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VAN DER CORPUT LEMMA WITH BESSEL FUNCTIONS

In this article, we study analogues of the van der Corput lemmas [19] involving Bessel functions. In harmonic analysis, one of the most important estimates is the van der Corput lemma, which is an estimate of the oscillatory integrals. This estimate was first obtained by the Dutch mathematician Johannes Gaultherus van der Corput. Van der Corput interested in the behavior for large positive λ of the oscillatory integral $\int\limits_a^b e^{i\lambda\phi(x)}\psi(x)dx$, where ϕ is a real-valued smooth function (the phase) and ψ is complex valued smooth function (amplitude). In case $a=-\infty, b=+\infty$, it is assumed that ψ has a compact support in $\mathbb{R}$. In our case we replace the exponential function with the Bessel functions, to study oscillatory integrals appearing in the analysis of wave equation with singular damping. More specifically, we study integral of the form $I(\lambda)=\int_a^b J_n(\lambda\phi(x))\psi(x)dx$ for the range $n=0$, where $\psi\in\mathcal{C}$ and smooth, and λ is a positive real number that can vary.The generalisations of the van der Corput lemma is proved. As an application of the above results, the generalised Riemann-Lebesgue lemma is considered.
Key words: van der Corput lemma, Bessel function, asymptotic estimate,wave equation,oscillatory integrals.

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Бессель функциялары қатысқан Ван дер Корпут леммасы

Бұл мақалада біз Ван дер Корпуттың Бессель функцияларын қамтитын леммасының аналогтарын зерттейміз. Гармоникалық талдауда ең маңызды бағалаулардың бірі - Ван дер Корпут леммасы, ол тербелмелі интегралдарды бағалау болып табылады. Бұл бағалауды алғаш рет голланд математигі Иоганнес Голтерус Ван дер Корпут алған. Ван дер Корпут $\int\limits_a^b e^{i\lambda\phi(x)}\psi(x)dx$ тербеліс интегралының λ үлкен оң болғандағы әрекетіне қызығушылық танытты. ϕ- нақты тегіс функция (фаза), ал ψ- күрделі тегіс функция (амплитуда). $a=-\infty, b=+\infty$ жағдайында, ψ- $\mathbb{R}$ ішінде компактілі үйірткілі болады деп болжанады. Біздің жағдайда экспоненциалды функцияны Бессель функцияларымен ауыстырамыз, сингулярлы сөнген толқындық теңдеудің талдауында пайда болатын тербелмелі интеграл зерттеледі. Нақтырақ айтқанда, $n=0$ диапазоны үшін $I(\lambda)=\int_a^b J_n(\lambda\phi(x))\psi(x)dx$ түріндегі тербелмелі интегралды зерттейміз, мұндағы $\psi\in\mathcal{C}$ және тегіс, ал λ- өзгере алатын оң нақты сан. Ван дер Корпут леммасының жалпылауы дәлелденеді. Жоғарыда алынған нәтижелердің қолданысы ретінде жалпыланған Риман-Лебег леммасы қарастырылады.
Түйін сөздер: Ван дер Корпут леммасы, Бессель функциясы, асимптотикалық бағалау,толқындық теңдеу, тербелмелі интегралдар.

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Лемма Ван дер Корпута с функциями Бесселя

В данной статье мы изучаем аналоги лемму Ван дер Корпута [19] с функциями Бесселя. В гармоническом анализе одной из важнейших оценок является лемма Ван дер Корпута, которая является оценкой осциллирующих интегралов. Эта оценка впервые была получена голландским математиком Йоханнесом Голтерусом ван дер Корпутом. Ван дер Корпут интересовался поведением при больших положительных λ осциллирующего интеграла $\int_a^b e^{i\lambda\phi(x)}\psi(x)dx$,, где ϕ - вещественная гладкая функция (фаза), а ψ - комплексная гладкая функция (амплитуда). В случае $a = -\infty, b = +\infty$ предполагается, что ψ имеет компактный носитель в $\mathbb{R}$. В нашем случае показательная функция заменяется функциями Бесселя, чтобы изучить осциллирующие интегралы, возникающие при анализе волнового уравнения с сингулярным затуханием. В частности, мы изучаем интеграл вида $I(\lambda) = \int_a^b J_n(\lambda\phi(x))\psi(x)dx$ для диапазона $n = 0$, где $\psi \in \mathcal{C}$ и гладкие, а λ - положительное действительное число, которое может меняться. Доказаны обобщения леммы Ван дер Корпута. В качестве приложения полученных результатов рассматривается обобщенная лемма Римана-Лебега.

Ключевые слова: лемма Ван дер Корпута, функция Бесселя, асимптотическая оценка, волновое уравнение, осциллирующие интегралы.

1 Introduction

In harmonic analysis, one of the most important estimates is the van der Corput lemma, which is an estimate of the oscillatory integrals.

This estimate was first obtained by the Dutch mathematician **Johannes Gaultherus van der Corput** (4 September 1890 – 16 September 1975) and named in his honour. While the paper [1] was published in Mathematische Annalen in 1921. Johannes Gaultherus van der Corput introduced the van der Corput lemma, a technique for creating an upper bound on the measure of a set drawn from harmonic analysis, and the van der Corput theorem on equidistribution modulo 1. He became member of the Royal Netherlands Academy of Arts and Sciences in 1929, and foreign member in 1953. He was a Plenary Speaker of the ICM in 1936 in Oslo.

Van der Corput interested in the behavior for large positive λ of the oscillatory integral

$$\int_a^b e^{i\lambda\phi(x)}\psi(x)dx,$$

where ϕ is a real-valued smooth function (the phase) and ψ is complex valued smooth function (amplitude). In case $a = -\infty, b = +\infty$, it is assumed that ψ has a compact support in $\mathbb{R}$.

Such integrals arise in the study of decay estimates of solutions of the Schrödinger and the wave equations.

Indeed, the estimate obtained by van der Corput, following Proposition 1 and Proposition 2 in Chapter VIII of [2], can be stated as follows:

Lemma 1 *Suppose ϕ is a real-valued and smooth function in $[a, b]$. If ψ is a smooth function, ϕ' is monotonic, $|\phi'| \geq 1$ for all $x \in (a, b)$, then*

$$\left|\int_a^b e^{i\lambda\phi(x)}\psi(x)dx\right| \leq C\lambda^{-1}, \ \ \lambda > 0.$$

Various generalizations of the van der Corput lemmas have been investigated over the years. One dinemsional and multidimensional analogues of the van der Corput lemmas were studied in [3-8] while in [9] the multi-dimensional van der Corput lemma was obtained with constants independent of the phase and amplitude. We also note that in [10-12] optimal constants were found for various versions of van der Corput's lemmas. The main goal of the present paper is to study van der Corput lemmas for the oscillatory integral defined by

$$I(\lambda) = \int_a^b J_0(\lambda\phi(x))\psi(x)dx, \tag{1}$$

where $\psi \in C$ and smooth, and λ is a positive real number that can vary.

Recently, the attention of many mathematicians has been attracted by Van der Corput's estimates for integrals with special functions [13-18]. For example, in [13] mainly focus on numerical evaluation of highly oscillatory Bessel transforms and presented based on the multiple integral, the schemes for computing this class of the transform. In work [14] presented van der Corput-type lemmas for Bessel and Airy transforms with shifted Jacobi weight functions. These results on the asymptotic orders of the highly oscillatory integrals on the frequency are optimal. Furthermore, from these estimates, the convergence rates on Filon-type methods are easily derived.

2 Material and methods

In this section we consider $I(\lambda)$,defined by (1), that is

$$I(\lambda) = \int\limits_a^b J_0(\lambda\phi(x))\psi(x)dx.$$

From the Chapter II, §11 (1) of [20] the Bessel function is

$$J_\alpha(\lambda\phi(x)) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+\alpha)!}\left(\frac{\lambda}{2}\right)^{2m+\alpha} \phi^{2m+\alpha}(x), \alpha \in Z. \tag{2}$$

The behavior of the function $J_0(\lambda\phi(x))$ which we derived from Chapter III, §36 (1) of [20] for any values of $\phi(x)$ ($\phi(x) = 0$ excepted) and for all $\lambda \in C$ is

$$|J_0(\lambda\phi(x))| \leq 1. \tag{3}$$

As for small λ the integral (1) is just bounded, we consider the case $\lambda \geq 1$.

Lemma 2 *Let the function $\phi(x)$ is in $\in C^2[a,b]$, and $\phi(x) \neq 0, \phi'(x) \neq 0$, then*

$$J_0(\lambda\phi(x)) = -\frac{1}{\lambda^2} \cdot \frac{1}{(\phi(x)} \cdot \frac{1}{(\phi'(x)} \cdot \frac{\mathrm{d}}{\mathrm{d}x}(\frac{\phi(x)}{\phi'(x)}\frac{\mathrm{d}}{\mathrm{d}x}J_0(\lambda\phi(x))]. \tag{4}$$

Proof. We consider the function (2) in the case $\alpha = 0$

$$J_0(\lambda\phi(x)) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{\lambda\phi(x)}{2}\right)^{2m},$$

and we will differentiate the function $J_0(\lambda\phi(x))$ once

$$\begin{aligned} \frac{d}{dx} J_0(\lambda\phi(x)) &= 2\sum_{m=1}^{\infty} \frac{(-1)^m}{m!(m-1)!} \phi^{2m-1}(x)\phi'(x) \left(\frac{\lambda}{2}\right)^{2m} \\ &= 2\sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{(m+1)!m!} \phi^{2m+1}(x)\phi'(x) \left(\frac{\lambda}{2}\right)^{2m+2} \\ &= -\frac{\lambda^2}{2}\phi'(x) \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)!m!} \phi^{2m+1}(x) \left(\frac{\lambda}{2}\right)^{2m}. \end{aligned} \tag{5}$$

Then, transforming the (5), we will differentiate the function bellow

$$\frac{d}{dx}\left(\frac{\phi(x)}{\phi'(x)} \frac{d}{dx} J_0(\lambda\phi(x))\right) = -\frac{\lambda^2}{2} \cdot 2\sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)!m!} (m+1)\phi^{2m+1}(x)\phi'(x) \left(\frac{\lambda}{2}\right)^{2m}$$

$$= -\lambda^2\phi'(x)\phi(x) \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \phi^{2m+1}(x) \left(\frac{\lambda}{2}\right)^{2m}$$

$$= -\lambda^2\phi'(x)\phi(x)J_0(\lambda\phi(x)).$$

Thus, by differentiating twice we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\phi(x)}{\phi'(x)} \frac{\mathrm{d}}{\mathrm{d}x} J_0(\lambda\phi(x))\right) + \lambda^2\phi'(x)\phi(x)J_0(\lambda\phi(x)) = 0.$$

The proof is complete.

3 Results and discussion

We formulate our result in the form of theorem and show its application bellow

Theorem 1 *Let $-\infty \le a < b \le \infty$ and $\alpha = 0$. Let $\phi(x) \in C^2[a,b]$ and $\psi \in C^1[a,b]$. If $|\phi(x)| > 1, |\phi'(x)| > 1, |\phi''(x)| > 1$ for all $x \in [a,b]$, then we shall now prove that*

$$|I(\lambda)| \le \frac{1}{\lambda^2}\left|\frac{\phi(b)}{\phi'(b)}D(b) - \frac{\phi(a)}{\phi'(a)}D(a)\right|.$$

Proof. We put the function $J_0(\lambda\phi(x))$ to the integral (1) and integrating it by parts we find that

$$I(\lambda) = \int_a^b J_0(\lambda\phi(x))\psi(x)dx = -\frac{1}{\lambda^2}\int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\phi(x)}{\phi'(x)} \frac{\mathrm{d}}{\mathrm{d}x} J_0(\lambda\phi(x)\right) \cdot \frac{\psi(x)}{\phi(x)\phi'(x)}dx$$

$$= -\frac{1}{\lambda^2}\left[\frac{\psi(x)}{\phi'^2(x)}\frac{\mathrm{d}}{\mathrm{d}x}J_0(\lambda\phi(x))|_{x=a}^{x=b} - \int_a^b \frac{\mathrm{d}}{\mathrm{d}x}J_0(\lambda\phi(x))\frac{\phi(x)}{\phi'(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\psi(x)}{\phi(x)\phi'(x)}\right)dx\right]$$

$$= -\frac{C}{\lambda^2}\int_a^b J_0(\lambda\phi(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\phi(x)}{\phi'(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\psi(x)}{\phi(x)\phi'(x)}\right)\right)dx,$$

where

$$C = \left[\frac{\psi(x)}{\phi'^2(x)}\frac{\mathrm{d}}{\mathrm{d}x}J_0(\lambda\phi(x)) - \frac{\phi(x)}{\phi'(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\psi(x)}{\phi(x)\phi'(x)}\right)J_0(\lambda\phi(x))\right]|_{x=a}^{x=b}.$$

Applying the expression (3) and suppose that $\phi(x) \in C^2[a,b]$, $\psi \in C^1[a,b]$,$|\phi(x)| > 1, |\phi'(x)| > 1, |\phi''(x)| > 1$ for all $x \in R$ we have

$$|I(\lambda)| = \left|-\frac{C}{\lambda^2}\int_a^b J_0(\lambda\phi(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\phi(x)}{\phi'(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\psi(x)}{\phi(x)\phi'(x)}\right)\right)dx\right|$$

$$= \frac{1}{\lambda^2}\int_a^b |J_0(\lambda\phi(x)|\left|\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\phi(x)}{\phi'(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\psi(x)}{\phi(x)\phi'(x)}\right)\right)\right|dx$$

$$\leq \frac{C}{\lambda^2}\left|\int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\phi(x)}{\phi'(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\psi(x)}{\phi(x)\phi'(x)}\right)dx\right)\right|$$

$$\leq \frac{C}{\lambda^2}\left|\frac{\phi(b)}{\phi'(b)}D(b) - \frac{\phi(a)}{\phi'(a)}D(a)\right|,$$

where

$$D(b) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\psi(x)}{\phi(x)\phi'(x)}\right)'|_{x=b}.$$

and

$$D(a) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\psi(x)}{\phi(x)\phi'(x)}\right)'|_{x=a}.$$

The proof is complete.

Remark 1 *If $[a,b] = R := (-\infty, +\infty)$, then assume that $\psi \in C_0^1(R)$ is the function with compact support.*

3.1 Aplication. Generalised Rieman-Lebesgue lemma

The Riemann-Lebesgue lemma is the classical result of harmonic and asymptotic analysis. The simplest form of the Riemann-Lebesgue lemma states that for a function $f \in C^1([a,b])$ we obtain

$$\int_a^b e^{ikx} f(x)dx = O\left(\frac{1}{k}\right), \quad \text{at} \quad k \to \infty.$$

We consider the following integral of Fourier-Bessel transform

$$\int_a^b J_0(kx)f(x)dx.$$

If $f \in C^2([a,b])$, then from the van der Corput lemma and by Theorem 1 we have

$$\int_a^b J_0(kx)f(x)dx = O\left(k^{-2}\right).$$

4 Conclusion

Thus, in this paper, we consider analogues of the van der Corput lemma involving Bessel functions. The main result of the work is to study oscillatory integrals appearing in the analysis of wave equation with singular damping. We have proved the behavior of the oscillatory integral for large positive λ. Therefore, the estimates, which we got, can be used to for proofs of generalised Riemann-Lebesgue lemma.

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