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CONVOLUTIONS GENERATED BY THE DIRICHLET PROBLEM OF THE STURM-LIOUVILLE OPERATOR

This paper is devoted to approximations of the product of two continuous functions on a finite segment by some special convolutions. The accuracy of the approximation depends on the length of the segment on which the functions are defined. These convolutions are generated by the Sturm-Liouville boundary value problems. The paper indicates that each boundary value problem for a second order differential equation generates its own individual convolution and its own individual Fourier transform. At that the Fourier transform of the convolution is equal to the product of the Fourier transforms. The latter property makes it possible to approximately solve nonlinear Burgers-type equations by first replacing the nonlinear term with a convolution of two functions. Similar methods of studying nonlinear partial differential equations can be found in the works of A. Y. Kolesov, N. H. Rozov, V. A. Sadovnichy.
In this paper, we construct a concrete convolution generated by the Dirichlet boundary value problem for twofold differentiation. The properties of the constructed convolution and their connection with the corresponding Fourier transform are derived. In the final part of the paper, the convergence of convolution is proved $(g(x)\sin(x)) * (f(x)\sin(x))$ defined on a segment $C[0,b]$ to the product $g(x)f(x)$ with b tending to zero for any two continuous functions $f(x)$ and $g(x)$.
Key words: approximation, convolution, boundary value problems, Dirichlet problem, Fourier transform.

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Штурм-Лиувилль операторының Дирихле есебінен туындайтын үйірткілер

Бұл жұмыс ақырлы кесіндіде анықталған арнайы үйірткілері бар екі үзіліссіз функцияның көбейтіндісінің аппроксимациясына арналған. Берілген функцияның жуықтау дәлдігі кесіндінің ұзындығына байланысты. Бұл үйірткілер Штурм-Лиувилль шеттік есебінен туындайды. Жұмыста екінші ретті дифференциалдық теңдеу үшін әрбір шеттік есептің өзінің жеке үйірткісі мен Фурье түрлендіруінің туындатылатынын көрсетеді. Сонымен қатар, бұл үйірткіден алынған Фурье түрлендіруі Фурье түрлендірулерінің көбейтіндісіне тең. Соңғы қасиет екі функцияның үйірткісінің сызықты емес мүшесінің алдын-ала алмастыру арқылы Бюргерс типті сызықты емес теңдеулерді жуықтап шешуге мүмкіндік береді. Дербес туындылары бар сызықты емес дифференциалдық теңдеулерді зерттеудің ұқсас әдістерін А.Ю. Колесов, Н.Х. Розов, В.А. Садовничий-лердің еңбектерінен табуға болады.
Жұмыста екі еселенген дифференциал үшін Дирихле шеттік есебінен туындаған нақты үйірткі құрылады. Құрылған үйірткінің қасиеттері және олардың Фурье түрлендірулерімен байланысы көрсетілген. Жұмыстың соңғы бөлімінде $C[0,b]$ кесіндісінде анықталған $(g(x)\sin(x)) * (f(x)\sin(x))$ үйірткісі үшін кез келген екі үзіліссіз $f(x)$, $g(x)$ функцияларының $g(x)f(x)$ көбейтісінің b нөлге ұмтылғандағы жинақтылығы дәлелденген.
Түйін сөздер: жуықтау, үйірткі, шеттік есеп, Дирихле есебі, Фурье түрлендіруі.

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Свертки, порождаемые задачей Дирихле оператора Штурма-Лиувилля

Настоящая работа посвящена аппроксимаций произведения двух непрерывных на конечном отрезке функций некоторыми специальными свертками. Точность приближения зависит от длины отрезка на котором задаются функций. Эти свертки порождаются краевыми задачами Штурма-Лиувилля. В работе указывается, что каждая краевая задача для дифференциального уравнения второго порядка порождает свою индивидуальную свертку и свое индивидуальное преобразование Фурье. Причем преобразование Фурье от свертки равно произведению преобразований Фурье. Последнее свойство позволяет приближенно решать нелинейные уравнения типа Бюргерса, предварительно заменив нелинейный член сверткой двух функций. Подобные методы исследования нелинейных дифференциальных уравнений с частными производными можно найти в работах А. Ю. Колесова, Н. Х. Розова, В. А. Садовничего.
В работе строится конкретная свертка, порожденная краевой задачей Дирихле для двухкратного дифференцирования. Выведены свойства построенной свертки и связь их с соответствующим преобразованием Фурье. В заключительной части работы доказана сходимость свертки $(g(x)\sin(x)) * (f(x)\sin(x))$ определенной на отрезки $C[0,b]$ к произведению $g(x)f(x)$ при b стремящемся к нулю для любых двух непрерывных функций $f(x)$ и $g(x)$.
Ключевые слова: приближение, свертка, краевые задачи, задача Дирихле, преобразование Фурье.

1 Introduction

In this paper, we are interested in the approximation of nonlinear terms of differential operators by some special convolutions. To motivate our research, let's consider the Burgers equation for simplicity

$$\frac{\partial u(t,x)}{\partial t} + u(t,x)\frac{\partial u(t,x)}{\partial x} = \nu \frac{\partial^2 u(t,x)}{\partial x^2},\ 0 < x < b,\ t > 0 \tag{1}$$

on a finite segment $(0, b)$ with kinematic viscosity ν. Replacement

$$\xi = x\sqrt{\nu} \tag{2}$$

equation (1) leads to the form

$$\frac{\partial v(t,\xi)}{\partial t} + v(t,x)\sqrt{\nu}\frac{\partial v(t,\xi)}{\partial \xi} = \frac{\partial^2 v(t,\xi)}{\partial \xi^2},\ 0 < \xi < b\sqrt{\nu},\ t > 0. \tag{3}$$

Whereas as kinematic viscosity ν and the Reynolds number are mutually inverse, then there is a critical viscosity value ν_{cr}. When $\nu > \nu_{cr}$ the fluid flow will be steadily laminar. Movement at $\nu < \nu_{cr}$ becomes unstably turbulent. Thus, for small values ν there is a movement of the liquid acquiring a turbulent character. If $\nu \to 0$, then the length of the interval $[0, b\sqrt{\nu}]$ becomes a small quantity. In this case, it becomes possible to approximate of the nonlinear term $v(t,x)\dfrac{\partial v(t,\xi)}{\partial \xi}$ by some special convolution $v_1 * \left(\dfrac{\partial v}{\partial \xi} s_0(\xi)\right)$. Here $v_1(t,\xi) = v(t)s_0(\xi)$, where $s_0(\xi)$ - fixed function. By convolution we mean some two-dimensional, associative, bilinear operation consistent with the corresponding Fourier transform. In other words, if F is Fourier transform, then the equality

$$F(f * g) = Ff \cdot Fg \tag{4}$$

is rightly for the convolution we introduced. Then for small values ν equation (3) can be approximated by its approximation

$$\frac{\partial W(t,\xi)}{\partial t} + (W(t,\xi)s_0(\xi)) * \left(\frac{\partial W(t,\xi)}{\partial t}s_0(\xi)\right) = \frac{\partial^2 W(t,\xi)}{\partial \xi^2}. \tag{5}$$

Property (4) makes it possible to solve equation (5) efficiently by the method of separation of variables. Similar schemes used in works [3]-[14].

A wide set of convolutional operations generate boundary value problems for linear differential operators. In mathematical physics, the solution of an inhomogeneous equation $Au = f$ is written as a convolution of two functions $u = \varepsilon * f$, where ε is the corresponding fundamental solution [3]. Under the convolution is understood to be the bilinear a (possibly noncommutative) operation without the right annulators. When there is an inverse operator A^{-1}, then the convolution associated with the linear operator A has nonzero divisors. If the operator A corresponds to a boundary value problem in a bounded domain, then the convolution may depend on its boundary conditions. For example, the convolution corresponding to the operator B_1 in the function space $L_2(0,1)$ has the following form

$$(f *_{B_1} g)(x) = \int_0^x f(x-t)g(t)dt + \frac{1}{h}\int_x^1 f(1+x-t)g(t)dt.$$

Here the operator B_1 corresponds to the boundary value problem

$$-i\frac{dy}{dt} = f(x),\ 0 < x < 1,\ y(1) = hy(0).$$

The resolvent of the B_1 operator has a convolutional representation

$$(B_1 - \lambda I)^{-1} f(x) = (\varepsilon_\lambda *_{B_1} f)(x),\ where\, \varepsilon_\lambda(t) = ih\frac{e^{i\lambda t}}{h - e^{i\lambda}}. \tag{6}$$

The convolution $*_{B_1}$ defined by formula (6) depends on the boundary parameter h. A more difficult example is given [4]. In the Hilbert space $L_2[0,1]$, we define the operator B_2 generated by the differential expression $lu = -\dfrac{d^2u(x)}{dx^2}$, $0 < x < 1$, and the domain of definition

$$D(B_2) = \left\{u \in W_2^2[0,1] : u(0) = 0,\ u^{'}(1) = u(1)\right\}.$$

The spectral properties of operator are studied in detail in the work of N. I. Ionkin [5]. The convolution generated by the operator B_2 is defined by the formula

$$(g *_{B_2} f)(x) = \frac{1}{2}\int_x^1 g(1+x-t)f(t)dt + \int_{1-x}^1 g(x-1+t)f(t)dt +$$

$$+\int_0^x g(x-t)f(t)dt - \frac{1}{2}\int_0^{1-x} g(1-x-t)f(t)dt + \frac{1}{2}\int_0^x g(1+t-x)f(t)dt.$$

In this case, the resolvent of operator B_2 has the convolutional representation

$$(B_2-\lambda I)^{-1}f(x) = (\varepsilon_\lambda *_{B_2} f)(x), where\, \varepsilon_\lambda(t) = \frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}(cos(\sqrt{\lambda}-1))}.$$

In [6, 7, 8, 9] we can find convolutions generated by first-order differential operators with integral boundary conditions. In the works of M. V. Ruzhansky and his co-authors [4, 10, 11, 12], convolutions generated by

1. Operators whose root elements form a Riesz basis in the corresponding space.

2. Riesz basis of the Hilbert space are investigated.

In [8], the construction of explicit convolution formulas uses representation of the Green function. Usually, the Green function $G(x,t)$ is a two-place function, while the fundamental solution $\varepsilon(t)$ is a one-place function. When deriving an explicit convolution formula, it is necessary to express the two-place function $G(x,t)$ linearly in terms of the one-place function $\varepsilon(t)$, and it is allowed to use integration and differentiation operations [13].

In the future, we will need a convolution generated by the periodic problem. For the operator B_3 corresponding to the periodic problem

$$-y - f(x),\ 0<x<1,\ y(0)=y(1),\ y'(0)=y'(1)$$

convolution $*_{B_3}$ has the following form

$$(f*_{B_3}g)(x) = \int_0^x f(x-t)g(t)dt + \int\ \_x^1 f(t-x)g(t)dt + \int_0^x f(1+t-x)g(t)dt + \int_x^1 f(1+x-t)g(t)dt.$$

The resolvent of the $*_{B_3}$ operator has a convolutional representation

$$(B_3-\lambda I)^{-1}f(x) = (\varepsilon_\lambda *_{B_3} f)(x),$$

where

$$\varepsilon_\mu(x) = -\frac{\sin(\sqrt{\lambda}x)}{2\sqrt{\lambda}(1-cos(\sqrt{\lambda}))}.$$

In the future, the convolution $*_{B_3}$ is re denoted by $*$.

2 Integral representation of the solution to the Dirichlet problem of the Sturm-Liouville operator

The main result of this section is stated in the following lemma.

Lemma 1 *In the function space $L_2(0,b)$ is studied the Dirichlet problem for the Sturm-Liouville equation*

$$-y''(x) = \lambda y(x) + f(x),\ 0<x<b \tag{7}$$

$$y(0) = 0,\ y(b) = 0. \tag{8}$$

The solution to problem (7)-(8) at $0 < x < b$ *has the representation*

$$y(x) = \frac{1}{2\sin\sqrt{\lambda}b}\left\{\int_0^x \sin\sqrt{\lambda}(b-x+\tau)d\tau\int_\tau^x f(t)dt + \int_0^x \sin\sqrt{\lambda}(b-x-\tau)d\tau\int_\tau^x f(t)dt\right\}+$$

$$+\frac{1}{2\sin\sqrt{\lambda}b}\left\{\int_0^x d\xi\int_x^b \sin\sqrt{\lambda}(b-t+\xi)f(t)dt + \frac{1}{2\sin\sqrt{\lambda}b}\int_0^x d\xi\int_x^b \sin\sqrt{\lambda}(b-t-\xi)f(t)dt\right\}. \tag{9}$$

In the functional space $L_2(0,b)$, we denote by B the Sturm-Liouville operator, which corresponds to the Dirichlet problem (7)-(8).Then the right part of formula (9) determines of the resolvent of operator B.

Proof of Lemma 1. First, let us check that the right part of relation (7) satisfies boundary conditions (8). For this, it is necessary to denote the right part of relation (9) by $u(x)$. Then direct substitution into $u(x)$ the values $x = 0$ and $x = b$ leads to the equalities

$$u(0) = 0,$$

$$u(b) = \frac{1}{2\sin\sqrt{\lambda}b}\left\{\int_0^b \sin\sqrt{\lambda}\tau d\tau\int_\tau^b f(t)dt - \int_0^b \sin\sqrt{\lambda}\tau d\tau\int_\tau^b f(t)dt\right\} = 0.$$

Now need to check that the function $u(x)$ is a solution of equation (7). For this we calculate the corresponding derivatives.

$$u'(x) = -\frac{1}{2\sin\sqrt{\lambda}b}\sqrt{\lambda}\left\{\int_0^x \cos\sqrt{\lambda}(b-x+\tau)d\tau\int_\tau^x f(t)dt - \int_0^x \cos\sqrt{\lambda}(b-x-\tau)d\tau\int_\tau^x f(t)dt\right\}+$$

$$+\frac{1}{2\sin\sqrt{\lambda}b}\left\{\int_x^b \sin\sqrt{\lambda}(b-t+x)f(t)dt + \int_x^b \sin\sqrt{\lambda}(b-t-x)f(t)dt\right\},$$

$$u''(x) = -\frac{1}{2\sin\sqrt{\lambda}b}\lambda\int_{-x}^x d\xi\int_x^b \sin\sqrt{\lambda}(b-t+\xi)f(t)dt-$$

$$-\frac{1}{2\sin\sqrt{\lambda}b}\left\{\int_0^x \sin\sqrt{\lambda}(b-x-\tau)d\tau\int_\tau^x f(t)dt - \int_0^x \sin\sqrt{\lambda}(b-x+\tau)\int_\tau^x f(t)dt\right\}-f(x).$$

The identity is used here

$$\int_0^x \sin\sqrt{\lambda}(b-x+\tau)d\tau\int_\tau^x f(t)dt - \int_0^x \sin\sqrt{\lambda}(b-x-\tau)d\tau\int_\tau^x f(t)dt = \int_{-x}^x d\xi\int_x^b \sin\sqrt{\lambda}(b-t+\xi)f(t)dt.$$

Then the equality follows required $u''(x) = -\lambda u(x) - f(x)$. Thus, Lemma 1 is completely proved.

3 Convolutions generated by the Dirichlet problem of the Sturm-Liouville operator

In this paragraph, the convolution formula is given, which corresponds to the Dirichlet problem of the Sturm-Liouville operator.

For any two functions $f(x), g(x) \in L_2(0,b)$ introduce a convolution, which at $0 < x < \frac{b}{2}$ is determined by the formula

$$(g*f)(x) = \int_0^x g(b-x+\tau)d\tau \int_\tau^x f(t)dt + \int_0^x g(b-x-\tau)d\tau \int_\tau^x f(t)dt +$$

$$+ \int_0^x d\xi \int_x^b g(b-t+\xi)f(t)dt + \int_0^x d\xi \int_x^{\frac{b}{2}} g(b-t-\xi)f(t)dt - \tag{10}$$

$$- \int_0^x d\xi \int_{\frac{b}{2}}^{b-\xi} g(b-t-\xi)f(t)dt - \int_0^x d\xi \int_{b-\xi}^b g(t+\xi-b)f(t)dt,$$

and at $\frac{b}{2} < x < b$ is determined by the formula

$$(g*f)(x) = \int_0^x g(b-x+\tau)d\tau \int_\tau^x f(t)dt + \int_0^{b-x} g(b-x-\tau)d\tau \int_\tau^x f(t)dt -$$

$$- \int_{b-x}^x g(x+\tau-b)d\tau \int_\tau^x f(t)dt + \int_0^x d\xi \int_x^b g(b-t+\xi)f(t)dt + \tag{11}$$

$$+ \int_0^{\frac{b}{2}} d\xi \int_x^{b-\xi} g(b-t-\xi)f(t)dt - \int_0^{\frac{b}{2}} d\xi \int_{b-\xi}^b g(t+\xi-b)f(t)dt - \int_{\frac{b}{2}}^x d\xi \int_x^b g(t+\xi-b)f(t)dt.$$

The convolution introduced by us is linear for each argument and has associativity properties, at the same time, this convolution is not commutative.

Definition 1 *We will say that * convolution is generated by the operator B if its resolvent $(B-\lambda I)^{-1}$ has the following convolutional representation*

$$(B-\lambda I)^{-1}f(x) = (\varepsilon_\lambda * f)(x),$$

where ε_λ is the corresponding fundamental solution.

In functional space $L_2(0,b)$ the Sturm-Liouville operator corresponding to the Dirichlet problem (7) - (8) is denoted by B.

Lemma 2 *The convolution given by formulas (10)-(11) is generated by the operator B.*

Proof of Lemma 2. According to Lemma 1, the resolvent of operator B is given using the right side of formula (9). In this paragraph, we will rewrite the right part of formula (9) in a convenient form for further research:

If $0 < x < \frac{b}{2}$, then

$$u(x) = \frac{1}{2\sin\sqrt{\lambda}b}\left\{\int_0^x \sin\sqrt{\lambda}(b-x+\tau)d\tau\int_\tau^x f(t)dt + \int_0^x \sin\sqrt{\lambda}(b-x-\tau)d\tau\int_\tau^x f(t)dt\right\}+$$

$$+\frac{1}{2\sin\sqrt{\lambda}b}\left\{\int_0^x d\xi\int_x^b \sin\sqrt{\lambda}(b-t+\xi)f(t)dt + \int_0^x d\xi\int_x^{\frac{b}{2}} \sin\sqrt{\lambda}(b-t-\xi)f(t)dt\right\}-$$

$$-\frac{1}{2\sin\sqrt{\lambda}b}\left\{\int_0^x d\xi\int_{\frac{b}{2}}^{b-\xi} \sin\sqrt{\lambda}(b-t-\xi)f(t)dt + \int_0^x d\xi\int_{b-\xi}^b \sin\sqrt{\lambda}(t+\xi-b)f(t)dt\right\}.$$

If $\frac{b}{2} < x < b$, then

$$u(x) = \frac{1}{2\sin\sqrt{\lambda}b}\left\{\int_0^x \sin\sqrt{\lambda}(b-x+\tau)d\tau\int_\tau^x f(t)dt + \int_0^{b-x} \sin\sqrt{\lambda}(b-x-\tau)d\tau\int_\tau^x f(t)dt\right\}-$$

$$-\frac{1}{2\sin\sqrt{\lambda}b}\left\{\int_{b-x}^x \sin\sqrt{\lambda}(x+\tau-b)d\tau\int_\tau^x f(t)dt - \int_0^x d\xi\int_x^b \sin\sqrt{\lambda}(b-t+\xi)f(t)dt\right\}+$$

$$+\frac{1}{2\sin\sqrt{\lambda}b}\left\{\int_0^{\frac{b}{2}} d\xi\int_x^{b-\xi} \sin\sqrt{\lambda}(b-t-\xi)f(t)dt - \int_0^{\frac{b}{2}} d\xi\int_{b-\xi}^b \sin\sqrt{\lambda}(t+\xi-b)f(t)dt\right\}-$$

$$-\frac{1}{2\sin\sqrt{\lambda}b}\int_{\frac{b}{2}}^x d\xi\int_x^b \sin\sqrt{\lambda}(t+\xi-b)f(t)dt.$$

It is easy to notice that the value of the resolvent $(B-\lambda I)^{-1}f(x)$ coincides with $u(x)$. On the other hand, the solution has a convolutional representation

$$u(x) = (g * f)(x),$$

where $g(x) = \dfrac{\sin\sqrt{\lambda}x}{2\sin\sqrt{\lambda}b}$. Lemma 2 is completely proved.

4 The Fourier transform generated by the operator B

The poles of the resolvent $(B-\lambda I)^{-1}$ determine the eigenvalues of the operator B. Since

$$(B-\lambda I)^{-1}f(x) = (g * f)(x), \tag{12}$$

where $g(x) = \dfrac{\sin\sqrt{\lambda}x}{2\sin\sqrt{\lambda}b}$ is the appropriate fundamental solution. It follows from the representation (12) that the poles of the resolvent $(B-\lambda I)^{-1}$ are zeros of the function $\sin\sqrt{\lambda}b = 0$. It follows that zeros have the form

$$\lambda_k = \frac{\pi^2}{b^2}k^2, k = 0, \pm 1, ... \tag{13}$$

A more detailed analysis shows that λ_0 is the eliminable singular point of the resolvent $(B-\lambda I)^{-1}$.

Thus, the resolvent $(B-\lambda I)^{-1}$ has only simple poles $\lambda_k = \dfrac{\pi^2}{b^2}k^2, k = 1, 2, ...$ In order to find their corresponding eigenfunctions of the operator B, we need to calculate

$$\underset{\lambda_k}{res}(B-\lambda I)^{-1}f(x) = -\frac{4}{b}\sin\frac{\pi kx}{b}\int_0^b f(x)\sin\frac{\pi kt}{b}dt.$$

The direct calculation of the residue at the point $\lambda = \lambda_k$ leads to the formula

$$\underset{\lambda_k}{res}(B-\lambda I)^{-1}f(x) = -\frac{4}{b}\sin\frac{\pi kx}{b}\int_0^b f(x)\sin\frac{\pi kt}{b}dt =$$

$$= \frac{(-1)^{2k+1}2\pi k}{b^2}\left\{\int_0^x \sin\frac{\pi k}{b}(x-\tau)d\tau\int_\tau^x f(t)dt + \int_0^x \sin\frac{\pi k}{b}(x+\tau)d\tau\int_\tau^x f(t)dt\right\} +$$

$$+\frac{(-1)^{2k+1}2\pi k}{b^2}\left\{\int_0^x d\xi\int_x^b \sin\frac{\pi k}{b}(t-\xi)f(t)dt + \int_0^x d\xi\int_x^b \sin\frac{\pi k}{b}(t+\xi)f(t)dt\right\} =$$

$$= -\frac{2\pi k}{b^2}\left[\int_0^x f(t)dt\left\{\int_0^t \sin\frac{\pi k}{b}(x-\tau)d\tau + \int_0^t \sin\frac{\pi k}{b}(x+\tau)d\tau\right\}\right] -$$

$$-\frac{2\pi k}{b^2}\left[\int_x^b f(t)dt\left\{\int_0^x \sin\frac{\pi k}{b}(t-\xi)d\xi + \int_0^x \sin\frac{\pi k}{b}(t+\xi)d\xi\right\}\right] =$$

$$= -\frac{2\pi k}{b^2}\left[\int_0^x f(t)dt\int_0^t 2\sin\frac{\pi k}{b}x\cos\frac{\pi k}{b}\tau d\tau + \int_x^b f(t)dt\int_0^x 2\sin\frac{\pi k}{b}t\cos\frac{\pi k}{b}\xi d\xi\right] =$$

$$= -\frac{2\pi k}{b^2}\left[\frac{2b}{\pi k}\int_0^x f(t)\sin\frac{\pi kx}{b}\sin\frac{\pi kt}{b}dt + \frac{2b}{\pi k}\int_x^b f(t)\sin\frac{\pi kx}{b}\sin\frac{\pi kt}{b}dt\right] =$$

$$= -\frac{4}{b}\int_0^b f(t)\sin\frac{\pi kx}{b}\sin\frac{\pi kt}{b}dt = -\frac{4}{b}\sin\frac{\pi kx}{b}\int_0^b f(t)\sin\frac{\pi kt}{b}dt.$$

It follows from this that the system of eigenfunctions of the operator B has the form $\{\sin\dfrac{\pi x}{b}, \sin\dfrac{2\pi x}{b}, \sin\dfrac{3\pi x}{b}, ...\}$, corresponding Fourier coefficients are calculated by the formulas

$$b_k(f) = -\underset{\lambda_k}{res}(B-\lambda I)^{-1}f(x) = \frac{4}{b}\int_0^b f(t)\sin\frac{\pi k}{b}tdt, k = 1, 2, ...$$

5 The relation of the Fourier transform and convolution generated by the operator B

Take two functions f and g from $L_2(0,b)$ and decompose them according to the system of eigenfunctions $\{\sin\frac{\pi x}{b}, \sin\frac{2\pi x}{b}, \sin\frac{3\pi x}{b}, ...\}$, as a result , we have

$$f(x)=\sum_{k=1}^{\infty} b_k(f)\sin\frac{\pi kx}{b}$$

$$g(x)=\sum_{j=1}^{\infty} b_j(g)\sin\frac{\pi jx}{b}.$$

Now calculate the convolution $(g*f)(x)$, to do this, we formulate an auxiliary statement.

Lemma 3 *For any k and j, the equality is true*

$$\sin\frac{\pi kx}{b}*\sin\frac{\pi jx}{b}=0, k\neq j,$$

$$\sin\frac{\pi kx}{b}*\sin\frac{\pi jx}{b}=1, k=j.$$

Proof of Lemma 3. By definition, at $0<x<\frac{b}{2}$

$$\sin\frac{\pi kx}{b}*\sin\frac{\pi jx}{b}=\int_0^x \sin\frac{\pi k}{b}(b-x+\tau)d\tau\int_\tau^x \sin\frac{\pi jt}{b}dt+$$

$$+\int_0^x \sin\frac{\pi k}{b}(b-x-\tau)d\tau\int_\tau^x \sin\frac{\pi jt}{b}dt+\int_0^x d\xi\int_x^b \sin\frac{\pi k}{b}(b-t+\xi)\sin\frac{\pi jt}{b}dt+$$

$$+\int_0^x d\xi\int_x^{\frac{b}{2}} \sin\frac{\pi k}{b}(b-t-\xi)\sin\frac{\pi jt}{b}dt-\int_0^x d\xi\int_{\frac{b}{2}}^{b-\xi} \sin\frac{\pi k}{b}(b-t-\xi)\sin\frac{\pi jt}{b}dt-$$

$$-\int_0^x d\xi\int_{b-\xi}^{b} \sin\frac{\pi k}{b}(t+\xi-b)\sin\frac{\pi jt}{b}dt.$$

For $\frac{b}{2}<x<b$, the statement of Lemma 3 is checked similarly. Lemma 3 is fully proved. Immediately follows by Lemma 3

$$(g*f)(x)=\sum b_k(f)b_k(g).$$

6 Approximation of multiplication by convolution

Let f, g be two arbitrary functions both defined and continuous on the segment $[0, b]$. Denote by $g_1(x)$ and $f_1(x)$

$$g_1(x) \equiv g(x)\sin(x),\ f_1(x) \equiv f(x)\sin(x).$$

Theorem 1 *For any two functions f and g continuous on $[0, b]$ is the limiting relation rightly*

$$\lim_{b\to 0}[(g_1 * f_1)(x) - g(x)f(x)] = 0, \forall x \in [0, b].$$

Consider the difference

$$R(x) \equiv (g_1 * f_1)(x) - g(x)f(x)$$

Proof of Theorem 1. Introduce the notation

$$R_1(x) = \int_0^x g(b-x+\tau)\sin(b-x+\tau)d\tau \int_\tau^x f(t)\sin(t)dt - \int_0^x g(x)\sin(b-x+\tau)d\tau \int_\tau^x f(x)\sin(t)dt,$$

$$R_2(x) = \int_0^x g(b-x-\tau)\sin(b-x-\tau)d\tau \int_\tau^x f(t)\sin(t)dt - \int_0^x g(x)\sin(b-x-\tau)d\tau \int_\tau^x f(x)\sin(t)dt,$$

$$R_3(x) = \int_0^x d\xi \int_x^b g(b-t+\xi)sin(b-t+\xi)f(t)\sin(t)dt - \int_0^x g(x)d\xi \int_x^b f(x)\sin(b-t+\xi)\sin(t)dt,$$

$$R_4(x) = \int_0^x d\xi \int_x^{\frac{b}{2}} g(b-t-\xi)\sin(b-t-\xi)f(t)\sin(t)dt - \int_0^x g(x)d\xi \int_x^{\frac{b}{2}} f(x)\sin(b-t-\xi)\sin(t)dt,$$

$$R_5(x) = \int_0^x d\xi \int_{\frac{b}{2}}^{b-\xi} g(b-t-\xi)\sin(b-t-\xi)f(t)sin(t)dt - \int_0^x g(x)d\xi \int_{\frac{b}{2}}^{b-\xi} f(x)\sin(b-t-\xi)\sin(t)dt,$$

$$R_6(x) = \int_0^x d\xi \int_{b-\xi}^b g(t+\xi-b)\sin(t+\xi-b)f(t)\sin(t)dt - \int_0^x g(x)d\xi \int_{b-\xi}^b f(x)\sin(t+\xi-b)\sin(t)dt.$$

Note that

$$R(x) = R_1(x) + R_2(x) + R_3(x) + R_4(x) + R_5(x) + R_6(x).$$

For the upper estimate of $R(x)$, it is necessary to estimate the values of $R_1(x)$, $R_2(x)$, ... , $R_6(x)$ from above. Now we evaluate the module of the function $|R_1(x)|$ from above

$$|R_1(x)| = \left|\int_0^x g(b-x+\tau)\sin(b-x+\tau)d\tau \int_\tau^x f(t)\sin(t)dt - \int_0^x g(x)\sin(b-x+\tau)d\tau \int_\tau^x f(x)\sin(t)dt\right| =$$

$$= |\int_0^x g(b-x+\tau)\sin(b-x+\tau)d\tau \int_\tau^x (f(t)\sin(t) - f(x)\sin(t))\, dt +$$

$$+\int_0^x g(b-x+\tau)\sin(b-x+\tau)d\tau \int_\tau^x f(x)\sin(t)dt - \int_0^x g(x)\sin(b-x+\tau)d\tau \int_\tau^x f(x)\sin(t)dt|,$$

$$|R_1(x)| = |\int_0^x g(b-x+\tau)\sin(b-x+\tau)d\tau \int_\tau^x (f(t)\sin(t) - f(x)\sin(t))\, dt -$$

$$-\int_0^x (g(b-x+\tau) - g(x))\sin(b-x+\tau)d\tau \int_\tau^x f(x)\sin(t)dt| \leq$$

$$\leq \left| \int_0^x g(b-x+\tau)\sin(b-x+\tau)d\tau \int_\tau^x (f(t)\sin(t) - f(x)\sin(t))\,dt \right| +$$
$$+\left| \int_0^x (g(b-x+\tau) - g(x))\sin(b-x+\tau)d\tau \int_\tau^x f(x)\sin(t)dt \right| \leq$$
$$\leq \int_0^x |g(b-x+\tau)|\,|\sin(b-x+\tau)|\,d\tau \int_\tau^x |f(t)-f(x)|\,|\sin(t)|\,dt+$$
$$+\int_0^x |g(b-x+\tau)-g(x)|\,|\sin(b-x+\tau)|\,d\tau \int_\tau^x |f(x)|\,|\sin(t)|\,dt \leq$$
$$\leq \int_0^x |g(b-x+\tau)|\,d\tau \int_\tau^x |f(t)-f(x)|\,dt + \int_0^x |g(b-x+\tau)-g(x)|\,d\tau \int_\tau^x |f(x)|\,dt.$$

If $0 < x < \dfrac{b}{2}$

$$|R_1(x)| \leq \int_0^x |g(b-x+\tau)|\,d\tau \int_\tau^x |f(t)-f(x)|\,dt + \int_0^x |g(b-x+\tau)-g(x)|\,d\tau \int_\tau^x |f(x)|\,dt.$$

Since $f, g \in C[0,b]$, then $|f(x)| \leq M_f$, $|g(x)| \leq M_g$. Therefore

$$|R_1(x)| \leq M_g \int_0^x d\tau \int_\tau^x |f(t)-f(x)|\,dt + M_f \int_0^x |g(b-x+\tau)-g(x)|\,(x-\tau)d\tau.$$

We consider that the length of the segment $[0,b]$ very small, then the inequalities are fulfilled $|f(t)-f(x)| < \varepsilon$, $|g(b-x+\tau)-g(x)| < \varepsilon$, $\forall\, 0 < \tau < x < \dfrac{b}{2}$.

$$|R_1(x)| \leq \varepsilon M_g \int_0^x d\tau \int_\tau^x dt + \varepsilon M_f \int_0^x (x-\tau)d\tau = \varepsilon(M_g + M_f)\frac{x^2}{2},$$

if the value b enough small. In an anological way , the values are estimated from above $|R_2(x)|$, $|R_3(x)|$, $|R_4(x)|$,$|R_5(x)|$,$|R_6(x)|$. Thus Theorem 1 is proved.

7 Conclusion

The article presents the convolution generated by the Dirichlet problem for the operator of twofold differentiation. This convolution makes it possible to approximate nonlinear expressions depending on two continuous functions. The accuracy of the approximation depends on the length of the segment on which these two functions are defined. Replacing nonlinear expressions with convolution allows applying the Fourier method to nonlinear partial differential equations.

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