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M.B. Borikhanov[1,2], **A.G. Smadiyeva**[3*]
[1]Khoja Akhmet Yassawi International Kazakh–Turkish University
Turkistan, Kazakhstan
[2]Ghent University, Ghent, Belgium
[3]Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan
[*]e-mail: smadiyeva@math.kz

CAUCHY PROBLEMS FOR THE TIME-FRACTIONAL DEGENERATE DIFFUSION EQUATIONS

This paper is devoted to the Cauchy problems for the one-dimensional linear time-fractional diffusion equations with ∂_t^α the Caputo fractional derivative of order $\alpha \in (0,1)$ in the variable t and time-degenerate diffusive coefficients t^β with $\beta > -\alpha$. The solutions of Cauchy problems for the one-dimensional time-fractional degenerate diffusion equations with the time-fractional derivative ∂_t^α of order $\alpha \in (0,1)$ in the variable t, are shown. In the "Problem statement and main results"section of the paper, the solution of the time-fractional degenerate diffusion equation in a variable coefficient with two different initial conditions are considered. In this work, a solution is found by using the Kilbas-Saigo function $E_{\alpha,m,l}(z)$ and applying the Fourier transform F and inverse Fourier transform $\mathcal{F}^{-1}$. Convergence of solution of problem 1 and problem 2 are proven using Plancherel theorem. The existence and uniqueness of the solution of the problem are confirmed.
Key words: Time-fractional diffusion equation, Fourier transform, the Kilbas-Saigo function.

М.Б. Бөріханов[1,2], А.Г. Смадиева[3*]
[1]Қожа Ахмет Яссауи атындағы халықаралық қазақ–түрік университеті
Түркістан қ., Қазақстан
[2]Гент университеті, Гент қ., Бельгия
[3]Математика және Математикалық Модельдеу институты, Алматы қ., Қазақстан
[*]e-mail: smadiyeva@math.kz

Бөлшек ретті туындылы өзгешеленген диффузия теңдеулері үшін Коши есебі

Бұл жұмыста t^β, $\beta > -\alpha$ диффузиялық коэффициенттері бар, t айнымалысы бойынша $\alpha \in (0,1)$ үшін бөлшек ретті Капуто туындысы ∂_t^α бар бір өлшемді сызықты бөлшек ретті туындылы өзгешеленген диффузия теңдеулері үшін Коши есептерді шешуге бағытталған. $\alpha \in (0,1)$ үшін бөлшек ретті туындысы ∂_t^α бар бір өлшемді сызықты бөлшек ретті туындылы өзгешеленген диффузия теңдеулеріне қойылған Коши есептерінің шешімдері көрсетілген. Жұмыстың "Есептің қойылымы және негізгі нәтижелер"бөлімінде екі түрлі бастапқы шарттары бар t айнымалысы бойынша $\alpha \in (0,1)$ үшін бөлшек ретті туындысы ∂_t^α бар бір өлшемді сызықты бөлшек ретті туындылы өзгешеленген диффузия теңдеулерінің шешімі қарастырылған. Бұл жұмыста шешім Килбас-Сайго $E_{\alpha,m,l}(z)$ функциясы арқылы берілген, Фурье түрлендіруін $\mathcal{F}$ және кері Фурье түрлендіруін $\mathcal{F}^{-1}$ қолдану арқылы шешім табылған. 1-ші есеп пен 2-ші есептің шешімдерінің жинақтылығы Баншерел теоремасы арқылы дәлелденді. Есептің шешімінің бар болуы мен жалғыздығы дәлелденген.

Түйін сөздер: Бөлшек ретті туындылы өзгешеленген диффузия теңдеуі, Фурье түрлендіруі, Килбас-Сайго функциясы.

М.Б. Бориханов[1,2], А.Г. Смадиева[3*]

[1]Международный казахско–турецкий университет им. Кожа Ахмета Яссауи
Туркистан, Казахстан
[2]Университет Гент, г. Гент, Бельгия
[3]Институт математики и математического моделирования, Алматы, Казахстан
*e-mail: smadiyeva@math.kz

Задача Коши для дробных вырожденных диффузионных уравнений

Данная работа посвящена задачам Коши для одномерных дробных вырожденных линейных диффузионных уравнений с ∂_t^α дробной производной Капуто порядка $\alpha \in (0,1)$ по переменной t и с вырождающимся коэффициентом диффузии t^β при $\beta > -\alpha$. Решения задачи Коши для одномерных уравнений вырождающейся диффузии с дробной по времени производной ∂_t^α порядка $\alpha \in (0,1)$ по переменной t показаны. В разделе "Постановка задачи и основные результаты"статьи рассматривается решение дробных вырожденных линейных диффузионных уравнений с переменным коэффициентом при двух различных начальных условиях. В этой работе решение представлено с помощью функции Килбаса-Сайго $E_{\alpha,m,l}(z)$ и путем применения преобразования Фурье $\mathcal{F}$ и обратного преобразования Фурье $\mathcal{F}^{-1}$. Сходимость решения задачи 1 и задачи 2 доказывается с помощью теоремы Планшереля. Доказаны существование и единственность решения задач.

Ключевые слова: Дробно-вырожденное диффузионное уравнение, преобразование Фурье, функция Килбаса-Сайго.

Introduction

Differential operators of fractional order can be defined in different ways, and therefore, when solving boundary value problems and partial differential equations of fractional order, depending on the proposed operators they should have different approaches.

Many researchers have studied the existence and uniqueness of solutions to fractional diffusion equations. Authors in [1] investigate two inverse source problems for non-homogeneous diffusion equation in rectangular domains. Various types of fractional derivatives and their properties have been investigated by the authors in the [2]-[5]. Fractional calculus is used in many branches of science, such as physics, mechanics, mathematics and etc. [6]-[10]. In [11] obtained results on the existence and uniqueness of a solution to the fractional nonlinear and linear Cauchy problem containing the Riemann-Liouville derivative in $\mathbb{R}^n$. In [12] authors proved a positive maximum (negative minimum) principle diffusion equation with the Caputo derivative of order $\alpha \in (0,1)$. Luchko and Yamamoto proved the existence of a suitably defined generalized solution for the general time-fractional diffusion equations with the Riemann-Liouville and the Caputo type derivatives in [13].

Recently, in [14] the initial-boundary value problems of Dirichlet and Neumann for the Caputo type time-fractional diffusion equation are considered. Also, the regular solution of a mixed problem for the Hilfer type nonlinear partial differential equation in three-dimensional domain is studied in [15].

One of the common definitions of a fractional derivative is Riemann-Liouville, which is applicable, and initial conditions with a fractional derivative are required. In such initial value problems solutions are practically useless, because there is no clear physical interpretation of this type of initial condition [5].

Another definition that can be used to compute a fractional derivative was introduced by Caputo [16] in 1967. The benefit of using the Caputo fractional differentiation operator is that it is applicable; standard initial conditions in terms of integer order derivatives are involved.

Therefore, we separately considered two Cauchy problems for the diffusion equation in a variable coefficient with Caputo and Riemann-Liouville fractional derivatives.

1 Problem statement and main result

1.1 Problem 1.

In this paper we study the time-fractional diffusion equation

$$\partial_t^\alpha u(t,x) - t^\beta u_{xx}(t,x) = 0, \ \ (t,x) \in (0,\infty) \times \mathbb{R}, \tag{1}$$

with the initial data

$$u(0,x) = \phi(x), \ \ x \in \mathbb{R}, \tag{2}$$

where $\beta > -\alpha$ and ∂_t^α is the Caputo fractional derivative of order $\alpha \in (0,1)$ in the variable t, which defined by

$$\partial_t^\alpha u(t,x) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \partial_s u(s,x) ds.$$

We denote $H^2(\mathbb{R})$ as a Hilbert space

$$H^2(\mathbb{R}) = \left\{ f : \ \ f \in L^2(\mathbb{R}); \ \ f_{xx} \in L^2(\mathbb{R}) \right\},$$

endowed with the norm

$$||f||^2_{H^2(\mathbb{R})} = \int_{-\infty}^{+\infty} (1+\xi^2) |\tilde{f}(\xi)|^2 d\xi,$$

where $\tilde{f}(\xi)$ is the inverse Fourier transform.

Definition 1. We say that the solution to the problem (1)-(2) is a function $u \in C([0,\infty); L^2(\mathbb{R}))$, such that $t^{-\beta}\partial_t^\alpha u$, $u_{xx} \in C([0,\infty); L^2(\mathbb{R}))$.

Theorem 1. *Let $\phi \in H^2(\mathbb{R})$, then the unique solution $u \in C([0,\infty); L^2(\mathbb{R}))$ to problem (1)-(2) has the form*

$$u(t,x) = \int_{-\infty}^{+\infty} e^{-ix\xi} \tilde{\phi}(\xi) E_{\alpha, 1+\frac{\beta}{\alpha}, \frac{\beta}{\alpha}}(-\xi^2 t^{\beta+\alpha}) d\xi, \ \ (t,x) \in (0,\infty) \times \mathbb{R},$$

where

$$\tilde{\phi}(\xi) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-is\xi} \phi(s) ds$$

and $E_{\alpha,m,l}(z)$ is the Kilbas-Saigo function [3, Remark 5.1] defined as

$$E_{\alpha,m,l}(z) = \sum_{k=0}^{\infty} c_k z^k, \ \ c_0 = 1, \ \ c_k = \Pi_{j=0}^{k-1} \frac{\Gamma(\alpha(jm+l)+1)}{\Gamma(\alpha(jm+l+1)+1)}, \ \ k \geq 1. \tag{3}$$

In addition, the solution u satisfies the following estimates

$$||u(t,\cdot)||_{C([0,\infty);L^2(\mathbb{R}))} \le ||\phi||_{L^2(\mathbb{R})},$$

$$\sup_{t\in(0,+\infty)} ||t^{-\beta}\partial_t^\alpha u(t,\cdot)||_{L^2(\mathbb{R})} \le ||\phi||_{H^2(\mathbb{R})},$$

$$\sup_{t\in(0,+\infty)} ||u_{xx}(t,\cdot)||_{L^2(\mathbb{R})} \le ||\phi||_{H^2(\mathbb{R})}.$$

The proof of Theorem 1.

• **The existence of a solution.** *Applying the Fourier transform to problem (1)-(2), one obtains*

$$\partial_t^\alpha \tilde{u}(t,\xi) + \xi^2 t^\beta \tilde{u}(t,\xi) = 0, \ \ (t,\xi) \in (0,\infty)\times\mathbb{R}, \tag{4}$$

$$\tilde{u}(0,\xi) = \tilde{\phi}(\xi). \tag{5}$$

The solution of problem (4)-(5) has the form [4, p. 233]

$$\tilde{u}(t,\xi) = \tilde{\phi}(\xi) E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\xi^2 t^{\beta+\alpha}). \tag{6}$$

Further, taking into account the inverse Fourier transform, we get

$$u(t,x) = \int_{-\infty}^{+\infty} e^{-ix\xi}\tilde{\phi}(\xi) E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\xi^2 t^{\beta+\alpha}) d\xi.$$

• **Convergence of the solution.** *Regarding the estimate for the Kilbas-Saigo function with $\alpha \in (0,1)$, $\beta > -\alpha$ provided in [2, Theorem 2]*

$$E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\xi^2 t^{\beta+\alpha}) \le \frac{1}{1+\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\xi^2 t^{\beta+\alpha}}, \ \ t>0, \tag{7}$$

and the Plancherel theorem, we obtain

$$\sup_{t\in(0,+\infty)} \int_{-\infty}^{+\infty} |u(t,x)|^2 dx = \sup_{t\in(0,+\infty)} \int_{-\infty}^{+\infty} |\tilde{u}(t,\xi)|^2 d\xi$$

$$\le \sup_{t\in(0,+\infty)} \int_{-\infty}^{+\infty} |\tilde{\phi}(\xi)|^2 |E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\xi^2 t^{\beta+\alpha})|^2 d\xi$$

$$\le ||\tilde{\phi}||^2_{L^2(\mathbb{R})} = ||\phi||^2_{L^2(\mathbb{R})}.$$

Then, using the following a simple calculation

$$\partial_t^\alpha \left[E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\xi^2 t^{\beta+\alpha}) \right] = -\xi^2 t^\beta E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\xi^2 t^{\beta+\alpha}),$$

we deduce that

$$\partial_t^\alpha u(t,x) = \int_{-\infty}^{+\infty} e^{-ix\xi}\tilde{\phi}(\xi)\partial_t^\alpha \left[E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\xi^2 t^{\beta+\alpha}) \right] d\xi$$

$$= -t^{\beta}\int_{-\infty}^{+\infty} e^{-ix\xi}\tilde{\phi}(\xi)\xi^2 E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\xi^2 t^{\beta+\alpha})d\xi.$$

Therefore, it follows that

$$\sup_{t\in(0,+\infty)} ||t^{-\beta}\partial_t^{\alpha}u(t,\cdot)||^2_{L^2(\mathbb{R})} \leq \sup_{t\in(0,+\infty)} \int_{-\infty}^{+\infty} |\xi|^4|\tilde{\phi}(\xi)|^2|E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\xi^2 t^{\beta+\alpha})|^2 d\xi$$

$$\leq \int_{-\infty}^{+\infty} |\xi^2\tilde{\phi}(\xi)|^2 d\xi \leq ||\phi||_{H^2(\mathbb{R})}.$$

Similarly, for $u_{xx}(t,x)$ one obtains

$$\sup_{t\in(0,+\infty)} ||u_{xx}(t,\cdot)||^2_{L^2(\mathbb{R})} \leq \sup_{t\in(0,+\infty)} \int_{-\infty}^{+\infty} |\xi|^4|\tilde{\phi}(\xi)|^2|E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\xi^2 t^{\beta+\alpha})|^2 d\xi$$

$$\leq \int_{-\infty}^{+\infty} |\xi^2\tilde{\phi}(\xi)|^2 d\xi \leq ||\phi||_{H^2(\mathbb{R})}.$$

• **Uniqueness of the solution.** *Suppose that there exist two solutions u_1 and u_2 to the problem (1)-(2), respectively. Then we choose $u = u_1 - u_2$ in such a way, that u satisfies the equation (1) with the initial condition (2).*

Let us consider the function

$$\tilde{u}(t,\xi) = \int_{-\infty}^{+\infty} e^{-ix\xi}u(t,x)dx, \quad t > 0, \quad \xi \in \mathbb{R}. \tag{8}$$

Applying the Caputo derivative ∂_t^{α} in (8) and remaining (1), we deduce that

$$\partial_t^{\alpha}\tilde{u}(t,\xi) = \int_{-\infty}^{+\infty} e^{-ix\xi}\partial_t^{\alpha}u(t,x)dx$$

$$= t^{\beta}\int_{-\infty}^{+\infty} e^{-ix\xi}u_{xx}(t,x)dx = -t^{\beta}\mathcal{F}[\mathcal{F}^{-1}(\xi^2\tilde{u}(t,x))]$$

$$= -t^{\beta}\xi^2\tilde{u}(t,\xi), \quad t > 0.$$

From (5), it yields

$$\tilde{u}(0,\xi) = 0.$$

In view of the equality (6), we can conclude that $\tilde{u}(t,\xi) = 0, \ t > 0, \ \xi \in \mathbb{R}$. Consequently,

$$\tilde{u}(t,\xi) = \int_{-\infty}^{+\infty} e^{-ix\xi}u(t,x)dx = 0, \quad t > 0, \quad \xi \in \mathbb{R}.$$

Therefore, using the inverse Fourier transform we can get $u \equiv 0$, which guarantees the uniqueness of the solution $u_1 = u_2$.

1.2 Problem 2.

Next, we consider the one-dimensional time-fractional diffusion equation

$$D_t^\alpha u(t,x) - t^\beta u_{xx}(t,x) = 0, \ \ (t,x) \in (0,\infty)\times\mathbb{R}, \tag{9}$$

supplemented with the initial condition

$$I_t^{1-\alpha} u(0,x) = \phi(x), \ \ x\in\mathbb{R}, \tag{10}$$

where $\beta > -\alpha$, $I_t^{1-\alpha}$ *and* D_t^α *are Riemann-Liouville fractional integral and derivative of order* $\alpha \in (0,1)$ *in the variable* t*, respectively, which defined by [4, p. 79-80]*

$$D_t^\alpha f(t) = \frac{d}{dt} I_t^{1-\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt}\int_0^t \frac{f(s)ds}{(t-s)^\alpha}, \tag{11}$$

and

$$I_t^{1-\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f(s)ds}{(t-s)^\alpha}. \tag{12}$$

Definition 2. *We say that the solution to the problem (9)-(10) is a function* $t^{1-\alpha}u \in C((0,\infty); L^2(\mathbb{R}))$*, such that* $t^{1-\alpha-\beta} D_t^\alpha u, \ t^{1-\alpha} u_{xx} \in C((0,\infty); L^2(\mathbb{R}))$.

Theorem 2. *Let* $\phi(x) \in H^2(\mathbb{R})$*, then the unique solution* u *of problem (9)-(10) has represented by*

$$u(t,x) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_{-\infty}^{+\infty} e^{-ix\xi} \tilde{\phi}(\xi) E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}(-\xi^2 t^{\beta+\alpha}) d\xi, \ \ (t,x)\in(0,\infty)\times\mathbb{R},$$

where

$$\tilde{\phi}(\xi) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-is\xi}\phi(s)ds,$$

and $E_{\alpha,m,l}(z)$ *is the Kilbas-Saigo function defined by (3) such that satisfies the estimate [2, Proposition 2]*

$$E_{\alpha,m,m-\frac{1}{\alpha}}(-\xi^2 t^{\beta+\alpha}) \le \frac{1}{\left(1 + \frac{\Gamma(1+\alpha m)}{\Gamma(1+\alpha(m+1))}\xi^2 t^{\beta+\alpha}\right)^{1+\frac{1}{m}}}, \ \ t>0, \tag{13}$$

with $\alpha\in(0,1)$, $\beta > -\alpha$, $m = 1 + \frac{\beta}{\alpha} > 0$.

Moreover, the next inequalities are valid for the u *solution*

$$\sup_{t\in(0,+\infty)} ||t^{1-\alpha}u(t,\cdot)||_{L^2(\mathbb{R})} \le ||\phi||_{L^2(\mathbb{R})},$$

$$\sup_{t\in(0,+\infty)} ||t^{1-\alpha-\beta} D_t^\alpha u(t,\cdot)||_{L^2(\mathbb{R})} \le ||\phi||_{H^2(\mathbb{R})},$$

$$\sup_{t\in(0,+\infty)} t^{1-\alpha}||u_{xx}(t,\cdot)||_{L^2(\mathbb{R})} \le ||\phi||_{H^2(\mathbb{R})}.$$

Proof of Theorem 2.

- **The existence of a solution.** *In view of the Fourier transform we can verify that*

$$D_t^\alpha \tilde{u}(t,\xi) + \xi^2 t^\beta \tilde{u}(t,\xi) = 0, \quad (t,\xi) \in (0,\infty) \times \mathbb{R}, \tag{14}$$

with the initial data

$$I_t^{1-\alpha} \tilde{u}(0,\xi) = \tilde{\phi}(\xi). \tag{15}$$

The solution of the problem (14)-(15) can be given explicitly by [4, p. 227]

$$\tilde{u}(t,\xi) = \tilde{\phi}(\xi) t^{\alpha-1} E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}(-\xi^2 t^{\beta+\alpha}). \tag{16}$$

Consequently, from the inverse Fourier transform, we have

$$u(t,x) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_{-\infty}^{+\infty} e^{-ix\xi} \tilde{\phi}(\xi) E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}(-\xi^2 t^{\beta+\alpha}) d\xi.$$

- **Convergence of the solution.** *Thanks to the estimate (13) and Plancherel's theorem, we deduce that*

$$\sup_{t\in(0,+\infty)} \int_{-\infty}^{+\infty} |t^{1-\alpha} u(t,x)|^2 dx = \sup_{t\in(0,+\infty)} \int_{-\infty}^{+\infty} |t^{1-\alpha} \tilde{u}(t,\xi)|^2 d\xi$$

$$\leq \sup_{t\in(0,+\infty)} \left| \frac{1}{\Gamma(\alpha)} \right|^2 \int_{-\infty}^{+\infty} |\tilde{\phi}(\xi)|^2 |E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}(-\xi^2 t^{\beta+\alpha})|^2 d\xi$$

$$\leq ||\tilde{\phi}||^2_{L^2(\mathbb{R})} = ||\phi||^2_{L^2(\mathbb{R})}.$$

In view of the next simple identity

$$D_t^\alpha \left[t^{\alpha-1} E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}(-\xi^2 t^{\beta+\alpha}) \right] = -\xi^2 t^{\beta+\alpha-1} E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}(-\xi^2 t^{\beta+\alpha}),$$

one obtains

$$D_t^\alpha u(t,x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{+\infty} e^{-ix\xi} \tilde{\phi}(\xi) D_t^\alpha \left[t^{\alpha-1} E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}(-\xi^2 t^{\beta+\alpha}) \right] d\xi$$

$$= -\frac{t^{\beta+\alpha-1}}{\Gamma(\alpha)} \int_{-\infty}^{+\infty} e^{-ix\xi} \tilde{\phi}(\xi) \xi^2 E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}(-\xi^2 t^{\beta+\alpha}) d\xi.$$

Hence, it yields

$$\sup_{t\in(0,+\infty)} ||t^{1-\alpha-\beta} D_t^\alpha u(t,\cdot)||^2_{L^2(\mathbb{R})}$$

$$\leq \sup_{t\in(0,+\infty)} \left| \frac{1}{\Gamma(\alpha)} \right|^2 \int_{-\infty}^{+\infty} |\xi|^4 |\tilde{\phi}(\xi)|^2 |E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}(-\xi^2 t^{\beta+\alpha})|^2 d\xi$$

$$\leq \int_{-\infty}^{+\infty} |\xi^2 \tilde{\phi}(\xi)|^2 d\xi \leq ||\phi||_{H^2(\mathbb{R})}.$$

Similarly, for u_{xx} *we can get the following estimate*

$$\sup_{t\in(0,+\infty)} t^{1-\alpha}||u_{xx}(t,\cdot)||^2_{L^2(\mathbb{R})}$$

$$\leq \sup_{t\in(0,+\infty)} \left|\frac{1}{\Gamma(\alpha)}\right|^2 \int_{-\infty}^{+\infty} |\xi|^4|\tilde{\phi}(\xi)|^2|E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}(-\xi^2t^{\beta+\alpha})|^2d\xi$$

$$\leq \int_{-\infty}^{+\infty} |\xi^2\tilde{\phi}(\xi)|^2d\xi \leq ||\phi||_{H^2(\mathbb{R})}.$$

• **Uniqueness of the solution.** *Suppose that there are two solutions* u_1 *and* u_2 *of problem (9)-(10) then* $u = u_1 - u_2$ *in such a way, that* u *satisfies the equation (9) and the condition (10).*

Next, we consider the function

$$\tilde{u}(t,\xi) = \int_{-\infty}^{+\infty} e^{-ix\xi}u(t,x)dx, \;\; t > 0, \;\; \xi \in \mathbb{R}. \tag{17}$$

Applying the operator D_t^α *to the function (17) and recalling (9), we have*

$$D_t^\alpha\tilde{u}(t,\xi) = \int_{-\infty}^{+\infty} e^{-ix\xi}D_t^\alpha u(t,x)dx$$

$$= -t^\beta \int_{-\infty}^{+\infty} e^{-ix\xi}u_{xx}(t,x)dx$$

$$= t^\beta\mathcal{F}[\mathcal{F}^{-1}(\xi^2\tilde{u}(t,x))]$$

$$= t^\beta\xi^2\tilde{u}(t,\xi), \;\; t > 0, \;\; \xi \in \mathbb{R}.$$

From (15) we obtain

$$I_t^{1-\alpha}\tilde{u}(0,\xi) = 0.$$

Then using the expression (16) we conclude that $I_t^{1-\alpha}\tilde{u}(t,\xi) = 0$*, which verifies* $\tilde{u}(t,\xi) = 0$*. Finally, applying the inverse transform we have* $u(t,x) \equiv 0$*, consequently* $u_1 = u_2$*.*

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