IRSTI 27.31.44 DOI: https://doi.org/10.26577/JMMCS.2022.v113.i1.04

**A.G. Smadiyeva**
Institute of Mathematics and Mathematical Modeling, Kazakhstan, Almaty
e-mail: smadiyeva@math.kz

# Initial-boundary value problem for the time-fractional degenerate diffusion equation

In this paper the initial-boundary value problems for the one-dimensional linear time-fractional diffusion equations with the time-fractional derivative $\partial_t^\alpha$ of order $\alpha \in (0,1)$ in the variable $t$ and time-degenerate diffusive coefficients $t^\beta$ with $\beta \geq 1-\alpha$ are studied. The solutions of initial-boundary value problems for the one-dimensional time-fractional degenerate diffusion equations with the time-fractional derivative $\partial_t^\alpha$ of order $\alpha \in (0,1)$ in the variable $t$, are shown. The second section present Dirichlet and Neumann boundary value problems, and in the third section has shown the solutions of the Dirichlet and Neumann boundary value problem for the one-dimensional linear time-fractional diffusion equation. The solutions of these fractional diffusive equations are presented using the Kilbas-Saigo function $E_{\alpha,m,l}(z)$. The solution of the problems is discovered by the method of separation of variables, through finding two problems with one variable. The existence and uniqueness to the solution of the problem are confirmed. In addition, the convergence of the solution has been proven using the estimate for the Kilbas-Saigo function $E_{\alpha,m,l}(z)$ from [13] and Parseval's identity.
**Key words:** Time-fractional diffusion equation, the method of separation variables, the Kilbas-Saigo function.

А.Г. Смадиева
Математика және Математикалық Модельдеу институты, Қазақстан, Алматы қ.
e-mail: smadiyeva@math.kz

**Бөлшек ретті туындылы өзгешеленген диффузия теңдеулері үшін бастапқы шеттік есебі**

Бұл жұмыста $t^\beta$, $\beta \geq 1-\alpha$ диффузиялық коэффициенттері бар бір өлшемді сызықты $\alpha \in (0,1)$ үшін $\partial_t^\alpha$ бөлшек ретті туындылы өзгешеленген диффузия теңдеулеріне қойылған бастапқы - шеттік есептері қарастырылған. Бір өлшемді сызықты $t$ айнымалысына тәуелді $\alpha \in (0,1)$ үшін $\partial_t^\alpha$ бөлшек ретті туындылы өзгешеленген диффузия теңдеулеріне қойылған бастапқы - шеттік есептерінің шешімдері көрсетілген. Екінші бөлімінде Дирихле және Нейман шеттік есептері берілген, ал үшінші бөлімінде бір өлшемді сызықты бөлшек ретті туындылы өзгешеленген диффузия теңдеулері үшін Дирихле және Нейман шеттік есептерінің шешімдері көрсетілген. Бұл бөлшек ретті туындылы өзгешеленген диффузиялық теңдеулердің шешімдері $E_{\alpha,m,l}(z)$ Килбас-Сайго функциясы арқылы берілген. Есептердің шешімдері айнымалысын ажырату әдісін қолданып, бір айнымалысы бар екі есепті шешу арқылы табылған. Есептің шешімінің бар болуы мен жалғыздығы дәлелденген. Шешімнің жинақтылығы Килбас-Сайго $E_{\alpha,m,l}(z)$ функциясының [13] көрсетілгендей бағалауын және Парсевал теңдігін қолдану арқылы дәлелденді.

**Түйін сөздер:** Бөлшек ретті туындылы өзгешеленген диффузия теңдеуі, айнымалыларын ажырату әдісі, Килбас-Сайго функциясы.

А.Г. Смадиева
Иститут Математики и Математического Моделирования, Казахстан, г. Алматы
e-mail: smadiyeva@math.kz

**Начально-краевая задача для дробных вырожденных диффузионных уравнений**

В данной работе рассматриваются начально-краевые задачи для одномерных дробных вырожденных линейных диффузионных уравнений с дробной производной $\partial_t^\alpha$ порядка $\alpha \in (0,1)$ по переменной $t$ и с вырождающимися коэффициентами диффузии $t^\beta$ при $\beta \geq 1-\alpha$. Показаны решения начально-краевых задач для одномерных уравнений вырождающейся диффузии с дробной по времени производной $\partial_t^\alpha$ порядка $\alpha \in (0,1)$ по переменной $t$. Во второй части даны краевые задачи Дирихле и Неймана, а в третьей части показаны решения краевых задач Дирихле и Неймана для одномерного дробного вырожденного линейного диффузионного уравнения. Решения этих дробных диффузионных уравнений представлены с помощью функции Килбаса-Сайго $E_{\alpha,m,l}(z)$. Решение задач получено с помощью метода разделения переменных, путем нахождения двух задач с одной переменной. Доказаны существование и единственность решения задач. Сходимости решения доказано с помощью оценки функции Килбаса-Сайго $E_{\alpha,m,l}(z)$ из [13] и тождество Парсеваля.

**Ключевые слова:** Дробно-вырожденное диффузионное уравнение, метод разделения переменных, функция Килбаса-Сайго.

## 1 Introduction

Over the past several millennia, fractional partial differential equations have begun to play an important role. They are used in modeling anomalous phenomena and in the theory of complex systems [1-6].

In the book [7], it is written about various applications of differential equations of fractional order in chemistry, technology, physics, etc. It contains research related to the equation of fractional diffusion in time. It is obtained from the classical diffusion equation by replacing the first-order time derivative with a fractional derivative.

In [8-11], the correctness and numerical modeling of thermal and wave equations with nonlocal conditions in time were studied. In [12] paper, authors consider the initial-boundary value problems of Dirichlet and Neumann for the diffusion equation in a variable coefficient. In [14], Nakhusheva proved a positive maximum principle for a nonlocal parabolic equation with Riemann– Liouville derivative. In [16], Luchko proved the maximum principle for the generalized diffusion equation with a fractional time derivative using the maximum principle for the Caputo fractional derivative. Then the maximum principle was applied to show some results of uniqueness and existence for the initial-boundary value problem of the fractional diffusion equation. In [15] Luchko studied initial-boundary value problems for a generalized diffusion equation with a distributed order. And [17] studied initial-boundary value problems for a fractional-fold diffusion equation in time. Thus, he obtained results on the existence of generalized solutions in [15, 17]. In [18], the generalized solution of the initial-boundary value problem for the diffusion equation with fractional time was shown as a regular solution. In [19], Gorenflo and Mainardi studied the one-dimensional diffusion-wave equation with fractional time.

In this paper, the initial-boundary value problems of Dirichlet and Neumann for the time-fractional diffusion equation in a variable coefficient are considered. The solution of the problems has been found by using the Kilbas-Saigo function and by the method of separation of variables. The existence and uniqueness of the solution are also proved.

## 2 Material and methods

### 2.1 Cauchy-Dirichlet problem

Let us consider the one-dimensional time-fractional diffusion equation

$$\partial_t^\alpha u(x,t) - t^\beta u_{xx}(x,t) = 0, \;\; (x,t) \in (0,1) \times (0,\infty), \tag{1}$$

with the Dirichlet boundary condition

$$u(0,t) = u(1,t) = 0, \;\; t \geq 0, \;\; x \in [0,1], \tag{2}$$

and the Cauchy initial condition

$$u(x,0) = \phi(x), \;\; x \in [0,1], \tag{3}$$

where $\partial_t^\alpha$ is the time-fractional derivative of order $\alpha \in (0,1)$ in the variable $t$ and $\beta \geq 1-\alpha$

$$\partial_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \partial_s u(x,s) ds.$$

Let $X_k(x) = sin\pi kx$ are the orthonormal eigenfunctions and $\lambda_k = (\pi k)^2$ the corresponding eigenvalues of the Sturm-Liouville operator with the Dirichlet boundary and the Cauchy initial conditions.

$H^2(0,1)$ is a Hilbert space defined by the initial-boundary

$$H^2(0,1) = \{u : \;\; u \in L^2(0,1); \;\; u_{xx} \in L^2(0,1)\},$$

endowed with the norm

$$||u||^2_{H^2(0,1)} = \sum_{k=1}^{\infty} \lambda_k^2 |(u, X_k(x))|^2 < \infty.$$

Definition 1. The solution of problem (1)-(3) is $u(x,t) \in C(L^2(0,1), R_+)$, such that satisfies $t^{-\beta}\partial_t^\alpha u, \;\; u_{xx} \in C(L^2(0,1), R_+)$.

### 2.2 Cauchy-Neumann problem

Let us consider the time-fractional diffusion equation

$$\partial_t^\alpha u(x,t) - t^\beta u_{xx}(x,t) = 0, \;\; (x,t) \in (0,1) \times (0,\infty), \tag{4}$$

with the Neumann boundary condition

$$u_x(0,t) = u_x(1,t) = 0, \;\; t \geq 0, \;\; x \in [0,1], \tag{5}$$

supplemented with the initial data

$$u(x,0) = \phi(x), \;\; x \in [0,1], \tag{6}$$

where $\partial_t^\alpha$ is the time-fractional fractional derivative of order $\alpha \in (0,1)$ in the variable $t$ and $\beta \geq 1-\alpha$.

Definition 2. The solution of problem (4)-(6) is $u(x,t) \in C(L^2(0,1), R_+)$, which satisfies $t^{-\beta}\partial_t^\alpha u, \;\; u_{xx} \in C(L^2(0,1), R_+)$.

## 3 Main results

**Theorem 1** *Let $\phi(x) \in H^2(0,1)$, then the unique solution of problem (1)-(3) is the function $u(x,t) \in C(L^2(0,1), R_+)$, which has the form*

$$u(x,t) = \sum_{k=1}^{\infty} \phi_k E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\pi^2 k^2 t^{\beta+\alpha}) sin\pi kx, \tag{7}$$

*where*

$$\phi_k = 2\int_0^1 \phi(x) sin\pi kx dx,$$

*and $E_{\alpha,m,l}(z)$ is the Kilbas-Saigo function([5] defined as*

$$E_{\alpha,m,l}(z) = \sum_{k=1}^{\infty} c_k z^k, \;\; c_0 = 1, \;\; c_k = \Pi_{j=0}^{k-1} \frac{\Gamma(\alpha(jm+l)+1)}{\Gamma(\alpha(jm+l+1)+1)}, \;\; k \geq 1. \tag{8}$$

*And for the function $E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\pi^2 k^2 t^{\beta+\alpha})$ the following estimate holds ([13])*

$$E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\pi^2 k^2 t^{\beta+\alpha}) \leq \frac{1}{1+\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\pi^2 k^2 t^{\beta+\alpha}}, \;\; t > 0. \tag{9}$$

**Theorem 2** *Let $\phi(x) \in H^2(0,1)$, then the unique solution of problem (4)-(6) is the function $u(x,t) \in C(L^2(0,1), R_+)$, which given by*

$$u(x,t) = \phi_0 + \sum_{k=1}^{\infty} \phi_k E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\pi^2 k^2 t^{\beta+\alpha}) cos\pi kx, \tag{10}$$

*where $\phi_0 = \int_0^1 \phi(x)dx$ and $\phi_k = 2\int_0^1 \phi(x) cos\pi kx dx$, $k \in N$ and $E_{\alpha,m,l}(z)$ is the Kilbas-Saigo function, which is defined by the formula (8)-(9).*

### 3.1 Proofs

Proof of Theorem 1

The existence of a solution. Since the Sturm-Liouville operator has eigenvalues $\{\lambda_k \geq 0, \;\; k \in N\}$ on $L^2(0,1)$ and the corresponding ortonormal eigenfunctions $\{X_k(x), \;\; k \in N\}$ in $L^2(0,1)$ and $\phi(x) \in H^2(0,1)$, then we can give the solution of problem (1)-(3) as follows

$$u(x,t) = \sum_{k=1}^{\infty} T_k(t) X_k(x), \;\; (x,t) \in (0,1) \times R_+, \tag{11}$$

$$\phi(x) = \sum_{k=1}^{\infty} \phi_k X_k(x), \;\; x \in (0,1), \tag{12}$$

where

$$\phi_k = 2\int_0^1 \phi(x) X_k(x) dx.$$

Substituting (11) in to the diffusion equation (1)-(3), we obtain the next problems

$$\partial_t^\alpha T_k(t) + \lambda_k t^\beta T_k(t) = 0, \;\; t > 0, \tag{13}$$

$$T_k(0) = \phi_k. \tag{14}$$

$$X_k''(x) + \lambda_k X_k(x) = 0, \tag{15}$$

$$X_k(0) = X_k(1) = 0. \tag{16}$$

The orthonormal eigenfunctions and the corresponding eigenvalues of the Dirichlet problem (15)-(16) are $X_k(x) = sin\pi kx$ and $\lambda_k = (\pi k)^2$, respectively.

The general solution of the Cauchy problem (13)-(14) is

$$T_k(t) = \phi_k E_{\alpha, 1+\frac{\beta}{\alpha}, \frac{\beta}{\alpha}}(-\pi^2 k^2 t^{\beta+\alpha}), \tag{17}$$

where

$$\phi_k = 2\int_0^1 \phi(x) sin\pi kx dx.$$

Substituting $X_k(x) = sin\pi kx$ orthonormal eigenfunctions and (17) to (11), we obtain the solution of problem (1)-(3) as

$$u(x,t) = \sum_{k=1}^{\infty} \phi_k E_{\alpha, 1+\frac{\beta}{\alpha}, \frac{\beta}{\alpha}}(-\pi^2 k^2 t^{\beta+\alpha}) sin\pi kx, \;\; (x,t) \in (0,1) \times [0,\infty). \tag{18}$$

Convergence of the solution. Using (9) to (17), we get

$$T_k(t) \le \frac{|\phi_k|}{1 + \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\pi^2 k^2 t^{\beta+\alpha}}.$$

By Parseval's identity, it follows from (18) that

$$\begin{aligned}
\sup_{t\ge 0} ||u(\cdot,t)||^2_{L^2(0,1)} &= \sup_{t\ge 0} \sum_{k=1}^{\infty} |\phi_k|^2 \left| E_{\alpha, 1+\frac{\beta}{\alpha}, \frac{\beta}{\alpha}}(-\pi^2 k^2 t^{\beta+\alpha}) \right|^2 ||sin\pi kx||^2_{L^2(0,1)} \\
&\le \sup_{t\ge 0} \sum_{k=1}^{\infty} \frac{|\phi_k|^2}{\left(1 + \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\pi^2 k^2 t^{\beta+\alpha}\right)^2} \\
&\le \sup_{t\ge 0} \frac{1}{\left(1 + \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\pi^2 t^{\beta+\alpha}\right)^2} \sum_{k=1}^{\infty} |\phi_k|^2 \\
&\le \sum_{k=1}^{\infty} |\phi_k|^2 = ||\phi(\cdot)||^2_{L^2(0,1)}.
\end{aligned} \tag{19}$$

Applying the operators $\partial_t^\alpha u$ and $u_{xx}$ to the identity (18) we obtain

$$\partial_t^\alpha u(x,t) = \sum_{k=1}^{\infty} \phi_k \partial_t^\alpha E_{\alpha, 1+\frac{\beta}{\alpha}, \frac{\beta}{\alpha}}(-\pi^2 k^2 t^{\beta+\alpha}) sin\pi kx$$

$$= -t^{\beta}\sum_{k=1}^{\infty}\pi^2k^2\phi_k E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\pi^2k^2t^{\beta+\alpha})sin\pi kx,\ \ (x,t)\in(0,1)\times[0,\infty), \tag{20}$$

and

$$u_{xx}(x,t)=\sum_{k=1}^{\infty}\phi_k E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\pi^2k^2t^{\beta+\alpha})sin''\pi kx$$

$$= -\sum_{k=1}^{\infty}\pi^2k^2\phi_k E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\pi^2k^2t^{\beta+\alpha})sin\pi kx,\ \ (x,t)\in(0,1)\times[0,\infty). \tag{21}$$

Applying (19)-(21) we get

$$\sup_{x\geq 0}||t^{-\beta}\partial_t^{\alpha}u(\cdot,t)||^2_{L^2(0,1)}\leq\sum_{k=1}^{\infty}\lambda_k^2|\phi_k|^2=||\phi(\cdot)||^2_{H^2(0,1)}<\infty,$$

and

$$\sup_{x\geq 0}||u_{xx}(\cdot,t)||^2_{L^2(0,1)}\leq\sum_{k=1}^{\infty}\lambda_k^2|\phi_k|^2=||\phi(\cdot)||^2_{H^2(0,1)}<\infty.$$

Uniqueness of the solution. Suppose that $u_1(x,t)$ and $u_2(x,t)$ are solutions to problem (1)-(3). We choose $u(x,t)=u_1(x,t)-u_2(x,t)$ in such a way, that $u(x,t)$ satisfies the diffusion equation (1) and boundary, initial condition (2), (3), respectively. Let us consider

$$T_k(t)=\int_0^1 u(x,t)sin\pi kxdx,\ \ k\in N, t\in[0,\infty). \tag{22}$$

Applying the operator $\partial_t^{\alpha}$ to the left-side of (22) equation by using (1) we obtain

$$\partial_t^{\alpha}T_k(t)=\int_0^1\partial_t^{\alpha}u(x,t)sin\pi kxdx$$

$$=t^{\beta}\int_0^1 u_{xx}(x,t)sin\pi kxdx$$

$$=t^{\beta}\int_0^1 u(x,t)sin''\pi kxdx$$

$$=-t^{\beta}\pi^2k^2\int_0^1 u(x,t)sin\pi kxdx$$

$$=-t^{\beta}\pi^2k^2T_k(t),\ \ k\in N, t\in[0,\infty).$$

Due to (2) and (3) we have

$$T_k(0)=0.$$

From the equation we get that $T_k(0)=0$, which means $u(x,t)\equiv 0$. Hence $u_1(x,t)=u_2(x,t)$, therefore the diffusion problem (1)-(3) has a unique solution.

Proof of Theorem 2

The existence of a solution. Since the Sturm-Liouville operator has eigenvalues $\{\lambda_k \geq 0,\ \ k \in N\}$ on $L^2(0,1)$ and the corresponding ortonormal eigenfunctions $\{X_k(x),\ \ k \in N\}$ in $L^2(0,1)$ and $\phi(x) \in H^2(0,1)$, then we write the solution of problem (4)-(6) as follows

$$u(x,t) = \sum_{k=1}^{\infty} T_k(t)X_k(x),\ \ (x,t) \in (0,1) \times R_+, \tag{23}$$

$$\phi(x) = \sum_{k=1}^{\infty} \phi_k X_k(x),\ \ x \in (0,1), \tag{24}$$

where

$$\phi_k = 2\int_0^1 \phi(x)X_k(x)dx$$

and

$$\phi_0 = \int_0^1 \phi(x)dx.$$

Substituting (23) in to the diffusion equation (4)-(6), we obtain the next problems

$$\partial_t^{\alpha} T_k(t) + \lambda t^{\beta} T_k(t) = 0,\ \ t > 0, \tag{25}$$

$$T_k(0) = \phi_k. \tag{26}$$

$$X_k''(x) + \lambda_k X_k(x) = 0, \tag{27}$$

$$X_k'(0) = X_k'(1) = 0. \tag{28}$$

The orthonormal eigenfunctions and the corresponding eigenvalues of the Neumann problem (27)-(28) are $X_k(x) = cos\pi kx$ and $\lambda_k = (\pi k)^2$, respectively.

The general solution of the Cauchy problem (25)-(26) is

$$T_k(t) = \phi_0 + \phi_k E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\pi^2 k^2 t^{\beta+\alpha}), \tag{29}$$

where

$$\phi_k = 2\int_0^1 \phi(x)cos\pi kx dx$$

and

$$\phi_0 = \int_0^1 \phi(x)dx.$$

Substituting $X_k(x) = cos\pi kx$ orthonormal eigenfunctions and (29) to (23), we obtain the solution of problem (4)-(6) as

$$u(x,t) = \phi_0 + \sum_{k=1}^{\infty} \phi_k E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\pi^2 k^2 t^{\beta+\alpha})cos\pi kx,\ \ (x,t) \in (0,1) \times [0,\infty). \tag{30}$$

Convergence of the solution. Using (9) to (29), we get

$$T_k(t) \leq |\phi_0| + \frac{|\phi_k|}{1 + \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\pi^2 k^2 t^{\beta+\alpha}}.$$

By Parseval's identity, it follows from (30) that

$$\sup_{t\geq 0}||u(\cdot,t)||^2_{L^2(0,1)} = \sup_{t\geq 0}\sum_{k=0}^{\infty}|\phi_k|^2\left|E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\pi^2k^2t^{\beta+\alpha})\right|^2||cos\pi kx||^2_{L^2(0,1)}$$

$$\leq \sup_{t\geq 0}\sum_{k=0}^{\infty}\frac{|\phi_k|^2}{\left(1+\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\pi^2k^2t^{\beta+\alpha}\right)^2}$$

$$\leq \sup_{t\geq 0}\frac{1}{\left(1+\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\pi^2t^{\beta+\alpha}\right)^2}\sum_{k=0}^{\infty}|\phi_k|^2$$

$$\leq \sum_{k=0}^{\infty}|\phi_k|^2 = ||\phi(\cdot)||^2_{L^2(0,1)}. \tag{31}$$

Applying the operators $\partial_t^\alpha u$ and $u_{xx}$ to the identity (30) we get

$$\partial_t^\alpha u(x,t) = \sum_{k=0}^{\infty}\phi_k\partial_t^\alpha E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\pi^2k^2t^{\beta+\alpha})cos\pi kx$$

$$= -t^\beta\sum_{k=0}^{\infty}\pi^2k^2\phi_kE_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\pi^2k^2t^{\beta+\alpha})cos\pi kx,\ \ (x,t)\in(0,1)\times[0,\infty), \tag{32}$$

and

$$u_{xx}(x,t) = \sum_{k=0}^{\infty}\phi_kE_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\pi^2k^2t^{\beta+\alpha})cos''\pi kx$$

$$= -\sum_{k=0}^{\infty}\pi^2k^2\phi_kE_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\pi^2k^2t^{\beta+\alpha})cos\pi kx,\ \ (x,t)\in(0,1)\times[0,\infty). \tag{33}$$

Applying (31)-(33) we get

$$\sup_{x\geq 0}||t^{-\beta}\partial_t^\alpha u(\cdot,t)||^2_{L^2(0,1)} \leq \sum_{k=0}^{\infty}\lambda_k^2|\phi_k|^2 = ||\phi(\cdot)||^2_{H^2(0,1)} < \infty,$$

and

$$\sup_{x\geq 0}||u_{xx}(\cdot,t)||^2_{L^2(0,1)} \leq \sum_{k=0}^{\infty}\lambda_k^2|\phi_k|^2 = ||\phi(\cdot)||^2_{H^2(0,1)} < \infty.$$

Uniqueness of the solution. Suppose that $u_1(x,t)$ and $u_2(x,t)$ are solutions to problem (4)-(6). We choose $u(x,t) = u_1(x,t) - u_2(x,t)$ in such a way, that $u(x,t)$ satisfies the diffusion equation (4) and boundary, initial condition (5), (6), respectively. Let us consider

$$T_k(t) = \int_0^1 u(x,t)cos\pi kxdx,\ \ k\in N, t\in[0,\infty). \tag{34}$$

Applying $\partial_t^\alpha$ to left-side (34) equation by using (4) we obtain

$$\partial_t^\alpha T_k(t) = \int_0^1 \partial_t^\alpha u(x,t) cos\pi kx dx$$

$$= t^\beta \int_0^1 u_{xx}(x,t) cos\pi kx dx$$

$$= t^\beta \int_0^1 u(x,t) cos''\pi kx dx$$

$$= -t^\beta \pi^2 k^2 \int_0^1 u(x,t) cos\pi kx dx$$

$$= -t^\beta \pi^2 k^2 T_k(t), \quad k \in N, t \in [0,\infty).$$

Due to (5) and (6) we have

$$T_k(0) = 0.$$

From the equation we get that $T_k(0) = 0$, which means that $u(x,t) \equiv 0$. Hence $u_1(x,t) = u_2(x,t)$, therefore the diffusion problem (4)-(6) has a unique solution.

## 4 Conclusions

In this research considered the initial-boundary value problems of Dirichlet and Neumann for the time-fractional diffusion equation in a variable coefficient. The solution of the problems has been found by using the Kilbas-Saigo function and by the method of separation of variables. The existence, uniqueness and convergence of solution were confirmed.

## 5 Acknowledgements

The author was supported by Ministry of Education and Science of the Republic of Kazakhstan (Grant № AP08052046).

### References

[1] Chechkin A.V., Gorenflo R., Sokolov I.M., *Fractional diffusion in inhomogeneous media,* [ J. Phys.] A 38 (2005) 679–684.

[2] Dubbeldam J.L.A., Milchev A., Rostiashvili V.G., Vilgis T.A., *Polymer translocation through a nanopore: A showcase of anomalous diffusion,* [Phys. Rev.] E 76 (2007) 010801 (R).

[3] Freed A., Diethelm K., Luchko Yu., Fractional-Order Viscoelasticity (FOV): Constitutive Development Using the Fractional Calculus, NASA's Glenn Research Center, Ohio, 2002.

[4] Gorenflo R., Mainardi F., "Random walk models for space-fractional diffusion processes", *Fract. Calc. Appl. Anal.* 1 (1998) 167–191.

[5] Kilbas A.A., Srivastava H.M., Trujillo J.J., "Theory and Applications of Fractional Differential Equations", *Elsevier, North-Holland. Mathematics studies,* 2006. 539 p.

[6] Podlubny I., "Fractional Differential Equations, Academic Press,", *San Diego,* 1999.

[7] Uchaikin V.V., "Method of Fractional Derivatives,", *Artishok, Ul'janovsk,* 2008 (in Russian).

[8] Byszewski L., Existence and uniqueness of solutions of nonlocal problems for hyperbolic equation uxt = F(x, t, u, ux), *J. Appl. Math. Stoch. Anal.* 3 (1990), 163–168.

[9] Chabrowski J., "On non-local problems for parabolic equations." , *Nagoya Math.* J. 93 (1984), 109–131.

[10] Dehghan M., "Implicit collocation technique for heat equationwith nonclassic initial condition." , *Appl. Math. Comput.* 147 (2004), 321–331.

[11] . Dehghan M, "Scattering solutions in networks of thin fibers: Small diameter asymptotics" , *Int. J. Nonlin. Sci. Numer. Simul.* 7 (2006), 447–450.

[12] Ruzhansky M., Tokmagambetov N., Torebek B.T., "On a non-local problem for a multi-term fractional diffusion-wave equation" , *Fractional Calculus and Applied Analysis,* 23:2 (2020), 324– 355.

[13] Boudabsa L., Simon T., Vallois P., "Fractional extreme distributions" , *ArXiv.* (2019). 1–46. arXiv: 1908.00584v1.

[14] Nakhusheva V. A., "The extremum principle for a nonlocal parabolic equation and the mixed problem for the generalized wave equation" , *Dokl. Adyg. (Cherkes.) Mezhdun. Akad. Nauk.* 2 (1996), 26–28 (in Russian)

[15] Luchko Y., "Boundary value problems for the generalized time-fractional diffusion equation of distributed order" , *Fract. Calc. Appl. Anal.* 12 (2009), 409–422

[16] Luchko Y., "Some uniqueness and existence results for the initial-boundary-value problems for the generalized time-fractional diffusion equation" , *Comput. Math. Appl.* 59 (2010), 1766–1772

[17] Luchko Y., "Initial-boundary problems for the generalized multi-term time-fractional diffusion equation" , *J. Math. Anal. Appl.* 374 (2011), 538–548

[18] Luchko Y., "Maximum principle and its application for the time-fractional diffusion equations" , *Fract. Calc. Appl. Anal.* 14 (2011), 110–124

[19] Gorenflo R., Mainardi F., "Signalling problem and Dirichlet-Neumann map for time-fractional diffusion-wave equation" , *Matimyas Mat.* 21 (1998), 109–118.