3.2 *Let $n-1<\alpha<n$ $(n\in N)$ and let $0\leqq\gamma<1$ be such that $\gamma\leqq\alpha$. Also let $\lambda\in\mathbb{R}$. If $f(x)\in C_{q,\gamma}[0,a]$, the Cauchy problem (1)-(2) has a unique solution $y(x)\in C_{q,\gamma}^{\alpha,n-\alpha}[0,a]$ and this solution is given by (15).*

In particular, if $\gamma=0$ and $f(x)\in C_q[0,a]$, then the solution $y(x)$ in (15) belongs to the space $C_q^{\alpha,n-\alpha}[0,a]$ defined in (13).

The Cauchy problem 2 involving the homogeneous q-difference equation (1)

$$\left({}^cD_{q,0+}^{\alpha}y\right)(x)-\lambda y(x)=0 \quad (0\leqq x\leqq a; n-1<\alpha<n; n\in\mathbb{N}; \lambda\in\mathbb{R}) \tag{20}$$

has a unique solution $y(x)\in C_{q,\gamma}^{\alpha,n-\alpha}[0,a]$ of the from

$$y(x)=\sum_{k=0}^{n-1}b_kx^kE_{\alpha,k+1,0}\left[\lambda x^{\alpha};q\right]. \tag{21}$$

3.2 The Cauchy problem for the more general homogeneous fractional q-difference equation with the Caputo fractional q-derivative

Now we consider the Cauchy problem for the more general homogeneous fractional q-difference equation (3) with the initial conditions (4).

Theorem 3.3 *Let $n-1<\alpha<n;\quad(n\in\mathbb{N})$,and let $0\leq\gamma<1$,be such that $\gamma\leq\alpha$. Also let $\lambda\in\mathbb{R}$ and $\beta\geqq 0$. If $f\in C_{q,\gamma}[0,a]$, then the Cauchy problem (3)-(4) has a unique solution $y(x)$ in the space $C_{q,n-\alpha}^{\alpha}[0,a]$ and this solution is given by*

$$y(x) \;=\; \sum_{j=0}^{n-1}\frac{b_j}{[j]_q!}x^jE_{\alpha,1+\frac{\beta}{\alpha},1+\frac{(\beta+j)}{\alpha}}\left[\lambda x^{\alpha+\beta};q\right]. \tag{22}$$

Proof. With $\beta>-\alpha$. Note again that, in accordance with Theorem 3.1, the problem (3)-(4) is equivalent in the space $C_{q,n-1}[0,a]$ to the following Volterra q-integral equation of the second kind:

$$y(x)=\sum_{j=0}^{n-1}\frac{b_j}{\Gamma_q(j+1)}x^j+\frac{\lambda x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x t^{\beta}\left(qt/x;q\right)_{\alpha-1}y(t)d_qt. \tag{23}$$

Similarity, we again apply the method of successive approximations to solve this q-integral equation (23). We assume that $y_0(x)=\sum\limits_{j=0}^{n-1}\frac{b_j}{[j]_q!}x^j$ and

$$y_m(x)=y_0(x)+\frac{\lambda x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x t^{\beta}\left(qt/x;q\right)_{\alpha-1}y_{m-1}(t)d_qt. \tag{24}$$

Using the same arguments as above, by using (5), (6) and (24) we find $y_1(x)$:

$$\begin{aligned}
y_1(x) &= y_0(x) + \frac{\lambda x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x t^{\beta} \left(qt/x;q\right)_{\alpha-1} y_0(x) d_qt \\
&= y_0(x) + \frac{\lambda}{\Gamma_q(\alpha)} \sum_{j=0}^{n-1} \frac{b_j x^{\alpha-1}}{[j]_q!} \int_0^x t^{\beta+j} \left(qt/x;q\right)_{\alpha-1} d_qt \\
&= y_0(x) + \frac{\lambda}{\Gamma_q(\alpha)} \sum_{j=0}^{n-1} \frac{b_j x^{\alpha+\beta+j}}{[j]_q!} \int_0^1 y^{\beta+j} \left(qy;q\right)_{\alpha-1} d_qy \\
&= \sum_{j=0}^{n-1} \frac{b_j}{[j]_q!} x^j + \frac{\lambda}{\Gamma_q(\alpha)} \sum_{j=0}^{n-1} \frac{b_j x^{\alpha+\beta+j}}{[j]_q!} B_q\left(\beta+j+1,\alpha\right) \\
&= \sum_{j=0}^{n-1} \frac{b_j}{[j]_q!} x^j \\
&+ \lambda \sum_{j=0}^{n-1} \frac{b_j x^{\alpha+\beta+j}}{\Gamma_q(j+1)} \frac{\Gamma_q(\beta+j+1)}{\Gamma_q(\alpha+\beta+j+1)}.
\end{aligned} \tag{25}$$

Similarly, for $m=2$ using (6), (24) and taking (25) into account, we derive

$$\begin{aligned}
y_2(x) &= y_0(x) + \lambda \left(I^{\alpha}_{q,0+} y_1\right)(x) \\
&= y_0(x) + \frac{\lambda}{\Gamma_q(\alpha)} \sum_{j=0}^{n-1} \frac{b_j}{[j]_q!} x^{\alpha-1} \int_0^x t^{\beta+j} \left(qt/x;q\right)_{\alpha-1} d_qt \\
&+ \frac{\lambda^2}{\Gamma_q(\alpha)} \sum_{j=0}^{n-1} \frac{b_j}{[j]_q!} \frac{\Gamma_q(\beta+j+1)}{\Gamma_q(\alpha+\beta+j+1)} x^{\alpha-1} \int_0^x t^{\alpha+2\beta+j} \left(qt/x;q\right)_{\alpha-1} d_qt \\
&= y_0(x) + \frac{\lambda}{\Gamma_q(\alpha)} \sum_{j=0}^{n-1} \frac{b_j}{[j]_q!} x^{\beta+j+\alpha} \int_0^1 y^{\beta+j} \left(qy;q\right)_{\alpha-1} d_qy \\
&+ \frac{\lambda^2}{\Gamma_q(\alpha)} \sum_{j=0}^{n-1} \frac{b_j}{[j]_q!} x^{2\alpha+2\beta+j} \frac{\Gamma_q(\beta+j+1)}{\Gamma_q(\alpha+\beta+j+1)} \int_0^1 y^{\alpha+2\beta+j} \left(qy;q\right)_{\alpha-1} d_qt \\
&= y_0(x) + \frac{\lambda}{\Gamma_q(\alpha)} \sum_{j=0}^{n-1} \frac{b_j}{[j]_q!} x^{\beta+j+\alpha} B_q\left(\beta+j+1,\alpha\right) \\
&+ \frac{\lambda^2}{\Gamma_q(\alpha)} \sum_{j=0}^{n-1} \frac{b_j}{[j]_q!} x^{2\alpha+2\beta+j} \frac{\Gamma_q(\beta+j+1)}{\Gamma_q(\alpha+\beta+j+1)} B_q\left(\alpha+2\beta+j+1,\alpha\right) \\
&= \sum_{j=0}^{n-1} \frac{b_j x^j}{[j]_q!} \left[1 + \lambda x^{\alpha+\beta} \frac{\Gamma_q(\beta+j+1)}{\Gamma_q(\alpha+\beta+j+1)}\right] \\
&+ \sum_{j=0}^{n-1} \frac{b_j x^j}{[j]_q!} \left[\lambda^2 x^{2\alpha+2\beta} \frac{\Gamma_q(\beta+j+1)}{\Gamma_q(\alpha+\beta+j+1)} \frac{\Gamma_q(\alpha+2\beta+j+1)}{\Gamma_q(2\alpha+2\beta+j+1)}\right].
\end{aligned}$$

The same arguments as in Section 3.1 lead to the following expression for $y_m(x)$ $m \in \mathbb{N}$:

$$y_m(x) = \sum_{j=0}^{n-1} \frac{b_j}{\Gamma_q(j+1)} t^j \left[1 + \sum_{k=1}^{m} d_k \left(\lambda t^{\alpha+\beta} \right)^k \right], \tag{26}$$

where

$$d_k = \prod_{r=1}^{k} \frac{\Gamma_q[r(\alpha+\beta)-\alpha+j+1]}{\Gamma_q[r(\alpha+\beta)+j+1]}; \quad (k \in \mathbb{N}). \tag{27}$$

Taking the limit as $m \to \infty$, we obtain the following explicit solution $y(x)$ to the q-integral equation (24) and hence to the Cauchy type problem (3)-(4):

$$y(x) = \sum_{j=0}^{n-1} \frac{b_j}{\Gamma_q(j+1)} t^j \left[1 + \sum_{k=1}^{\infty} d_k \left(\lambda t^{\alpha+\beta} \right)^k \right], \tag{28}$$

According to the relations (8), we rewrite this solution in terms of the generalized Mittag-Leffler q-function $E_{\alpha,m,l}[z;q]$:

$$y(x) = \sum_{j=0}^{n-1} \frac{b_j}{\Gamma_q(j+1)} x^j E_{\alpha,1+\frac{\beta}{\alpha},\frac{(\beta+j)}{\alpha}} \left[\lambda x^{\alpha+\beta}; q \right]. \tag{29}$$

If $\beta \geq 0$, then $f[x,y] = \lambda(t)^\beta$ satisfies the Lipschitz condition for any $x_1, x_2 \in (a,b]$ and any $y \in G$, where G is any open set of $\mathbb{C}$. If $\gamma \geq n-\alpha$, then, by Property 3.l(b) and Remark 3.18, there exists a unique solution to the Cauchy type problem (1)-(2) in the space $C^{\alpha}_{n-\alpha}$, and thus this solution has the form (29). This leads to the following result.

4 A Set of Examples

Example 1 *Let $b \in \mathbb{R}$. Then the solution to the Cauchy type problem*

$$\left({}^cD^{\alpha}_{q,0+} y \right)(x) - \lambda y(x) = f(x), y(0+) = b, \tag{30}$$

with $0 < \alpha < 1$ and $\lambda \in \mathbb{R}$ has the form:

$$\begin{aligned} y(x) &= bE_{\alpha,0}\left[\lambda x^{\alpha}; q\right] \\ &+ x^{\alpha-1} \int_0^x (qt/x;q)_{\alpha-1} E_{\alpha,\alpha,t}\left[\lambda x^{\alpha}(q^{\alpha}t/x;q)_{\alpha}; q\right] f(t) d_q t \end{aligned} \tag{31}$$

while the solution to the problem

$$\left({}^cD^{\alpha}_{q,0+} y \right)(x) - \lambda y(x) = 0, y(0+) = b, \tag{32}$$

is given by

$$y(x) = bE_{\alpha,0}\left[\lambda x^{\alpha}; q\right] \tag{33}$$

In particular, the Cauchy type problem

$$\left({}^{c}D_{q,0+}^{\frac{1}{2}}y\right)(x) - \lambda y(x) = f(x), y(0+) = b, \tag{34}$$

has the solution given by

$$\begin{aligned} y(x) &= bE_{\frac{1}{2},0}\left[\lambda x^{\frac{1}{2}}; q\right] \\ &+ \int_0^x E_{\frac{1}{2},\frac{1}{2},t}\left[\lambda x^{\frac{1}{2}}(q^{\frac{1}{2}}t/x;q)_{\frac{1}{2}}; q\right] \frac{f(t)}{x^{\frac{1}{2}}(qt/x;q)_{\frac{1}{2}}} d_qt \end{aligned} \tag{35}$$

and the solution to the problem

$$\left({}^{c}D_{q,0+}^{\frac{1}{2}}y\right)(x) - \lambda y(x) = 0, y(0+) = b, \tag{36}$$

is given by

$$y(x) = bE_{\frac{1}{2},0}\left[\lambda x^{\frac{1}{2}}; q\right] \tag{37}$$

Example 2 *Let $b, d \in \mathbb{R}$. Then the solution to the Cauchy type problem*

$$\left({}^{c}D_{q,0+}^{\alpha}y\right)(x) - \lambda y(x) = f(x), y(0+) = b, y'(0+) = d, \tag{38}$$

with $1 < \alpha < 2$ and $\lambda \in \mathbb{R}$ has the form:

$$\begin{aligned} y(x) &= bE_{\alpha,0}\left[\lambda x^{\alpha}; q\right] + dxE_{\alpha,2,t}\left[\lambda x^{\alpha}; q\right] \\ &+ x^{\alpha-1}\int_0^x (qt/x;q)_{\alpha-1} E_{\alpha,\alpha,t}\left[\lambda x^{\alpha}(q^{\alpha}t/x;q)_{\alpha}\right] f(t) d_qt \end{aligned} \tag{39}$$

In particular, the solution to the problem $(1 < \alpha < 2)$

$$\left({}^{c}D_{q,0+}^{\alpha}y\right)(x) - \lambda y(x) = 0, y(0+) = b, y'(0+) = d, \tag{40}$$

is given by

$$y(x) = bE_{\alpha,0}\left[\lambda x^{\alpha}; q\right] + dxE_{\alpha,2,t}\left[\lambda x^{\alpha}; q\right]. \tag{41}$$

Example 3 *let $b \in \mathbb{R}$. Then the solution to the Cauchy type problem*

$$\left({}^{c}D^{\alpha}_{q,0+}y\right)(x) - \lambda x^{\beta} y(x) = 0, y(0+) = b, \tag{42}$$

with $0 < \alpha < l$, $\beta \in \mathbb{R}(\beta > -\alpha)$ and $\lambda \in \mathbb{R}$ is given by

$$y(x) = bE_{\alpha,1+\frac{\beta}{\alpha},1+\frac{(\beta)}{\alpha}}\left[\lambda x^{\alpha+\beta}; q\right]. \tag{43}$$

In particular, the Cauchy type problem

$$\left({}^{c}D^{\frac{1}{2}}_{q,0+}y\right)(x) - \lambda x^{\beta} y(x) = 0, y(0+) = b, \tag{44}$$

with $\beta > -\frac{1}{2}$ is given by

$$y(x) = bE_{\frac{1}{2},2\beta+1,2\beta}\left[\lambda x^{\beta+\frac{1}{2}}; q\right]. \tag{45}$$

Example 4 *Let $b, d \in \mathbb{R}$. Then the solution to the Cauchy type problem*

$$\left({}^{c}D^{\alpha}_{q,0+}y\right)(x) - \lambda x^{\beta} y(x) = 0, y(0+) = b, y'(0+) = d, \tag{46}$$

with $1 < \alpha < 2$, $\beta > -\alpha$ and $\lambda \in \mathbb{R}$ has the form

$$\begin{aligned} y(x) &= bE_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left[\lambda x^{\alpha+\beta}; q\right] \\ &+ dxE_{\alpha,1+\frac{\beta}{\alpha},1+\frac{(\beta+1)}{\alpha}}\left[\lambda x^{\alpha+\beta}; q\right]. \end{aligned} \tag{47}$$

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