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ON QUASI-IDENTITIES OF FINITE MODULAR LATTICES

In 1970 R. McKenzie proved that any finite lattice has a finite basis of identities. However the similar result for quasi-identities is not true. That is, there is a finite lattice that has no finite basis of quasi-identities. The problem "Which finite lattices have finite bases of quasi-identities?" was suggested by V.A. Gorbunov and D.M. Smirnov. In 1984 V.I. Tumanov found a sufficient condition consisting of two parts under which a locally finite quasivariety of lattices has no finite (independent) basis of quasi-identities. Also he conjectured that a finite (modular) lattice has a finite basis of quasi-identities if and only if a quasivariety generated by this lattice is a variety. In general, the conjecture is not true. W. Dziobiak found a finite lattice that generates a finitely axiomatizable proper quasivariety. Tumanov's problem is still unsolved for modular lattices. We construct a finite modular lattice that does not satisfy one of Tumanov's conditions but the quasivariety generated by this lattice is not finitely based.
Key words: Lattice, quasivariety, finite basis of quasi-identities.

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Соңғы модулярлық торлардың квази-сәйкестіктері туралы

1970 жылы Р. Маккензи кез-келген соңғы тордың түпкілікті сәйкестендіру негізі бар екенін дәлелдеді. Алайда, квази-сәйкестендіру үшін ұқсас нәтиже дұрыс емес. Яғни, квази-сәйкестендірудің түпкілікті негізі жоқ соңғы тор бар. Мәселе "Квази-сәйкестендірудің соңғы негіздері қандай соңғы торларға ие?" В.А. Горбунов және Д.М. Смирнов ұсынды.
1984 жылы В.И. Туманов екі бөліктен тұратын жеткілікті жағдайды тапты: жергілікті түрде, соңғы квазикөпбейне торларда квази-сәйкестендірудің соңғы (тәуелсіз) негізі жоқ.
Сондай-ақ, ол ақырғы (модулярлық) тордың квази-сәйкестендірудің соңғы негізі бар деп ұсынды содан кейін және тек осы тордан пайда болған квазикөпбейне бұл көпбейне. Жалпы жағдайда гипотеза дұрыс емес. В. Дзебяк ақырлы торды тапты, ол аксиоматизацияланатын өзіндік квазикөпбейнені тудырады. Тумановтың мәселесі әлі де модулярлық торлар үшін шешілген жоқ. Біз Тумановтың бір жағдайын қанағаттандырмайтын соңғы модулярлық торды саламыз, бірақ осы тордан пайда болған квазикөпбейненің түпкі негізі жоқ.
Түйін сөздер: Тор, квазикөпбейне, квази-сәйкестіктердің соңғы базисі.

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О квазитождествах конечных модулярных решеток

В 1970 году Р. Маккензи доказал, что любая конечная решетка имеет конечный базис тождеств. Однако аналогичный результат для квазитождеств неверен. То есть существует конечная решетка, которая не имеет конечного базиса квазитождеств. Проблема "Какие конечные решетки имеют конечные базисы квазитождеств?" была предложена В.А. Горбуновым и Д.М. Смирновым.

В 1984 году В.И. Туманов нашел достаточное условие, состоящее из двух частей, при котором локально конечное квазимногообразие решеток не имеет конечного (независимого) базиса квазитождеств. Также он предположил, что конечная (модулярная) решетка имеет конечный базис квазитождеств тогда и только тогда, когда квазимногообразие, порожденное этой решеткой, является многообразием. В общем случае гипотеза неверна. В. Дзебяк нашел конечную решетку, которая порождает конечно аксиоматизируемое собственное квазимногообразие. Проблема Туманова до сих пор не решена для модулярных решеток. Мы строим конечную модулярную решетку, которая не удовлетворяет одному из условий Туманова, но квазимногообразие, порожденное этой решеткой, не является конечно базируемым.
Ключевые слова: Решетка, квазимногообразие, конечный базис квазитождеств.

1 Introduction

Questions concerning finite basability are among the most researched and relevant topics in universal algebra. It is well known that the finite based results begin with R.C. Lyndon, who in 1951 proved that the algebras on a two-element universe are always finitely based. R. McKenzie [1] in 1970 established that every finite lattice is finitely based, and generalizing this result, K.A. Baker in 1976 proved that every finite algebra generating a congruence-distributive variety is finitely based. There are two major directions in which Baker's theorem was generalized. In congruence-modular direction there was a series of results by R. Freese and R. McKenzie, the final result by McKenzie published in 1987 states that every finite algebra generating a congruence-modular residually finite variety is finitely based. In congruence meet-semidistributive direction, R. Willard in 2000 proved that every finite algebra generating a congruence meet-semidistributive residually strictly finite variety is finitely based.

Thus, according to R. McKenzie, any finite lattice has a finite basis of identities. The similar result for quasi-identities is not true, that was established by V.P. Belkin [2]. In 1979 he proved that there is a finite lattice that has no finite basis of quasi-identities. In particular, the smallest lattice that does not have a finite basis of quasi-identities is the ten-element modular lattice M_{3-3}. In this regard, the following question naturally arises. Which finite lattices have finite bases of quasi-identities? This problem was suggested by V.A. Gorbunov and D.M. Smirnov [3] in 1979. V.I. Tumanov [4] in 1984 found sufficient condition consisting of two parts under which the locally finite quasivariety of lattices has no finite (independent) basis for quasi-identities. Also he conjectured that a finite (modular) lattice has a finite basis of quasi-identities if and only if a quasivariety generated by this lattice is a variety. In general, the conjecture is not true. W. Dziobiak [5] found a finite lattice that generates finitely axiomatizable proper quasivariety. Also we would like to point out that Tumanov's problem is still unsolved for modular lattices.

The main goal of the paper is to present a finite modular lattice that does not satisfy one of Tumanov's conditions but the quasivariety generated by this lattice is not finitely based (has no finite basis of quasi-identities).

2 Material and methods

We recall some basic definitions and results for quasivarieties that we will refer to. For more information on the basic notions of general algebra introduced below and used throughout this paper, we refer to [6] and [7].

A *quasivariety* is a class of lattices that is closed with respect to subalgebras, direct products, and ultraproducts. Equivalently, a quasivariety is the same thing as a class of lattices axiomatized by a set of quasi-identities. A *quasi-identity* means a universal Horn sentence with the non-empty positive part, that is of the form

$$(\forall \bar{x})[p_1(\bar{x}) \approx q_1(\bar{x}) \wedge \dots \wedge p_n(\bar{x}) \approx q_n(\bar{x}) \rightarrow p(\bar{x}) \approx q(\bar{x})]$$

where $p, q,\ p_1, q_1, \dots, p_n, q_n$ are lattice's terms. A *variety* is a quasivariety which is closed under homomorphisms. According to Birkhoff theorem [8], a variety is a class of similar algebras axiomatized by a set of identities, where by an identity we mean a sentence of the form $(\forall \bar{x})[s(\bar{x}) \approx t(\bar{x})]$ for some terms $s(\bar{x})$ and $t(\bar{x})$.

By $\mathbf{Q}(\mathbf{K})$ ($\mathbf{V}(\mathbf{K})$) we denote the smallest quasivariety (variety) containing a class $\mathbf{K}$. If $\mathbf{K}$ is a finite family of finite algebras then $\mathbf{Q}(\mathbf{K})$ is called finitely generated. In case when $\mathbf{K} = \{\mathcal{A}\}$ we write $\mathbf{Q}(\mathcal{A})$ instead of $\mathbf{Q}(\{\mathcal{A}\})$.

Let $\mathbf{K}$ be a quasivariety. A congruence α on algebra $\mathcal{A}$ is called a $\mathbf{K}$-*congruence* or *relative congruence* provided $\mathcal{A}/\alpha \in \mathbf{K}$. The set $\mathrm{Con}_{\mathbf{K}}\mathcal{A}$ of all $\mathbf{K}$-congruences of $\mathcal{A}$ forms an algebraic lattice with respect to inclusion $\subseteq$ which is called a *relative congruence lattice*.

The least $\mathbf{K}$-congruence $\theta_{\mathbf{K}}(a, b)$ on algebra $\mathcal{A} \in \mathbf{K}$ containing pair $(a, b) \in A \times A$ is called a *principal* $\mathbf{K}$-*congruence* or a *relative principal congruence*. In case when $\mathbf{K}$ is a variety, relative congruence $\theta_{\mathbf{K}}(a, b)$ is usual principal congruence that we denote by $\theta(a, b)$.

An algebra $\mathcal{A}$ belonging to a quasivariety $\mathbf{K}$ is *(finitely) subdirectly irreducible relative to* $\mathbf{K}$, or *(finitely) subdirectly* $\mathbf{K}$-*irreducible*, if intersection of any (finite) number of nontrivial $\mathbf{K}$-congruences is again nontrivial; in other words, the trivial congruence 0_A is a (meet-irreducible) completely meet-irreducible element of $\mathrm{Con}_{\mathbf{K}}\mathcal{A}$.

Let $(a] = \{x \in L \mid x \leq a\}$ ($[a) = \{x \in L \mid x \geq a\}$) be a principal ideal (coideal) of a lattice $\mathcal{L}$. A pair $(a, b) \in L \times L$ is called *dividing* (*semi-dividing*) if $L = (a] \cup [b)$ and $(a] \cap [b) = \varnothing$ ($L = (a] \cup [b)$ and $(a] \cap [b) \neq \varnothing$).

For any semi-dividing pair (a, b) of a lattice $\mathcal{M}$ we define a lattice

$$\mathcal{M}_{a-b} = \langle \{(x, 0), (y, 1) \in M \times 2 \mid x \in (a], y \in [b)\}; \vee, \wedge \rangle \leq_s \mathcal{M} \times \mathbf{2},$$

where $\mathbf{2} = \langle \{0, 1\}; \vee, \wedge \rangle$ is a two element lattice.

Theorem 1 (Tumanov's theorem [4]) *Let* $\mathbf{M}$, $\mathbf{N}$ *(*$\mathbf{N} \subset \mathbf{M}$*) be locally finite quasivarieties of lattices satisfying the following conditions:*

a) in any finitely subdirectly $\mathbf{M}$-*irreducible lattice* $\mathcal{M} \in \mathbf{M} \backslash \mathbf{N}$ *there is a semi-dividing pair* (a, b) *such that* $\mathcal{M}_{a-b} \in \mathbf{N}$*;*

b) there exists a finite simple lattice $\mathcal{P} \in \mathbf{N}$ *which is not a proper homomorphic image of any subdirectly* $\mathbf{N}$-*irreducible lattice.*

Then the quasivariety $\mathbf{N}$ *has no coverings in the lattice of subquasivarieties of* $\mathbf{M}$*. In particular,* $\mathbf{N}$ *has no finite basis of quasi-identities provided* $\mathbf{M}$ *is finitely axiomatizable.*

In the next section, the algebra $\mathcal{L}$ and its carrier (its main set) L will be identified and denoted by the same way, namely L.

3 Results and discussion

Let T be a modular lattice displayed in Figure 1. And let $\mathbf{N} = \mathbf{Q}(T)$ and $\mathbf{M} = \mathbf{V}(T)$ be the quasivariety and variety generated by T, respectively. Since every subdirectly $\mathbf{N}$-irreducible lattice is a sublattice of T, we have that a class $\mathbf{N}_{si}$ of all subdirectly $\mathbf{N}$-irreducible lattices consists of the lattices $\mathbf{2}$, M_3, M_{3-3} and T (see Figures 1 and 2). It easy to see that M_3 is a unique simple lattice in $\mathbf{N}_{si}$ and is a homomorphic image of T. Thus, the condition a) of Tumanov's theorem is not valid for quasivarieties $\mathbf{N} \subset \mathbf{M}$. We show

Theorem 2 *Quasivariety $\mathbf{Q}(T)$ generated by the lattice T is not finitely based.*

To prove the theorem we modify the proof of the second part of Theorem 3.4 from [9].

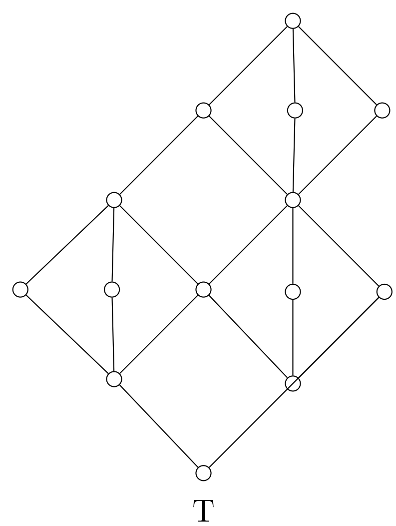

Figure 1: Lattice T

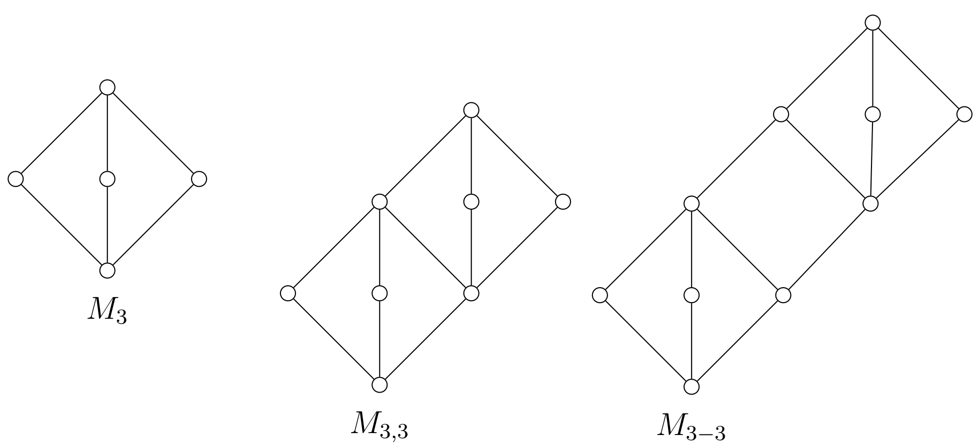

Figure 2: Lattices M_3, $M_{3,3}$ and M_{3-3}

Let S be a non-empty subset of a lattice L. Denote by $\langle S\rangle$ the sublattice of L generated by S.

We define a modular lattice L_n by induction:

$n = 1$. $L_1 \cong M_{3-3}$ and $L_1 = \langle\{a_1, b_1, c_1, e, d\}\rangle$ (see Figure 3);

$n = 2$. L_2 is a modular lattice generated by $L_1 \cup \{a_2, b_2, c_2, d\}$ such that $b_1 = c_2$, $\langle\{a_2, b_2, c_2, e, b_1\}\rangle \cong M_3$, and $a_2 \vee b_2 = e \wedge d_1$, $d \vee b_1 = d_1$, and $b_2 < d$ (see Figure 3).

$n > 2$. L_n is a modular lattice generated by the set $\{a_i, b_i, c_i \mid i \leq n\} \cup \{e, d\}$ such that a_i is not comparable with a_j and b_k for all $j \neq i$ and $k \leq n$, $b_{i-1} = c_i$, $\langle\{a_i, b_i, c_i\}\rangle \cong M_3$ for all $i < n$, $b_i \vee d = d_i$ for all $i < n$, and $b_n < d$ (see Figure 4).

One can see that L_n is a subdirect product of the lattices L_{n-1} and M_3 for any $n > 2$.

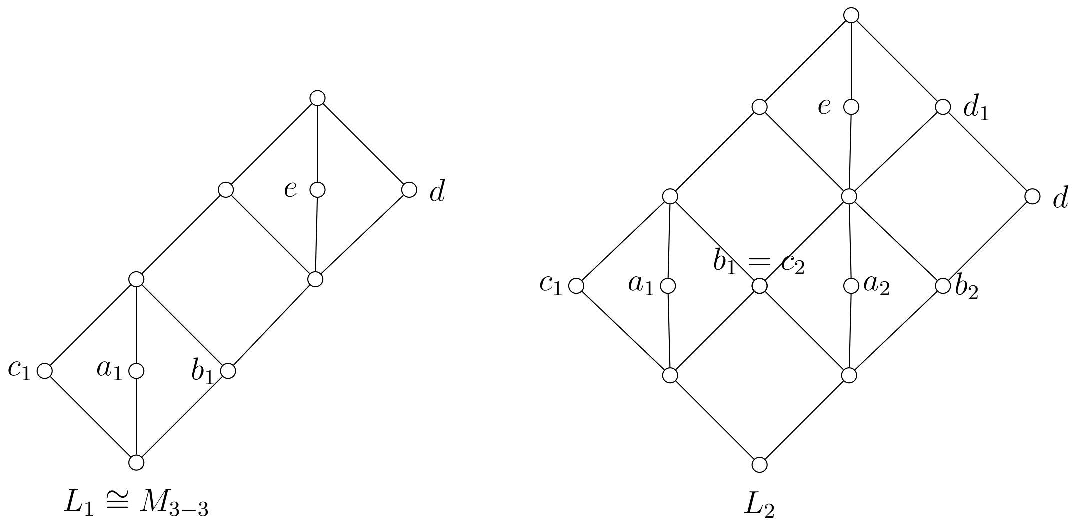

Figure 3: Lattices L_1, L_2

Let L_n^- be a sublattice of L_n generated by the set $\{a_i, b_i, c_i \mid i \leq n\}$.

Lemma 1 *For any $n > 1$ and a non-trivial congruence $\theta \in \mathrm{Con}L_n$ there is $1 < m < n$ such that $L_n/\theta \cong L_m$ or $L_n/\theta \cong M_{3,3}$ provided $(a_1, b_1) \notin \theta$, otherwise $L_n/\theta \cong L_m^-$.*

Proof of Lemma 1.

We prove by induction on $n > 2$. One can check that it is true for $n = 3$ because of $L_3/\theta \cong L_2$ or $L_3/\theta \cong M_{3,3}$ if $(a_1, b_1) \notin \theta$ and $L_3/\theta \cong L_2^-$ or $L_3/\theta \cong M_3$ for any non-trivial congruence $\theta \in \mathrm{Con}L_3$.

Let $n > 3$. And let u cover v in L_n and $\theta(u, v) \subseteq \theta$. By construction of L_n, we have $L_n/\theta(u, v) \cong L_{n-1}$ or $L_n/\theta(u, v) \cong L_{n-1}^-$.

Assume $(a_1, b_1) \notin \theta$. Since for every non-trivial congruence $\theta \in \mathrm{Con}L_n$ there are $u, v \in L_n$ such that u covers v and $\theta(u, v) \subseteq \theta$, we get

$$L_n/\theta \cong (L_n/\theta(u, v))/(\theta/\theta(u, v)).$$

Since $L_n/\theta(u, v) \cong L_{n-1}$ we obtain

$$L_n/\theta \cong (L_n/\theta(u, v))/(\theta/\theta(u, v)) \cong L_{n-1}/\theta',$$

for some $\theta' \in \mathrm{Con}(L_{n-1})$. And, by induction, $L_{n-1}/\theta' \cong L_m$ or $L_{n-1}/\theta' \cong M_{3,3}$ for some $m > 0$. Thus $L_n/\theta \cong L_m$ or $L_n/\theta \cong M_{3,3}$.

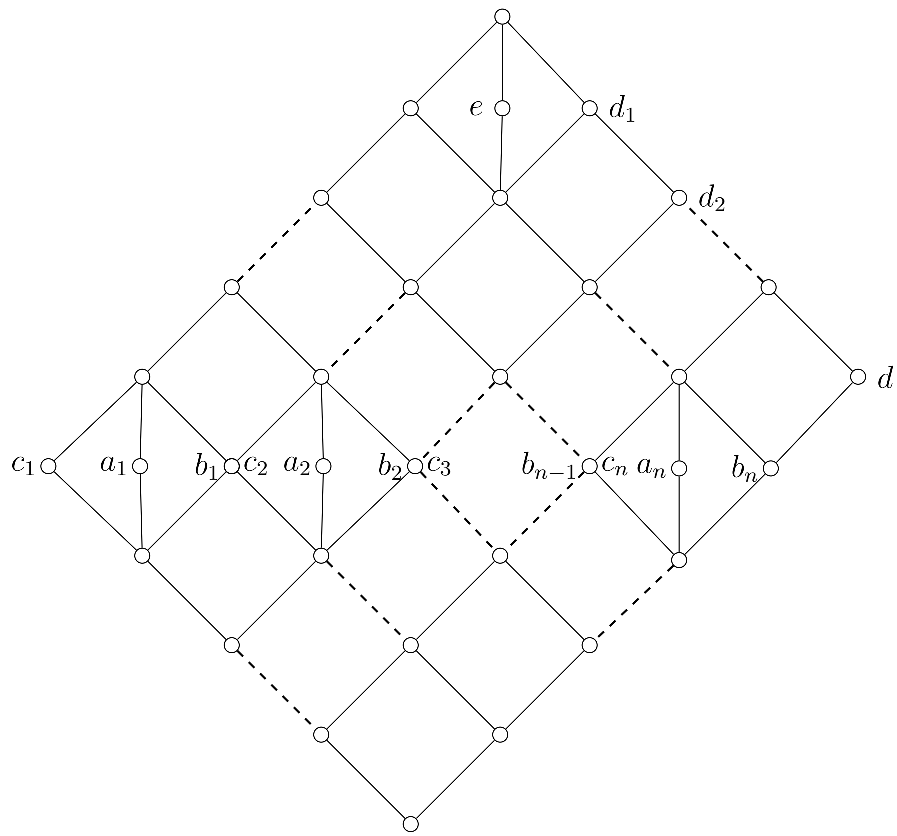

Figure 4: Lattice L_n, $n \geq 2$

Now assume $(a_1, b_1) \in \theta$. Then $\theta(a_1, b_1) = \theta(u, v)$ and $L_n/\theta(u, v) \cong L_n^-$. Hence

$$L_n/\theta \cong (L_n/\theta(u, v))/(\theta/\theta(u, v)) \cong L_n^-/\theta',$$

for some $\theta' \in \mathrm{Con}(L_n^-)$. It is not difficult to check that $L_n^-/\theta' \cong L_m^-$ for some $m > 0$ (see Lemma 3.1 [9]). Thus $L_n/\theta \cong L_m$ or $L_n/\theta \cong L_m^-$.

Corollary 1 *For all $n > 1$, there is no proper homomorphism from L_n to M_{3-3} and T.*

Proof of Corollary 1.

We provide the proof for a proper homomorphism from L_n into M_{3-3}. It is not difficult to check that the same arguments hold for a proper homomorphism from L_n into T.

Assume $h : L_n \to M_{3-3}$, $n > 1$, is a proper homomorphism. Hence $\ker h$ is not a trivial congruence on L_n. By Lemma 1, $L_n/\ker h \cong L_m$ or $L_n/\theta \cong M_{3,3}$ or $L_n/\ker h \cong L_m^-$ for some $m > 1$. Thus $L_m = h(L_n) \leq M_{3-3}$. It is impossible because, by definition of L_m, $|L_m| > |M_{3-3}|$ for all $m > 1$, hence L_n is not a sublattice of M_{3-3}. Obviously, $M_{3,3}$ and L_M^- are not sublattices of M_{3-3}. Thus there is no such homomorphism h.

Lemma 2 *For every $n > 2$, a lattice L_n has the following properties:*

i) $L_n \leq_s L_{n-1} \times L_{n-1}$;

ii) $L_n \in \mathbf{V}(M_{3,3}) = \mathbf{V}(T)$;

iii) $L_n \notin \mathbf{Q}(T)$;

iv) Every proper subalgebra of L_n *belongs to* $\mathbf{Q}(T)$.

Proof of Lemma 2.

i). One can check that $L_n/\theta(a_i, b_i) \cong L_{n-1}$ for all $1 < i \leq n$. Since $n > 2$ then $\theta(a_2, b_2), \theta(a_3, b_3) \in \mathrm{Con}L_n$ and $\theta(a_2, b_2) \cap \theta(a_3, b_3) = \Delta$. This means that $L_n \leq_s L_{n-1} \times L_{n-1}$.

ii). One can see that T is a subdirect product of M_3 and $M_{3,3}$. Hence $T \in \mathbf{V}(M_{3,3})$. On the other hand, by Jonsson lemma [10], every subdirectly irreducible lattice in $\mathbf{V}(T)$ is a homomorphic image of some sublattice of T. Hence $M_{3,3} \in \mathbf{V}(T)$. Thus $\mathbf{V}(M_{3,3}) = \mathbf{V}(T)$, and, by *i)* and induction on n, we get $L_n \in \mathbf{V}(T)$.

iii). Suppose $L_n \in \mathbf{Q}(T)$ for some $n > 1$. Then L_n is a subdirect product of subdirectly $\mathbf{Q}(T)$-irreducible algebras. Since every subdirectly $\mathbf{Q}(T)$-irreducible algebra is a subalgebra of T, we get that L_n is a subdirect product of subalgebras of T. By Lemma 1, there is no proper homomorphism from L_n onto T or M_{3-3}. Hence $L_n \in \mathbf{Q}(M_3)$ for all $n > 1$. It is impossible because $M_{3-3} \leq L_n$ and $M_{3-3} \notin \mathbf{Q}(M_3)$.

iv). We prove by induction on n. It is true for $n \leq 2$ by manual checking. Let $n > 2$ and let S be a maximal sublattice of L_n. Since the lattice L_n is generated by the set of double irreducible elements $\{a_1, \ldots, a_n, c_1, e, d\}$, there is $0 < i \leq n$ such that $a_i \notin S$ or $c_1 \notin S$ or $e \notin S$ or $d \notin S$.

Suppose $c_1 \notin S$. One can see that $\langle S \rangle \leq_s \mathbf{2} \times M_3 \times L^-_{n-1}$. Since $L_{n-1} \leq_s M_3^{n-1}$ we get $\langle S \rangle \in \mathbf{Q}(M_3) \subset \mathbf{Q}(T)$.

Suppose $e \notin S$. Then $\langle S \rangle \leq_s \mathbf{2} \times L^-_n \leq_s \mathbf{2} \times M_3^n \in \mathbf{Q}(M_3) \subset \mathbf{Q}(T)$.

Suppose $d \notin S$. Put $S_m = \{\{a_1, \ldots, a_m, c_1, e\}$, $m < n$, and $T_m = \langle S_m \rangle$. One can see that $T_m/\theta(a_i, b_i) \cong T_{m-1}$ for all $1 < i < m$. And $T_m/\theta(a_1, b_1) \cong L^-_{m-1}$. Since $\theta(a_1, b_1) \cap \theta(a_i, b_i) = \Delta$, by distributivity of $\mathrm{Con}T_m$, we have $\theta(a_1, b_1) \cap (\bigvee\{\theta(a_i, b_i) \mid 1 < i < m\}) = \Delta$. Since $T_m/(\bigvee\{\theta(a_i, b_i) \mid 1 < i < m\}) \cong T$ we obtain $\langle S_m \rangle \leq_s T \times L^-_{n-1} \leq_s T \times M_3^{n-1} \in \mathbf{Q}(T)$.

Suppose $a_i \notin S$. Since $n > 1$ and S is a maximal sublattice, then there are $i \neq k \neq l \neq i$ such that $\theta(b_k, c_k), \theta(b_l, c_l) \in \mathrm{Con}L_n$,

$$\theta(b_k, c_k) \cap \theta(b_l, c_l) = \Delta.$$

and

$$L_n/\theta(b_k, c_k) \cong L_n/\theta(b_l, c_l) \cong L_{n-1} \quad \text{or} \quad \{L_n/\theta(b_k, c_k), L_n/\theta(b_l, c_l)\} = \{L_{n-1}, L^-_{n-1}\}.$$

We provide the proof for the first case, $L_n/\theta(b_k, c_k) \cong L_n/\theta(b_l, c_l) \cong L_{n-1}$. These isomorphisms mean that $L_n \leq_s L_{n-1} \times L_{n-1}$ and $S \leq L_{n-1} \times L_{n-1}$. Let $h_k : L_n \to L_{n-1}$ and $h_l : L_n \to L_{n-1}$ are homomorphisms such that $\ker h_k = \theta(b_k, c_k)$ and $\ker h_l = \theta(b_l, c_l)$. Since $(a_i, b_i) \notin \theta(b_k, c_k) \cup \theta(b_l, c_l)$ then $h_k(S)$, $h_l(S)$ are proper sublattices of L_{n-1}. And, by induction, $h_k(S), h_l(S) \in \mathbf{Q}(T)$. As $b_k, c_k, b_l, c_l \in S$, the restrictions of congruences $\theta(b_k, c_k)|_S$ and $\theta(b_l, c_l)|_S$ on the algebra S are not trivial congruences on S. Moreover $\theta(b_k, c_k)|_S \cap \theta(b_l, c_l)|_S = \Delta$. It means $S \leq_s h_k(S) \times h_l(S)$. Hence $S \in \mathbf{Q}(T)$. Since every maximal proper subalgebra of L_n belongs to $\mathbf{Q}(T)$ then every proper subalgebra of L_n belongs to $\mathbf{Q}(T)$.

It is not difficult to check that for $\{L_n/\theta(b_k, c_k), L_n/\theta(b_l, c_l)\} = \{L_{n-1}, L^-_{n-1}\}$ the same arguments hold.

Now we prove the main result, Theorem 2.

We use the following folklore fact which provides non-finite axiomatizability: A locally finite quasivariety $\mathbf{K}$ is not finitely axiomatizable if for any positive integer $n \in N$ there is a finite algebra L_n such that $L_n \notin \mathbf{K}$ and every n-generated subalgebra of L_n belongs to $\mathbf{K}$.

We show that for quasivariety $\mathbf{Q}(T)$, the lattice L_n satisfies the conditions of this fact. Indeed, by Lemma 2(iii), $L_n \notin \mathbf{Q}(T)$ for all $n > 1$. Since L_n is generated by at least $n+1$ double irreducible elements then every n-generated subalgebra of L_n is a proper subalgebra. By Lemma 2(iv), every n-generated subalgebra of L_n belongs to $\mathbf{Q}(T)$. Hence $\mathbf{Q}(T)$ has no finite basis of quasi-identities.

We note that there is an infinite number of lattices similar to the lattice T.

The proof of Theorem 2 give us more general result:

Theorem 3 *Suppose L is a finite lattice such that $M_{3,3} \not\leq L$, $T \leq L$ and $L_n \not\leq L$ for all $n > 1$. Then the quasivariety $\mathbf{Q}(L)$ is not finitely based.*

4 Conclusion

There are three measures of the highest complexity of the structure of quasivariety lattices: Q-universality, property (N) or non-computability of the set of finite sublattices, and an existence of continuum of quasivarieties without covers in a given quasivariety lattice. The presence in the quasivariety lattices of a continuum of elements that do not have coverings indicates the complexity of the structure of these lattices; in this case, there is a continuum of subquasivarieties of a given quasivariety $\mathbf{K}$ that do not have an independent basis of quasi-identities with respect to $\mathbf{K}$. In [11] a sufficient condition for a quasivariety $\mathbf{K}$ to be Q-universal, to have continuum many subclasses with the property (N), continuum many Q-universal subquasivarieties and continuum many subquasivarieties with no upper covers in the lattice $Lq(\mathbf{K})$ was provided. In [12] a sufficient condition for a class $\mathbf{K}$ to have continuum many subclasses with the property (N) but which are not Q-universal was established. In [13] it was proved that almost all known Q-universal quasivarieties contain classes having property (N).

In this paper we construct a finite modular lattice that does not satisfy one of Tumanov's conditions but the quasivariety generated by this lattice is not finitely based. It has no finite basis of quasi-identities.

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