IRSTI 41.03.02 DOI: https://doi.org/10.26577/JMMCS.2022.v116.i4.04

M.Zh. Minglibayev , **A.B. Kosherbayeva***
al-Farabi Kazakh National University, Almaty, Kazakhstan
*e-mail: kosherbaevaayken@gmail.com

EVOLUTION EQUATIONS OF MULTI-PLANET SYSTEMS WITH VARIABLE MASSES

In celestial mechanics and astrodynamics, the study of the dynamical evolution of exoplanetary systems is the relevant topics. For today more than 3,000 exoplanetary systems are known. In this paper, we study the dynamic evolution of extrasolar systems, when the leading factor of evolution is the variability of the masses of gravitating bodies. The problem of $n+1$ spherically symmetric bodies with variable masses is considered in a relative coordinate system, this bodies inter-gravitating according to Newton's law. The quasi-elliptical motions of planets whose orbits do not intersect during evolution are investigated. It is believed that the mass of bodies under consideration varies isotropically by various known laws with different velocities. The mass of the parent star is considered to be the most massive than its planets and the origin of the relative coordinate system is in the center of the parent star. Due to the variability of the masses, the differential equations of motion become non-autonomous and the task is difficult. The problem is investigated by methods of perturbation theory. The canonical perturbation theory based on a periodic motion over a quasi-canonical section is used. Canonical equations of motion are obtained in analogues of the second Poincare system, which are effective in the case when the analogues of eccentricities and the analogues of the inclination of the orbital plane of planets are sufficiently small. The secular perturbations of the planets, which determine the behavior of the orbital parameters over long time intervals, are studied.
The evolutionary equations of many planetary systems with isotropically varying masses in analogues of the second system of Poincare variables are derived in an analytical form which are obtained using the Wolfram Mathematica computer algebra system. This takes into account the effects of the decreasing mass of the parent star and the growth of the masses of the planets due to the accretion of matter from the remnants of the protoplanetary disk. For the three-planetary problem of four bodies with variable masses, the evolutionary equations in dimensionless variables are obtained explicitly. In the future, these results will be used to study the dynamics of the three-planet system K2-3 in the non-stationary stage of its evolution.
Key words: variable mass, perturbation theory, evolutionary equations, exoplanetary systems, Poincare elements.

М.Дж. Минглибаев, А.Б. Кошербаева*
Әл-Фараби атындағы Қазақ Ұлттық Университеті, Алматы қ., Қазақстан
*e-mail: kosherbaevaayken@gmail.com

Массалары өзгермелі көп планеталы жүйелердің эволюциялық теңдеулері

Аспан механикасында және астродинамикада экзопланеталы жүйенің динамикалық эволюциясын зерттеу өзекті тақырып. Қазіргі таңда 3000-нан артық экзопланеталы жүйе белгілі. Бұл жұмыста гравитация арқылы әсерлесетін денелердің массаларының айнымалылығы эволюцияның жетекші факторы ретінде қарастырылған кезде күн жүйесі сыртындағы басқа да жүйелердің динамикалық эволюциясы зерттеледі. Салыстырмалы координаталар жүйесінде ньютон заңы бойынша өзара әсерлесетін айнымалы массалы сфералық симметриялы денелер мәселесі қарастырылады. Эволюция кезінде планеталардың орбиталары бір-бірімен қиылыспайтын квазиэллиптикалық қозғалыс зерттеледі. Қарастырылатын денелердің массалары әртүрлі жылдамдықпен белгілі әртүрлі заңдылықтар бойынша изотропты түрде өзгереді деп саналады. Орталық жұлдыздың массасы оның планеталарының массаларынан әлде-қайда үлкен деп алынады, және салыстырмалы координаталар жүйесінің бас нүктесі орталық жұлдыздың центрінде орналасады. Массалардың айнымалылыға есебінде диффе-

ренциалды қозғалыс теңдеулері автономды емес түрге енеді және есеп қиындайды. Мәселе ұйытқу теориясы әдістерімен зерттеледі. Квазиконустық қима бойынша апериодты қозғалыс негізінде канондық ұйытқу теориясы қолданылады. Экцентриситет аналогтары мен планета орбитасының көлбеулік бұрышының аналогтары жеткілікті деңгейде кіші шама болған кезде тиімді болып табылатын Пуанкаренің екінші жүйесінің аналогтары арқылы канондық қозғалыс теңдеуі алынды. Уақыттың үлкен интервалында орбита параметрлерінің өзгерісін анықтауға мүмкіндік беретін планетаның ғасырлық ұйытқуы зерттелінеді.
"Wolfram Mathematica" компьютерлік алгебра көмегімен массалары изотропты өзгеретін көп планеталы жүйенің эволюциялық теңдеулері Пуанкаре айнымалыларының екінші жүйесі аналогтары арқылы аналитикалық түрде келтірілген. Сонымен қатар, протопланеталы диск қалдықтары бөлшектерінің аккрециясы есебімен планета массасының өсуі және орталық жұлдыздың массасының азаю әсерлері есепке алынады. Айнымалы массалы төрт дененің үш планеталы мәселесі үшін өлшемсіз шама арқылы эволюциялық теңдеулер анық түрде алынды. Ендігі кезекте алынған нәтижелер К2-3 үш планеталы жүйесінің стационар емес эволюция кезеңінде динамикалық эволюциясын зерттеу үшін қолданылады.
Түйін сөздер: айнымалы масса, ұйытқу теориясы, эволюциялық теңдеулер, экзопланеталы жүйелер, Пуанкаре элементтері.

М.Дж. Минглибаев, А.Б. Кошербаева*
Казахский Национальный Университет имени аль-Фараби, г. Алматы, Казахстан
*e-mail: kosherbaevaayken@gmail.com

Эволюционные уравнения много планетных систем с переменными массами

В небесной механике и в астродинамике изучение динамическую эволюцию экзопланетных систем актуальная тема. На сегодняшний день известно более 3000 экзопланетные системы. В настоящей работе исследуется динамическая эволюция внесолнечных систем, когда ведущим фактором эволюции является переменность масс гравитирующих тел. Рассматривается в относительной системе координат задача сферический симметрических тел с переменными массами, взаимогравитирующие по закону ньютона. Исследуется квазиэллиптические движения планет орбиты которых в ходе эволюции не пересекаются. Считается, что масса рассматриваемых тел изменяется изотропно по различным известным законам с различными скоростями. Масса родительской звезды считается наиболее массивным чем её планеты и начало относительной системы координат находится в центре родительской звезды. Из-за переменности масс дифференциальные уравнения движения становится неавтономными и задача усложняется. Проблема исследуется методами теории возмущения. Используется каноническая теория возмущения на базе апериодического движения по квазиконическому сечению. Канонические уравнения движения получены в аналогах второй системы Пуанкаре, которые эффективны в случае, когда аналоги эксцентриситетов и аналоги наклонности орбитальной плоскости планет достаточно малы. Исследуются вековые возмущения планет, которые определяют поведение орбитальных параметров на больших интервалах времени.
В аналитическом виде приведены эволюционные уравнения много планетных систем с изотропно изменяющимися массами в аналогах второй системы переменных Пуанкаре, которые получены с использованием системы компьютерной алгебры "Wolfram Mathematica". При этом учитываются эффекты убывания массы родительской звезды и роста масс планет из-за аккреции вещества из остатков протопланетного диска. Для трех планетной задачи четырех тел с переменными массами, в явном виде, получены эволюционные уравнения в безразмерных переменных. В дальнейшем эти результаты будет использованы для изучения динамику трех планетной системы К2-3 в нестацонарной этапе ее эволюции.

Ключевые слова: переменная масса, теория возмущения, эволюционные уравнения, экзопланетные системы, элементы Пуанкаре.

1 Introduction

Multi-body problem is one of the center problem in celestial mechanics. Let us short review of more interesting work about this problem that are close to our topic. In paper [1] three body problem was researched and algorithm of solving equation in osculating elements was given, here perturbing acceleration smaller than main acceleration caused by the induced of the central body gravity. In article [2] integrability of the N body problem was described. In [3] the problem of deriving theory of motion of four planet around center star was considered. Here Hamiltonian was given in the Poisson series in the osculating elements of the second Poincare systems. The expansion in series was constructed up to third power of a small parameter. A relevant problem is the problem of formation planetary systems. In work [4] the orbital evolution of two planetary system of three bodies Sun-Jupiter-Saturn was investigated. The Hamiltonian written in osculating elements is represented in Poisson series expansion over all elements.

In [5] orbital evolution of asteroids Phaethon clusters was studied, taking into account perturbations from eight major planets, the dwarf planet Pluto, the influence of the Yarkovsky effect, the flattened Sun and relativistic effects. In article [6] dynamical evolution of orbits due to pressure of solar radiation was investigated. In [7] the authors analyzed dynamical evolution of young pairs of asteroids in close orbits. In work [8] evolution of planetary systems was studied. The averaged equations of motion was derived analytically up to third power of a small parameter for the case of a four planetary system. Here the system of Sun-Jupiter-Saturn-Neptune is considered.

In [9] and [10] the authors described a methodology for detection the initial orbits of exoplanet using the curve radial velocity of parent star and obtained an algorithm for solving the equations of two body problem in the form of series and proved that the serieses converges to solving the equations for small values of eccentricity.

In work [11] the orbital evolution of the three-planet exosystem as HD 39194 and the four-planet exosystems as HD 141399 and HD 160691 (μ Ara) are studied. In result the authors have derived an averaged semi-analytical theory of second-order motion by the masses of exoplanets. Here multi-planetary problem is considered. The equations of motion are given in the Jacobi coordinates and written in the elements of the second Poincare system.

In celestial mechanics and astrodynamics one of the relevant topics is the study dynamical evolution of non-stationary gravitational exoplanetary systems. For today 3677 exo-systems and 4903 confirmed exoplanets are known [12].

In this paper, in difference to the above-mentioned works, the dynamical evolution of multi-planetary systems is researched, when the leading factor of evolution is the variability of the masses of the celestial bodies themselves.

The particular case – two planetary problem of three bodies with variable masses was considered in work [13].

The motions are studied in a relative coordinate system, with the origin in the center of the parent star. The canonical perturbation theory is used, which elaborated on the base a periodic motion over quasi-canonical section [14]. Dimensionless evolutionary equations are obtained in analogues of the second Poincare system.

2 Materials and methods

2.1 The problem statement and differential equations of motion

We will consider the motion of $n+1$ $(n \geq 3)$ bodies, which inter-gravitating according to Newton's law, in a relative coordinate system with the origin in the center of the parent star, whose axes are parallel to the corresponding axes of the absolute coordinate system. The bodies will be considered spherical with isotropically varying masses. We introduce the following notation: S – the parent star of planetary system – the center body, P_i, $(i = 1, 2, \ldots, n)$ – planets. The positions of the planets are such that P_i – the inner planet relative to the P_{i+1} planets, but the outer, relative to P_{i-1}. We will assume that such positions of the planets are preserved during of the evolution and their orbits don't intersect.

The law of varying of mass is considered to be known and different:

$$m_0 = m_0(t), \quad m_1 = m_1(t), \ldots, m_n = m_n(t) \tag{1}$$

where, $m_0 = m_0(t)$ – mass of parent star S, $m_i = m_i(t)$, – mass of planet P_i.

The motion equations of n planets in the relative coordinate system are written as the following [14-15]:

$$\ddot{\vec{r}}_i = -f\frac{(m_0+m_i)}{r_i^3}\vec{r}_i + f\sum_{j=1}^{n}{}' m_j\left(\frac{\vec{r}_j-\vec{r}_i}{r_{ij}^3} - \frac{\vec{r}_j}{r_j^3}\right), \quad (i,j = 1,2,\ldots,n) \tag{2}$$

where, f – the gravitational constant, $\vec{r}_i(x_i, y_i, z_i)$ – the radius-vector of planet P_i, in summing the sign "stroke" means that $i \neq j$, r_{ij} – the mutual distances of the center of spherical bodies:

$$r_{ij} = \sqrt{(x_j-x_i)^2 + (y_j-y_i)^2 + (z_j-z_i)^2} = r_{ji} \tag{3}$$

We will use the methods of the canonical perturbation theory, elaborated on the basis of the aperiodic motion over a quasi-canonical section [14]. Canonical equations are convenient for studying non-stationary gravitating systems.

Based on differential equations of planetary motion written in the relative coordinate system (2), it is possible to write the canonical equations of motion in the osculating analogues of the second system of canonical Poincare variables [16-17]:

$$\Lambda_i, \quad \lambda_i, \quad \xi_i, \quad \eta_i, \quad p_i, \quad q_i \tag{4}$$

The system of canonical equations has the form

$$\begin{aligned}
\dot{\lambda}_i &= \frac{\partial R_i^*}{\partial \Lambda_i} = \frac{\mu_{i0}^2}{\gamma_i^2\Lambda_i^3} - \frac{\partial W_i}{\partial \Lambda_i}, & \dot{\Lambda}_i &= \frac{\partial R_i^*}{\partial \lambda_i} = \frac{\partial W_i}{\partial \lambda_i},\\
\dot{\eta}_i &= \frac{\partial R_i^*}{\partial \xi_i} = -\frac{\partial W_i}{\partial \xi_i}, & \dot{\xi}_i &= \frac{\partial R_i^*}{\partial \eta_i} = \frac{\partial W_i}{\partial \eta_i},\\
\dot{q}_i &= \frac{\partial R_i^*}{\partial p_i} = -\frac{\partial W_i}{\partial p_i}, & \dot{p}_i &= \frac{\partial R_i^*}{\partial q_i} = \frac{\partial W_i}{\partial q_i}.
\end{aligned} \tag{5}$$

where, the Hamilton function has the form

$$R_i^* = -\frac{\mu_{i0}^2}{2\Lambda_i^2} \cdot \frac{1}{\gamma_i^2(t)} - W_i\left(t, \Lambda_i, \xi_i, p_i, \lambda_i, \eta_i, q_i\right), \quad \gamma_i = \frac{m_0(t_0) + m_i(t_0)}{m_0(t) + m_i(t)} = \gamma_i(t) \tag{6}$$

here, $\mu_{i0} = f(m_0(t_0) + m_i(t_0)) = const$ – gravitational parameter of unperturbed motion at the initial moment of time t_0, $W_i(t, \Lambda_i, \xi_i, p_i, \lambda_i, \eta_i, q_i)$ – perturbing function.

In work [16] a scheme for expressing perturbing functions via osculating elements was presented (4). In the article [18] obviously expansion of the perturbing function in analogues of the second system of canonical Poincare variables were obtained up to the second power of small parameters including, for n – planetary systems with variable masses. The equations of secular perturbations in the general case are also obtained

$$\begin{aligned} \dot{\lambda}_i &= \frac{\mu_{i0}^2}{\gamma_i^2 \Lambda_i^3} - \frac{\partial W_i^{(\text{sec})}}{\partial \Lambda_i}, & \dot{\Lambda}_i &= 0, \\ \dot{\eta}_i &= -\frac{\partial W_i^{(\text{sec})}}{\partial \xi_i}, & \dot{\xi}_i &= \frac{\partial W_i^{(\text{sec})}}{\partial \eta_i}, \\ \dot{q}_i &= -\frac{\partial W_i^{(\text{sec})}}{\partial p_i}, & \dot{p}_i &= \frac{\partial W_i^{(\text{sec})}}{\partial q_i}. \end{aligned} \tag{7}$$

The obviously form of the obtained evolutionary equations of the problem of multi bodies with variable masses (7) is as following

$$\begin{aligned} \dot{\xi}_i &= f\sum_{s=1}^{i-1} m_s \left(\frac{\Pi_{ii}^{is}}{\Lambda_i}\eta_i + \frac{\Pi_{is}^{is}}{\sqrt{\Lambda_i\Lambda_s}}\eta_s\right) + f\sum_{k=i+1}^{n} m_k \left(\frac{\Pi_{kk}^{ik}}{\Lambda_i}\eta_i + \frac{\Pi_{ik}^{ik}}{\sqrt{\Lambda_i\Lambda_k}}\eta_k\right) - \frac{3\ddot{\gamma}_i\Lambda_i^3}{2\gamma_i\mu_{i0}^2}\eta_i, \\ \dot{\eta}_i &= -f\sum_{s=1}^{i-1} m_s \left(\frac{\Pi_{ii}^{is}}{\Lambda_i}\xi_i + \frac{\Pi_{is}^{is}}{\sqrt{\Lambda_i\Lambda_s}}\xi_s\right) - f\sum_{k=i+1}^{n} m_k \left(\frac{\Pi_{kk}^{ik}}{\Lambda_i}\xi_i + \frac{\Pi_{ik}^{ik}}{\sqrt{\Lambda_i\Lambda_k}}\xi_k\right) + \frac{3\ddot{\gamma}_i\Lambda_i^3}{2\gamma_i\mu_{i0}^2}\xi_i \\ \dot{p}_i &= -f\sum_{s=1}^{i-1} m_s B_1^{is} \left(\frac{q_i}{4\Lambda_i} - \frac{q_s}{4\sqrt{\Lambda_i\Lambda_s}}\right) - f\sum_{k=i+1}^{n} m_k B_1^{ik} \left(\frac{q_i}{4\Lambda_i} - \frac{q_k}{4\sqrt{\Lambda_i\Lambda_k}}\right) \\ \dot{q}_i &= f\sum_{s=1}^{i-1} m_s B_1^{is} \left(\frac{p_i}{4\Lambda_i} - \frac{p_s}{4\sqrt{\Lambda_i\Lambda_s}}\right) + f\sum_{k=i+1}^{n} m_k B_1^{ik} \left(\frac{p_i}{4\Lambda_i} - \frac{p_k}{4\sqrt{\Lambda_i\Lambda_k}}\right) \end{aligned} \tag{8}$$

$$\dot{\lambda}_i = \frac{\mu_{i0}^2}{\gamma_i^2 \Lambda_i^3} - \frac{\partial W_i^{(\text{sec})}}{\partial \Lambda_i}, \quad \dot{\Lambda}_i = 0 \tag{9}$$

here, index s – denotes the inner planet relative to the investigated planet, and the index k – the outer one.

For n planetary problem of multi-bodies with variable masses the system of canonical equations (8) represent $4n$-linear non-autonomous equations with complex coefficients. The explicit form of non-autonomous coefficients of equations (8) – (9) are cumbersome, for internal and external perturbing planets they are written separately. They are described in detail and given in the work [18]. These coefficients, in turn, depend on the Laplace coefficients. The Laplace coefficients can be calculated exactly and expressed throughout elliptic integrals of the first and second kind [19].

The resulting system of canonical equations (8) is divided into two separate subsystems [18]. The first subsystem defines the equations of secular perturbations for eccentric elements (ξ_i, η_i), and the second one for oblique elements (p_i, q_i). The linearity of the system of non-autonomous differential equations (8) significantly ease the study of the canonical system of differential equations in the formulation under consideration.

From the last equation (9) follows

$$\Lambda_i = const \quad \text{or} \quad a_i = const \tag{10}$$

Note that λ_i is calculated after integrating equations (8).

Remark that when the analogues of eccentricities and the analogues of the inclination of the orbital planes of planets are small enough, the equations of secular perturbations (8) – (9) are convenient for describing the dynamic evolution of planetary systems with variable masses.

2.2 Dimensionless differential equations of motion

For the calculation we use the following dimensionless quantities:

$$t^* = \tau = \omega_1 t, \quad \left(\frac{d}{d\tau}\right) = (\ )', \quad a_i^* = \frac{a_i}{a_1}, \quad m_i^* = \frac{m_i}{m_{00}}, \tag{11}$$

where, t^* – dimensionless time, a_i^* – dimensionless distance, m_i^* – dimensionless mass, $m_{00} = m_0(t_0) = const$ – the mass of the parent star at the initial moment of time, $a_1 = a_1(t_0) = const$ – the semi major axis of the planet P_1 at the initial moment of time, the value of ω_1 is defined as follows:

$$\omega_1 = \frac{\sqrt{fm_{00}}}{a_1^{3/2}} = const. \tag{12}$$

Accordingly, we write down the period of the planet P_1 at the initial moment of time in Earth years

$$T_1 = \frac{2\pi}{\omega_1} = \frac{2\pi}{\sqrt{fm_{00}}} a_1^{3/2} = const = k_1. \tag{13}$$

Then, taking into account the relations [14], [16]

$$\begin{gathered}
\Lambda_i = \sqrt{\mu_{i0}}\sqrt{a_i}, \quad \lambda_i = l_i + \pi_i \\
\xi_i = \sqrt{2\sqrt{\mu_{i0}}\sqrt{a_i}(1-\sqrt{1-e_i^2})}\cos\pi_i, \quad \eta_i = -\sqrt{2\sqrt{\mu_{i0}}\sqrt{a_i}(1-\sqrt{1-e_i^2})}\sin\pi_i, \\
p_i = \sqrt{2\sqrt{\mu_{i0}}\sqrt{a_i}\sqrt{1-e_i^2}(1-\cos i_i)}\cos\Omega_i, \quad q_i = -\sqrt{2\sqrt{\mu_{i0}}\sqrt{a_i}\sqrt{1-e_i^2}(1-\cos i_i)}\sin\Omega_i, \\
l_i = M_i = n_i[\phi_i(t) - \phi_i(\tau_i)], \ \pi_i = \Omega_i + \omega_i,
\end{gathered} \tag{14}$$

where

$$a_i, \quad e_i, \quad i_i, \quad \omega_i, \quad \Omega_i, \quad \phi_i(\tau_i) \tag{15}$$

the osculating elements of the aperiodic motion over the quasi conic section, we can write

$$\xi_i = \xi_i^*(fm_{00}a_1)^{1/4}, \quad \eta_i = \eta_i^*(fm_{00}a_1)^{1/4}, \quad p_i = p_i^*(fm_{00}a_1)^{1/4}, \quad q_i = q_i^*(fm_{00}a_1)^{1/4} \tag{16}$$

$$\Lambda_i = \sqrt{fm_{00}}\sqrt{a_1}\Lambda_i^*, \quad \frac{3\ddot{\gamma}_i\Lambda_i^3}{2\gamma_i\mu_{i0}^2} = \omega_1 \frac{3\gamma_i''}{2\gamma_i}\frac{\Lambda_i^{*3}}{\mu_{i0}^{*2}}. \tag{17}$$

At the same time, dimensionless eccentric and oblique elements have the form

$$\begin{aligned} \xi_i^* &= \sqrt{2\sqrt{\mu_{i0}^*}\sqrt{a_i^*}(1-\sqrt{1-e_i^2})}\cos\pi_i, \\ \eta_i^* &= -\sqrt{2\sqrt{\mu_{i0}^*}\sqrt{a_i^*}(1-\sqrt{1-e_i^2})}\sin\pi_i, \end{aligned} \tag{18}$$

$$\begin{aligned} p_i^* &= \sqrt{2\sqrt{\mu_{i0}^*}\sqrt{a_i^*}\sqrt{1-e_i^2}(1-\cos i_i)}\cos\Omega_i, \\ q_i^* &= -\sqrt{2\sqrt{\mu_{i0}^*}\sqrt{a_i^*}\sqrt{1-e_i^2}(1-\cos i_i)}\sin\Omega_i, \end{aligned} \tag{19}$$

$$\Lambda_i^* = \sqrt{\mu_{i0}^*}\sqrt{a_i^*}, \quad \mu_{i0}^* = 1 + \frac{m_{i0}}{m_{00}} = const. \tag{20}$$

Using the introduced notation (11) – (17) and the relations (18) – (20), we proceed to dimensionless variables.

In equations (8), by reducing the left and right sides of the equation by a common multiplier $\omega_1(fm_{00}a_1)^{1/4} = const$, we obtain the evolution equations in dimensionless quantities.

For the convenience of writing, omitting the symbol $(*)$, we rewrite the equations (8) in dimensionless variables in the following form

$$\begin{aligned} \xi_i' &= \sum_{s=1}^{i-1} m_s\left(\frac{\Pi_{ii}^{is}}{\Lambda_i}\eta_i + \frac{\Pi_{is}^{is}}{\sqrt{\Lambda_i\Lambda_s}}\eta_s\right) + \sum_{k=i+1}^{n} m_k\left(\frac{\Pi_{kk}^{ik}}{\Lambda_i}\eta_i + \frac{\Pi_{ik}^{ik}}{\sqrt{\Lambda_i\Lambda_k}}\eta_k\right) - \frac{3\gamma_i''}{2\gamma_i}\frac{\Lambda_i^3}{\mu_{i0}^2}\eta_i, \\ \eta_i' &= -\sum_{s=1}^{i-1} m_s\left(\frac{\Pi_{ii}^{is}}{\Lambda_i}\xi_i + \frac{\Pi_{is}^{is}}{\sqrt{\Lambda_i\Lambda_s}}\xi_s\right) - \sum_{k=i+1}^{n} m_k\left(\frac{\Pi_{kk}^{ik}}{\Lambda_i}\xi_i + \frac{\Pi_{ik}^{ik}}{\sqrt{\Lambda_i\Lambda_k}}\xi_k\right) + \frac{3\gamma_i''}{2\gamma_i}\frac{\Lambda_i^3}{\mu_{i0}^2}\xi_i, \\ p_i' &= -\sum_{s=1}^{i-1} m_s B_1^{is}\left(\frac{q_i}{4\Lambda_i} - \frac{q_s}{4\sqrt{\Lambda_i\Lambda_s}}\right) - \sum_{k=i+1}^{n} m_k B_1^{ik}\left(\frac{q_i}{4\Lambda_i} - \frac{q_k}{4\sqrt{\Lambda_i\Lambda_k}}\right), \\ q_i' &= \sum_{s=1}^{i-1} m_s B_1^{is}\left(\frac{p_i}{4\Lambda_i} - \frac{p_s}{4\sqrt{\Lambda_i\Lambda_s}}\right) + \sum_{k=i+1}^{n} m_k B_1^{ik}\left(\frac{p_i}{4\Lambda_i} - \frac{p_k}{4\sqrt{\Lambda_i\Lambda_k}}\right). \end{aligned} \tag{21}$$

At the same time, the expressions Π_{ii}^{is}, Π_{is}^{is}, Π_{kk}^{ik}, Π_{ik}^{ik} in equations (21) and the Laplace coefficients retain their form. But, they are already dimensionless quantities.

3 Results

3.1 Dimensionless evolutionary equations of the three-planetary problem of four bodies for numerical calculations

Now we will explicitly write dimensionless evolutionary equations for the special case when $n = 3$. The planet P_1 is affected only by the outer planets $(s = 0,\ k = 2, 3)$, and for planet P_2 we take into account the influence of one inner planet $(s = 1)$ and one outer planet $(k = 3)$. For planet P_3, there is only the influence of the outer planets $(s = 1, 2,\ k = 0)$.

The system of equations of eccentric elements consists of six equations

$$\begin{aligned}
&\xi_1' = (D_2^{1,2} + D_2^{1,3} + D_3^1)\cdot\eta_1 + D_1^{1,2}\cdot\eta_2 + D_1^{1,3}\cdot\eta_3,\\
&\eta_1' = -(D_2^{1,2} + D_2^{1,3} + D_3^1)\cdot\xi_1 - D_1^{1,2}\cdot\xi_2 - D_1^{1,3}\cdot\xi_3,\\
&\xi_2' = D_1^{2,1}\cdot\eta_1 + (D_2^{2,1} + D_2^{2,3} + D_3^2)\cdot\eta_2 + D_1^{2,3}\cdot\eta_3,\\
&\eta_2' = -D_1^{2,1}\cdot\xi_1 - (D_2^{2,1} + D_2^{2,3} + D_3^2)\cdot\xi_2 - D_1^{2,3}\cdot\xi_3,\\
&\xi_3' = D_1^{3,1}\cdot\eta_1 + D_1^{3,2}\cdot\eta_2 + (D_2^{3,1} + D_2^{3,2} + D_3^3)\cdot\eta_3,\\
&\eta_3' = -D_1^{3,1}\cdot\xi_1 - D_1^{3,2}\cdot\xi_2 - (D_2^{3,1} + D_2^{3,2} + D_3^3)\cdot\xi_3,
\end{aligned}\tag{22}$$

Similarly, we obtain a system of equations for oblique elements

$$\begin{aligned}
&p_1' = -(H_2^{1,2} + H_2^{1,3})\cdot q_1 + H_1^{1,2}\cdot q_2 + H_1^{1,3}\cdot q_3,\\
&q_1' = (H_2^{1,2} + H_2^{1,3})\cdot p_1 - H_1^{1,2}\cdot p_2 - H_1^{1,3}\cdot p_3,\\
&p_2' = H_1^{2,1}\cdot q_1 - (H_2^{2,1} + H_2^{2,3})\cdot q_2 + H_1^{2,3}\cdot q_3,\\
&q_2' = -H_1^{2,1}\cdot p_1 + (H_2^{2,1} + H_2^{2,3})\cdot p_2 - H_1^{2,3}\cdot p_3,\\
&p_3' = H_1^{3,1}\cdot q_1 + H_1^{3,2}\cdot q_2 - (H_2^{3,1} + H_2^{3,2})\cdot q_3,\\
&q_3' = -H_1^{3,1}\cdot p_1 - H_1^{3,2}\cdot p_2 + (H_2^{3,1} + H_2^{3,2})\cdot p_3,
\end{aligned}\tag{23}$$

The following notation is introduced in equations (22) and (23)

$$D_1^{ik} = \frac{m_k \Pi_{ik}^{ik}}{\sqrt{\Lambda_i \Lambda_k}},\quad D_2^{ik} = \frac{m_k \Pi_{kk}^{ik}}{\Lambda_i},\quad D_3^i = -\frac{3\Lambda_i^3}{2\mu_{i0}^2}\frac{\gamma_i''(t)}{\gamma_i},\tag{24}$$

$$H_1^{i,k} = \frac{1}{4}\frac{m_k B_1^{i,k}}{\sqrt{\Lambda_i \Lambda_k}},\quad H_2^{i,k} = \frac{1}{4}\frac{m_k B_1^{i,k}}{\Lambda_i},\tag{25}$$

$$\begin{aligned}
&\Pi_{ik}^{ik} = \frac{1}{8}\left(9B_0^{ik} + B_2^{ik}\right) - \frac{9\left(1+\alpha_{ik}^2\right)}{8\alpha_{ik}}C_0^{ik} + \frac{21}{16}C_1^{ik} + \frac{3\left(1+\alpha_{ik}^2\right)}{8\alpha_{ik}}C_2^{ik} + \frac{3}{16}C_3^{ik},\\
&\Pi_{kk}^{ik} = -\frac{3}{4\alpha_{ik}}B_0^{ik} - \frac{1}{2}B_1^{ik} + \frac{15\alpha_{ik}^2 + 6}{8\alpha_{ik}^2}C_0^{ik} - \frac{3}{2\alpha_{ik}}C_1^{ik} - \frac{9}{8}C_2^{ik},
\end{aligned}\tag{26}$$

$$B_0^{ik} = \frac{2a_i\gamma_i}{\pi\left(a_k\gamma_k\right)^2}\int\limits_0^\pi \frac{d\lambda}{\left(1+\alpha_{ik}^2 - 2\alpha_{ik}\cos\lambda\right)^{3/2}},\quad B_1^{ik} = \frac{2a_i\gamma_i}{\pi\left(a_k\gamma_k\right)^2}\int\limits_0^\pi \frac{\cos\lambda d\lambda}{\left(1+\alpha_{ik}^2 - 2\alpha_{ik}\cos\lambda\right)^{3/2}},$$

$$B_2^{ik} = \frac{2a_i\gamma_i}{\pi\left(a_k\gamma_k\right)^2}\int\limits_0^\pi \frac{\cos 2\lambda d\lambda}{\left(1+\alpha_{ik}^2-2\alpha_{ik}\cos\lambda\right)^{3/2}},$$

$$C_0^{ik} = \frac{2\left(a_i\gamma_i\right)^2}{\pi\left(a_k\gamma_k\right)^3}\int\limits_0^\pi \frac{d\lambda}{\left(1+\alpha_{ik}^2-2\alpha_{ik}\cos\lambda\right)^{5/2}},\quad C_1^{ik} = \frac{2\left(a_i\gamma_i\right)^2}{\pi\left(a_k\gamma_k\right)^3}\int\limits_0^\pi \frac{\cos\lambda d\lambda}{\left(1+\alpha_{ik}^2-2\alpha_{ik}\cos\lambda\right)^{5/2}},\tag{27}$$

$$C_2^{ik} = \frac{2\left(a_i\gamma_i\right)^2}{\pi\left(a_k\gamma_k\right)^3}\int\limits_0^\pi \frac{\cos 2\lambda d\lambda}{\left(1+\alpha_{ik}^2-2\alpha_{ik}\cos\lambda\right)^{5/2}},\quad C_3^{ik} = \frac{2\left(a_i\gamma_i\right)^2}{\pi\left(a_k\gamma_k\right)^3}\int\limits_0^\pi \frac{\cos 3\lambda d\lambda}{\left(1+\alpha_{ik}^2-2\alpha_{ik}\cos\lambda\right)^{5/2}},$$

where the following conditions are met for the outer planets $(i < k)$

$$\alpha_{ik} = \frac{\gamma_i a_i}{\gamma_k a_k} = \alpha_{ik}(t) < 1.\tag{28}$$

For the inner planets , the designations are as follows

$$D_1^{is} = \frac{m_s\Pi_{is}^{is}}{\sqrt{\Lambda_s\Lambda_i}},\quad D_2^{is} = \frac{m_s\Pi_{ii}^{is}}{\Lambda_i},\quad D_3^{i} = -\frac{3\Lambda_i^3}{2\mu_{i0}^2}\frac{\gamma_i''(t)}{\gamma_i},\tag{29}$$

$$H_1^{i,s} = \frac{1}{4}\frac{m_s B_1^{i,s}}{\sqrt{\Lambda_s\Lambda_i}},\quad H_2^{i,s} = \frac{1}{4}\frac{m_s B_1^{i,s}}{\Lambda_i},\tag{30}$$

$$\begin{aligned}\Pi_{is}^{is} &= \frac{1}{8}\left(9B_0^{is}+B_2^{is}\right) - \frac{9\left(1+\alpha_{is}^2\right)}{8\alpha_{is}}C_0^{is} + \frac{21}{16}C_1^{is} + \frac{3\left(1+\alpha_{is}^2\right)}{8\alpha_{is}}C_2^{is} + \frac{3}{16}C_3^{is},\\ \Pi_{ii}^{is} &= -\frac{3\alpha_{is}}{4}B_0^{is} - \frac{1}{2}B_1^{is} + \frac{15+6\alpha_{is}^2}{8}C_0^{is} - \frac{3\alpha_{is}}{2}C_1^{is} - \frac{9}{8}C_2^{is},\end{aligned}\tag{31}$$

$$B_0^{is} = \frac{2a_s\gamma_s}{\pi\left(a_i\gamma_i\right)^2}\int\limits_0^\pi \frac{d\lambda}{\left(1+\alpha_{is}^2-2\alpha_{is}\cos\lambda\right)^{3/2}},\quad B_1^{is} = \frac{2a_s\gamma_s}{\pi\left(a_i\gamma_i\right)^2}\int\limits_0^\pi \frac{\cos\lambda d\lambda}{\left(1+\alpha_{is}^2-2\alpha_{is}\cos\lambda\right)^{3/2}},$$

$$B_2^{is} = \frac{2a_s\gamma_s}{\pi\left(a_i\gamma_i\right)^2}\int\limits_0^\pi \frac{\cos 2\lambda d\lambda}{\left(1+\alpha_{is}^2-2\alpha_{is}\cos\lambda\right)^{3/2}},$$

$$C_0^{is} = \frac{2\left(a_s\gamma_s\right)^2}{\pi\left(a_i\gamma_i\right)^3}\int\limits_0^\pi \frac{d\lambda}{\left(1+\alpha_{is}^2-2\alpha_{is}\cos\lambda\right)^{5/2}},\quad C_1^{is} = \frac{2\left(a_s\gamma_s\right)^2}{\pi\left(a_i\gamma_i\right)^3}\int\limits_0^\pi \frac{\cos\lambda d\lambda}{\left(1+\alpha_{is}^2-2\alpha_{is}\cos\lambda\right)^{5/2}},\tag{32}$$

$$C_2^{is} = \frac{2\left(a_s\gamma_s\right)^2}{\pi\left(a_i\gamma_i\right)^3}\int\limits_0^\pi \frac{\cos 2\lambda d\lambda}{\left(1+\alpha_{is}^2-2\alpha_{is}\cos\lambda\right)^{5/2}},\quad C_3^{is} = \frac{2\left(a_s\gamma_s\right)^2}{\pi\left(a_i\gamma_i\right)^3}\int\limits_0^\pi \frac{\cos 3\lambda d\lambda}{\left(1+\alpha_{is}^2-2\alpha_{is}\cos\lambda\right)^{5/2}},$$

In formulas (29) – (32), the following conditions are met for the inner planets $(s < i)$

$$\alpha_{is} = \frac{\gamma_s a_s}{\gamma_i a_i} = \alpha_{is}(t) < 1\tag{33}$$

4 Discussion

The resulting system of canonical equations (19) is divided into two separate subsystems.

The first subsystem (22) defines the equations of secular perturbations for eccentric elements. The second subsystem (23) contains equations for oblique elements. The linearity of the obtained non-autonomous canonical systems of differential equations (22) – (23) significantly ease the study of the problem in the formulation under consideration.

Note that equations (9) determining the mean longitude λ_i of the planets is calculated after integrating equations (22) and (23).

The obtained systems of differential equations (22) – (23) in dimensionless variables will be further used to analyze the effects of mass variability on the dynamic evolution of specific planetary systems by numerical methods.

5 Conclusion

In the work using the symbolic computing system "Wolfram Mathematica" [20-21], evolutionary equations are obtained in explicit analytical form, in dimensionless variables for the three-planetary problem of four bodies system with isotropically varying masses. Differential equations are described in analogues of the second system of canonical Poincare elements.

The obtained evolutionary equations will be used to study the dynamic evolution of extrasolar planetary systems. This will take into account the effects of the decrease in the mass of the parent star and the increase in the mass of the planets due to the accretion of matter from the remnants of the protoplanetary disk.

Acknowledgments

This research has been funded by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP14869472).

References

[1] Sannikova T.N., Kholshevnikov K.V., "Motion in a Central Field in the Presence of a Constant Perturbing Acceleration in a Co-moving Coordinate System", *Astronomy Reports,* 59(8) (2015): 806–817.

[2] Sokolov L.L., Kholshevnikov K.V., "Ob integpipyemocti zadachi N tel [On the integrability of the N-body problem]", *Astronomy Letters,* 12(7) (1986): 557–561.

[3] Perminov A.S., Kuznetsov E.D., "The Implementation of Hori–Deprit Method to the Construction Averaged Planetary Motion Theory by Means of Computer Algebra System Piranha", *Mathematics in Computer Science,* 14(2) (2020): 305–316. DOI: 10.1007/s11786-019-00441-4.

[4] Kuznetsov E.D., Kholshevnikov K.V., "Orbital'naya evolyuciya dvuplanetnoj sistemy Solnce – YUpiter – Saturn [Orbital evolution of the Sun – Jupiter – Saturn bi - planetary system]", *Vestniks of Saint Petersburg University,* 1(1) (2009): 139–150.

[5] Kuznetsov E.D., "Orbital evolution of phaethon cluster", *Meteoritics & Planetary Science,* 56 (2021): 1.

[6] Belkina S.O., Kuznetsov E.D., "Orbital flips due to solar radiation pressure for space debris in near-circular orbits", *Acta Astronautica,* 178 (2021): 360–369. https://doi.org/10.1016/j.actaastro.2020.09.025.

[7] Kuznetsov E.D., Rosaev A.E., Plavalova E., Safronova V.S., Vasileva M.A., "A Search for Young Asteroid Pairs with Close Orbits", *Solar System Research,* 54(3) (2020): 236–252. DOI: 10.1134/s0038094620030077.

[8] Perminov A., Kuznetsov E., "The orbital evolution of the Sun–Jupiter–Saturn–Uranus–Neptune system on long time scales" , *Astrophysics and Space Science,* 365(8) (2020): 144. DOI: 10.1007/s10509-020-03855-w.

[9] Kholshevnikov K.V., Mullari A.A., Tolumbaeva D.A., Vavilov D.E., "Opredelenie pervonachal'nyh orbit vnesolnechnyh planet metodom luchevyh skorostej: zamknutye formuly [Determination of the initial orbits of extrasolar planets by the method of radial velocities: closed formulas]" , *Vestniks of Saint Petersburg University,* 1(3) (2011): 143–152.

[10] Kholshevnikov K.V., Tolumbaeva D.A., Mullari A.A., "Opredelenie pervonachal'nyh orbit vnesolnechnyh planet metodom luchevyh skorostej: stepennye ryady [Determination of the initial orbits of extrasolar planets by the method of radial velocities: power series]" , *Vestniks of Saint Petersburg University,* 1(1) (2011): 166–172.

[11] Perminov A.S., Kuznetsov E.D., "Orbital'naya evolyuciya vnesolnechnyh planetnyh sistem HD 39194, HD 141399 I HD 160691 [Orbital evolution of extrasolar planetary systems HD 39194, HD 141399 and HD160691]" , *The Astronomical Journal,* 96(10) (2019): 795–813. DOI: 10.1134/S1063772919090075.

[12] https://exoplanets.nasa.gov/.

[13] Prokopenya A., Minglibayev M., Shomshekova S., "Computing Perturbations in the Two-Planetary Three-Body Problem with Masses Varying Non-Isotropically at Different Rates" , *Mathematics in Computer Science,* 14(2) (2020): 241–251. https://doi.org/10.1007/s11786-019-00437-0.

[14] Minglibayev M.Zh., *Dinamika gravitiruyushchikh tel s peremennymi massami i razmerami [Dynamics of gravitating bodies with variable masses and sizes]* (LAP LAMBERT Academic Publishing, 2012): 224. Germany. ISBN:978-3-659-29945-2.

[15] Minglibayev M.Zh., Kosherbayeva A.B., "Differential equations of planetary systems" , *Reports of the National Academy of Sciences of the Republic of Kazakhstan,* 2(330) (2020): 14–20. https://doi.org/10.32014/2020.2518-1483.26.

[16] Minglibayev M.Zh., Kosherbayeva A.B., "Equations of planetary systems motion" , *News of NAS RK. Physical-mathematical series,* 6 (2020): 53–60.

[17] Minglibayev M. Zh., Mayemerova G.M., "Evolution of the orbital-plane orientations in the two-protoplanet three-body problem with variable masses" , *Astronomy Reports,* 58(9) (2014): 667–677. DOI: 10.1134/S1063772914090066.

[18] Prokopenya A.N., Minglibayev M.Zh., Kosherbayeva A.B., "Derivation of evolutionary equations in the multi-body problem with isotropically varying masses using computer algebra" , *Programming and Computer Software,* 48(2) (2022): 1–11.

[19] Charlier K., *Nebesnaya mekhanika [Celestial mechanics]* [Текст] / Perevod s nem. V.G. Demina; Pod red. prof. B.M. Shchigoleva [Translated from German by V.G. Demin; Edited by prof. B.M. Shchigolev] (Moscow: Nauka, 1966): 627.

[20] Wolfram S., *An elementary introduction to the Wolfram Language* (New York: Wolffram Media, Inc., 2017): 324. ISBN: 978-1-944183-05-9.

[21] Prokopenya A.N., *Reshenie fizicheskih zadach c ispolzovaniem sistemy Mathematica [Solving physical problems using the Mathematica system]* (BSTU Publishing, Brest., 2005): 260.