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ONE RESULT ON BOUNDEDNESS OF THE HILBERT TRANSFORM

In mathematics and in signal theory, the Hilbert transform is an important linear operator that takes a real-valued function and produces another real-valued function. The Hilbert transform is a linear operator which arises from the study of boundary values of the real and imaginary parts of analytic functions. Also, it is a widely used tool in signal processing. The Cauchy integral is a figurative way to motivate the Hilbert transform. The complex view helps us to relate the Hilbert transform to something more concrete and understandable. Moreover, the Hilbert transform is closely connected with many operators in harmonic analysis such as Laplace and Fourier transforms which have numerous application in partial and ordinary differential equations. In this paper, we study boundedness properties of the classical (singular) Hilbert transform acting on Marcinkiewicz spaces. More precisely, we obtain if and only if condition for boundedness of the Hilbert transform in Marcinkiewicz function spaces.
Key words: Symmetric (quasi-)Banach function space, Hilbert transform, Calderón operator, Marcinkiewicz space.

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Гильберт түрлендіруінің шенелгендігі туралы бір нәтиже

Математикада және сигнал теориясында Гильберт түрлендіруі нақты мәнді функцияны қабылдайтын және оған басқа бір нақты мәнді функцияны сәйкес қоятын маңызды сызықтық оператор болып табылады. Гильберт түрлендіруі - аналитикалық функциялардың нақты және жорамал бөліктерінің шекаралық мәндерін зерттеу нәтижесінде пайда болатын сызықтық оператор. Сондай-ақ, ол сигналды өңдеуде кеңінен қолданылатын құрал болып табылады. Коши интегралы Гильберт түрлендіруін қолдану үшін маңызды рөл атқарады. Комплекс тұрғыда бізге Гильберт түрлендіруін нақтырақ және түсінікті нәрсемен байланыстыруға болады. Сонымен қатар, Гильберт түрлендіруі гармоникалық талдаудың көптеген операторларымен тығыз байланысты, мысалы, қарапайым және дербес туындылы дифференциалдық теңдеулерде көп қолданылатын Лаплас және Фурье түрлендірулері. Бұл жұмыста біз Марцинкевич кеңістігіндегі функцияларға әсер ететін классикалық (сингулярлық) Гильберт түрлендіруінің шенелгендік қасиеттерін зерттедік. Дәлірек айтқанда, Марцинкевич функционалдық кеңістіктеріндегі Гильберт түрлендіруінің шенелген болу шартын алдық.

Түйін сөздер: Симметриялық(квази-) Банах кеңістігі, Гильберт түрлендіруі, Кальдерон операторы, Марцинкевич кеңістігі.

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Один результат об ограниченности преобразования Гильберта

В математике и теории сигналов преобразование Гильберта является важнейшим линейным оператором, который переводит функцию действительной переменной в другую функцию действительной переменной. Преобразование Гильберта - линейный оператор, возникающий при изучении граничных значений действительной и мнимой частей аналитических функций. Кроме того, это широко используемый инструмент в обработке сигналов. Интеграл Коши - образный способ мотивировать преобразование Гильберта. Комплексное представление помогает нам связать преобразование Гильберта с чем-то более конкретным и понятным. Более того, преобразование Гильберта тесно связано со многими операторами гармонического анализа, такими как преобразования Лапласа и Фурье, которые находят многочисленные применения в обыкновенных дифференциальных уравнениях и в уравнениях с частными производными. В данной работе изучаются свойства ограниченности классического (сингулярного) преобразования Гильберта, действующего на пространствах Марцинкевича. Точнее, мы получили необходимое и достаточное условие ограниченности преобразования Гильберта в функциональных пространствах Марцинкевича.

Ключевые слова: Симметричные(квази-) банаховы пространства, преобразование Гильберта, оператор Кальдерона, пространство Марцинкевича.

1 Introduction

The Hilbert transform is one of the powerful operators in the field of signal theory. Given a locally integrable measurable function f, its Hilbert transform, denoted by $\mathcal{H}(f)$, is calculated through the integral in the sense of principal value

$$(\mathcal{H}f)(t) = p.v.\frac{1}{\pi}\int_{\mathbb{R}}\frac{f(s)}{t-s}ds = \lim_{\varepsilon\to 0+}\frac{1}{\pi}\int_{|s-t|\geq\varepsilon}\frac{f(s)}{t-s}ds.$$

The Hilbert transform is named after German mathematician David Hilbert (1862-1943). Its first use dates back to 1905 in Hilbert's work concerning analytical functions in connection to the Riemann problem. In 1928 it was proved by Marcel Riesz (1886-1969) that the Hilbert transform is a bounded linear operator on $L_p(\mathbb{R})$ for $1 < p < \infty$. This result was generalized for the Hilbert transform in several dimensions (and singular integral operators in general) by Antoni Zygmund (1900-1992) and Alberto Calderón (1920-1998). Our investigation concerns with boundedness of the Hilbert transform in so called rearrangement invariant Banach function spaces which received a lot of attention since Boyd's pioneer work in 1966 [1] (see also [2]). We also refer the reader to recent papers [6–11] and references there in. In this note, we study boundedness of the Hilbert transform from one Marcinkiewicz space to another.

2 Materials and methods

2.1 Symmetric Banach function spaces and their Köthe dual spaces

Let (I, m) denote the measure space $I = \mathbb{R}_+, \mathbb{R}$, where $\mathbb{R}_+ := (0, \infty)$ and $\mathbb{R}$ is the set of real numbers, equipped with Lebesgue measure m. Let $L(I, m)$ be the space of all measurable real-valued functions on I equipped with Lebesgue measure m, i.e. functions which coincide almost everywhere are considered identical. Define $L_0(I)$ to be the subset of $L(I, m)$ which consists of all functions f such that $m(\{t : |f(t)| > s\})$ is finite for some $s > 0$.

For $f \in L_0(I)$ (where $I = \mathbb{R}_+$ or $\mathbb{R}$), we denote by f^* the decreasing rearrangement of the function $|f|$. That is,

$$f^*(t) = \inf\{s \geq 0 : \ m(\{|f| > s\}) \leq t\}, \quad t > 0.$$

Definition 1 *We say that* $(E(I), \|\cdot\|_{E(I)})$ *is a symmetric (quasi-)Banach function space on* I*, if the following conditions hold:*

(a) $E(I)$ *is a subset of* $L_0(I)$*;*

(b) $(E(I), \|\cdot\|_{E(I)})$ *is a (quasi-)Banach space;*

(c) *If* $f \in E(I)$ *and if* $g \in L_0(I)$ *are such that* $g^*(t) \leq f^*(t), t > 0$ *then* $g \in E(I)$ *and* $\|g\|_{E(I)} \leq \|f\|_{E(I)}$.

It is well known that $L_p(I)$, $(0 < p \leq \infty)$ is a classical example of symmetric (quasi-)Banach space of functions

We say that $g \in L_0(I)$ is submajorized by $f \in L_0(I)$ in the sense of Hardy–Littlewood–Pólya (written $g \prec\prec f$) if

$$\int_0^t g^*(s)ds \leq \int_0^t f^*(s)ds, \quad t \geq 0.$$

Let E be a symmetric Banach function space on I with Lebesgue measure m the Köthe dual space $E^\times$ on I is defined by

$$E(I)^\times = \left\{ g \in L_0(I) : \int_0^\infty |f(t)g(t)|dt < \infty, \ \forall f \in E(I) \right\}.$$

The space $E^\times$ is Banach with the norm

$$\|g\|_{E(I)^\times} := \sup \left\{ \int_0^\infty |f(t)g(t)|dt : f \in E(I), \ \|f\|_{E(I)} \leq 1 \right\}.$$

If E is a symmetric Banach function space, then $(E^\times, \|\cdot\|_{E^\times})$ is also a symmetric Banach function space (cf. [3, Section 2.4]). For more details on Köthe duality we refer to [3, 5].

2.2 Lorentz and Marcinkiewicz spaces

For the function $\varphi(t) := \log(1+t)$, $t > 0$, the Lorentz space $\Lambda_{\log}(I)$ is defined by setting

$$\Lambda_{\log}(I) := \left\{ f \in L_0(I) : \int_{\mathbb{R}_+} \frac{f^*(s)}{1+s} ds < \infty \right\}$$

equipped with the norm

$$\|f\|_{\Lambda_{\log}(I)} := \int_{\mathbb{R}_+} \frac{f^*(s)}{1+s} ds.$$

Definition 2 *[4, Definition II. 1.1, p. 49] A function* φ *on the semiaxis* $[0, \infty)$ *is said to be quasiconcave if*

(i) $\varphi(t) = 0 \Leftrightarrow t = 0$;

(ii) $\varphi(t)$ *is positive and increasing for* $t > 0$*;*

(iii) $\frac{\varphi(t)}{t}$ *is decreasing for* $t > 0$.

Observe that every nonnegative concave function on $[0,\infty)$ that vanishes only at origin is quasiconcave. The reverse, however, is not true. But, we may replace, if necessary, a quasiconcave function φ by its smallest concave majorant $\widetilde{\varphi}$ such that

$$\frac{1}{2}\widetilde{\varphi} \leq \varphi \leq \widetilde{\varphi}$$

(see [3, Proposition 5.10, p. 71]). Let $\phi : [0,\infty) \to [0,\infty)$ be a quasiconcave function for which $\lim\limits_{t\to 0+} \phi(t) = 0$ (or simply $\phi(0+) = 0$). Define the Marcinkiewicz space $M_\phi(I)$ as follows

$$M_\phi(I) := \left\{ f \in L_0(I) : \sup_{t>0} \frac{1}{\phi(t)} \int_0^t f^*(s)ds < \infty \right\}$$

with the norm

$$\|f\|_{M_\phi(I)} := \sup_{t>0} \frac{1}{\phi(t)} \int_0^t f^*(s)ds.$$

2.3 Weak-L_1 and $L_{1,\infty} + L_\infty$ spaces

Define the weak-L_1 space $L_{1,\infty}(I)$ by setting

$$L_{1,\infty}(I) = \{f \in L_0(I) : \sup_{t>0} tf^*(t) < \infty\}$$

and equip it with the quasi-norm

$$\|f\|_{L_{1,\infty}(I)} = \sup_{t>0} tf^*(t), \quad f \in L_{1,\infty}(I).$$

The space $L_{1,\infty}(I)$ is a quasi-Banach symmetric space.

Equip the vector space $L_0(I)$ on I with the topology of convergence in measure. The space $(L_{1,\infty} + L_\infty)(I) = L_{1,\infty}(I) + L_\infty(I)$ consists of functions for which

$$\begin{aligned} \|f\|_{(L_{1,\infty}+L_\infty)(I)} &= \inf\{\|f_1\|_{L_{1,\infty}(I)} + \|f_2\|_{L_\infty(I)} : f = f_1 + f_2, \\ &\quad f_1 \in L_{1,\infty}(I),\ f_2 \in L_\infty(I)\} < \infty. \end{aligned}$$

2.4 Calderón operator and Hilbert transform

For a function $f \in \Lambda_{\log}(\mathbb{R}_+)$, define the Calderón operator $S : \Lambda_{\log}(\mathbb{R}_+) \to (L_{1,\infty} + L_\infty)(\mathbb{R}_+)$ as follows

$$(Sf)(t) := \frac{1}{t}\int_0^t f(s)ds + \int_t^\infty f(s)\frac{ds}{s},\ \ t > 0. \tag{1}$$

For more details on this operator, see for instance [3], [7]. If $f \in \Lambda_{\log}(\mathbb{R})$, then the classical Hilbert transform $\mathcal{H}$ is defined by the principal-value integral

$$(\mathcal{H}f)(s) := p.v.\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(\eta)}{s-\eta}d\eta,\ \ \forall f \in \Lambda_{\log}(\mathbb{R}), \tag{2}$$

(see, e.g. [3, Chapter III. 4]).

We use standart methods of integration theory and the theory of rearrangement invariant Banach function spaces. We also use some results on Köthe duality of rearrangement invariant Banach function spaces.

3 Main results

Let $\phi : [0, \infty) \to [0, \infty)$ be a quasi-concave function. Define a function ψ by the following formula

$$\psi(t) := \inf_{t<s} \frac{s}{\phi(s)\log(\frac{t}{s})}. \tag{3}$$

We need the following lemmas.

Lemma 1 *If a function $\phi : [0, \infty) \to [0, \infty)$ is quasi-concave, then the function ψ defined by the formula* (3) *is also quasi-concave.*

Proof. First, it is easy to see that $\psi(x) = 0$ if and only if $x = 0$. By changing variables in (3), we obtain

$$\psi(u) := \inf_{\omega>1} \frac{u\omega}{\phi(u\omega)(1 + \log(\omega))}$$

Since ϕ is quasi-concave, it follows that

$$\begin{aligned}\psi(u_1) &:= \inf_{w>1} \frac{u_1 w}{\phi(u_1 w)(1 + \log(w))} = \inf_{w>1} \frac{1}{\frac{\phi(u_1 w)}{u_1 w} \cdot (1 + \log(w))} \\ &\leq \inf_{w>1} \frac{1}{\frac{\phi(u_2 w)}{u_2 w} \cdot (1 + \log(w))} = \inf_{w>1} \frac{u_2 w}{\phi(u_2 w)(1 + \log(w))} = \psi(u_2), \quad 0 < u_1 < u_2,\end{aligned}$$

which shows that ψ is increasing for any $u > 0$. Similarly, we can prove that $\frac{\psi(u)}{u}$ is decreasing for any $u > 0$, thereby completing the proof.

Let E and F be symmetric spaces on $\mathbb{R}_+$ with the Fatou property and let $E^\times$ and $F^\times$ their Köthe dual spaces on $\mathbb{R}_+$, respectively.

Lemma 2 *The operator S is self-adjoint with respect to L_1-pairing in the following sense*

$$\int_{\mathbb{R}_+} (Sf)(s)g(s)ds = \int_{\mathbb{R}_+} f(s)(Sg)(s)ds, \tag{4}$$

for all non-negative functions $x, y \in \Lambda_{\log}(\mathbb{R}_+)$.

If $E(\mathbb{R}_+), F^\times(\mathbb{R}_+) \subseteq \Lambda_{\log}(\mathbb{R}_+)$, then $S : E(\mathbb{R}_+) \to F(\mathbb{R}_+)$ if and only if $S : F^\times(\mathbb{R}_+) \to E^\times(\mathbb{R}_+)$, and we have

$$\|S\|_{E(\mathbb{R}_+)\to F(\mathbb{R}_+)} = \|S\|_{F^\times(\mathbb{R}_+)\to E^\times(\mathbb{R}_+)}. \tag{5}$$

Proof. The equality (4) follows from the formula (6.31) in [4, Chapter II.7, p.138].

If $S : E(\mathbb{R}_+) \to F(\mathbb{R}_+)$, then $S^\times : F^\times(\mathbb{R}_+) \to E^\times(\mathbb{R}_+)$. Since $S^\times = S$ by (4), it follows that $S : F^\times(\mathbb{R}_+) \to E^\times(\mathbb{R}_+)$.

Conversely, if $S : F^\times(\mathbb{R}_+) \to E^\times(\mathbb{R}_+)$, then $S^\times : E^{\times\times}(\mathbb{R}_+) \to F^{\times\times}(\mathbb{R}_+)$. Since $E(\mathbb{R}_+)$ and $F(\mathbb{R}_+)$ have Fatou property, we have $E^{\times\times}(\mathbb{R}_+) = E(\mathbb{R}_+)$ and $F^{\times\times}(\mathbb{R}_+) = F(\mathbb{R}_+)$. Therefore, again using $S^\times = S$ we obtain that $S : E(\mathbb{R}_+) \to F(\mathbb{R}_+)$. In this case, we have

$$\begin{aligned}\|S\|_{E(\mathbb{R}_+)\to F(\mathbb{R}_+)} &= \sup_{\|f\|_{E(\mathbb{R}_+)}\leq 1} \sup_{\|g\|_{F^\times(\mathbb{R}_+)}\leq 1} \left| \int_{\mathbb{R}_+} (Sf)(s)g(s)ds \right| \\ &\overset{(4)}{=} \sup_{\|g\|_{F^\times(\mathbb{R}_+)}\leq 1} \sup_{\|f\|_{E(\mathbb{R}_+)}\leq 1} \left| \int_{\mathbb{R}_+} f(s)(Sg)(s)ds \right| = \|S\|_{F^\times(\mathbb{R}_+)\to E^\times(\mathbb{R}_+)},\end{aligned}$$

which completes the proof.

Theorem 1 *Let ϕ and φ be increasing concave functions on $[0,\infty)$ vanishing at the origin. Suppose $M_\phi(\mathbb{R}_+) \subset \Lambda_{\log}(\mathbb{R}_+)$. We have*

$$S : M_\phi(\mathbb{R}_+) \to M_\varphi(\mathbb{R}_+)$$

if and only if

$$S(\phi') \prec\prec c_{\phi,\varphi} \cdot \varphi'.$$

Proof. Since $M_\phi(\mathbb{R}_+)^\times = \Lambda_\phi(\mathbb{R}_+)$, $M_\varphi(\mathbb{R}_+)^\times = \Lambda_\varphi(\mathbb{R}_+)$, and both spaces have the Fatou property, it follows from Lemma 2 that

$$S : M_\phi(\mathbb{R}_+) \to M_\varphi(\mathbb{R}_+)$$

if and only if

$$S : \Lambda_\varphi(\mathbb{R}_+) \to \Lambda_\phi(\mathbb{R}_+).$$

But, by Lemma 8 and Lemma 9 in [8] the latter one is equivalent to

$$\|S\chi_{(0,u)}\|_{\Lambda_\phi(\mathbb{R}_+)} \le c_{\phi,\varphi}\varphi(u), \quad u > 0,$$

which is, in fact, equivalent to

$$S(\phi') \prec\prec c_{\phi,\varphi} \cdot \varphi'. \tag{6}$$

Indeed, we have

$$\begin{aligned} \|S\chi_{(0,u)}\|_{\Lambda_\phi(\mathbb{R}_+)} &= \int_0^\infty S\chi_{(0,u)}(s)d\phi(s) = \int_0^\infty S\chi_{(0,u)}(s)\phi'(s)ds \\ &= \int_0^\infty \chi_{(0,u)}(s)(S\phi')(s)ds = \int_0^u (S\phi')(s)ds, \quad u > 0. \end{aligned} \tag{7}$$

For the left hand side of (6), since $\varphi(+0) = 0$, we have

$$c_{\phi,\varphi} \cdot \varphi'(u) = c_{\phi,\varphi} \cdot \int_0^u \varphi(s)ds, \quad u > 0. \tag{8}$$

Combining (7) and (8), we obtain (6), thereby completing the proof.

Corollary 1 *Let the assumptions of Theorem 1 hold. Then the Hilbert transform*

$$\mathcal{H} : M_\phi(\mathbb{R}) \to M_\varphi(\mathbb{R})$$

is bounded if and only if

$$S(\phi') \prec\prec c_{\phi,\varphi} \cdot \varphi'.$$

Proof. Following the argument in [8, Corollary 13] mutatis mutandi, we obtain that the Hilbert transform $\mathcal{H} : M_\phi(\mathbb{R}) \to M_\varphi(\mathbb{R})$ is bounded if and only if $S : M_\phi(\mathbb{R}_+) \to M_\varphi(\mathbb{R}_+)$. Hence, the assertion follows from Theorem 1.

Proposition 1 *Let $\phi : [0,\infty) \to [0,\infty)$ be a quasi-concave function and let ψ be the function defined by the formula (3). For every $u > 0$, there exists $y \in M_\phi(\mathbb{R}_+)$ such that $\|y\|_{M_\phi(\mathbb{R}_+)} \leq c_{abs} \cdot \psi(u)$ and*

$$\chi_{(0,u)} \leq S\mu(y).$$

Proof. Choose $w > 1$ such that

$$c_{abs} \cdot \psi(u) \geq \frac{uw}{\phi(uw)(1+\log(w))}, \quad u > 0. \tag{9}$$

By (1), we have

$$(S\chi_{(0,uw)})(u) = 1 + \log(w), \quad w > 1. \tag{10}$$

Set $y = \mu(y) = \frac{\chi_{(0,uw)}}{1+\log(w)}$. Then by (10), we obtain

$$1 = (Sy)(u) \leq (Sy)(t), \quad t < u,$$

and, therefore, $\chi_{(0,u)} \leq S\mu(y)$.

On the other hand, by (9) we obtain

$$\begin{aligned}
\|y\|_{M_\phi(\mathbb{R}_+)} &= \sup_{t>0} \frac{1}{\phi(t)} \int_0^t \mu(s,y)ds = \frac{1}{1+\log(w)} \cdot \sup_{t>0} \left\{ \frac{1}{\phi(t)} \int_0^{\min\{t,uw\}} ds \right\} \\
&= \frac{1}{1+\log(w)} \cdot \sup_{t>0} \left\{ \frac{1}{\phi(t)} \min\{t, uw\} \right\} \\
&= \frac{1}{1+\log(w)} \cdot \max \left\{ \sup_{t<uw} \frac{t}{\phi(t)}, \sup_{t \geq uw} \frac{uw}{\phi(t)} \right\} \\
&= \frac{1}{1+\log(w)} \cdot \frac{uw}{\phi(uw)} \leq c_{abs} \cdot \psi(u).
\end{aligned}$$

This concludes the proof.

4 Conclusions

In this paper, we studied boundedness properties of the classical Hilbert transform acting on Marcinkiewicz spaces. We obtained if and only if condition for boundedness of the Hilbert transform from one Marcinkiewicz space into another. We obtained results by using standart methods of integration theory and the theory of rearrangement invariant Banach function spaces. The results can be further used to identify the optimal range of the Hilbert transform acting on Marcinkiewicz spaces.

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References

[1] Boyd D. The Hilbert transformation on rearrangement-invariant Banach spaces // University of Toronto. - 1966.

[2] Boyd D. The Hilbert transform on rearrangement-invariant spaces // Can. J. Math. - 1967. - V. 19. - P. 599-616.

[3] Bennett C., Sharpley R. Interpolation of Operators // Pure and Applied Mathematics. - 2015. - V. 129. - P. 1-9.

[4] Krein S., Petunin Y., Semenov E. Spectral analysis of a differential operator with an Interpolation of linear operators // Amer. Math. Soc., Providence. - 1982. - V. 54. - P. 669-684.

[5] Lindenstrauss J., Tzafriri L. Classical Banach spaces // Springer-Verlag. - 1979. - V. I and II. - P. 33-46.

[6] Meyer-Nieberg P. Banach Lattices. // Springer-Verlag. - 1991. doi:10.1007/978-3-642-76724-1.

[7] Sukochev F.A., Tulenov K.S., Zanin D.V. The optimal range of the Calderón operator and its applications // J. Func. Anal. - 2019. - V. 277, №10. - P. 3513-3559.

[8] Sukochev F.A., Tulenov K.S., Zanin D.V. The boundedness of the Hilbert transformation from one rearrangement invariant Banach space into another and applications // Bulletin des Sciences Mathematiques. - 2019. - V. 167. doi:10.1016/2020/102943.

[9] Tulenov K. S. The optimal symmetric quasi-Banach range of the discrete Hilbert transform // Arch. Math. - 2019. - V. 113, №6. - P. 649-660.

[10] Tulenov K. S. Optimal Rearrangement-Invariant Banach function range for the Hilbert transform // Eurasian Math. J. - 2021. - V. 12, №2. - P. 90-103.

[11] Bekbayev N.T., Tulenov K. S. On boundedness of the Hilbert transform on Marcinkiewicz spaces // Bulletin of the Karaganda University. - 2020. - V. 100, №4. - P. 26-32.