IRSTI 27.31.15 DOI: https://doi.org/10.26577/JMMCS.2022.v114.i2.02

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BOUNDARY CONTROL OF ROD TEMPERATURE FIELD WITH A SELECTED POINT

In this paper, we study the issue of boundary control of the temperature field of a rod with a selected point. The main purpose of the work is to clarify the conditions for the existence of a boundary control that ensures the transition of the temperature field from the initial state to the final state. Relations connecting the boundary controls with the initial and final states, as well as with the external temperature field are found. Such boundary controls, generally speaking, constitute an infinite set. For an unambiguous choice of the boundary control, a strictly convex objective functional is chosen. We are looking for a boundary control that minimizes the selected target functional. To do this, we first investigate the existence and uniqueness of solutions to the initial boundary value problem and the conjugate problem. And also, we present the derivation of a system of linear Fredholm integral equations of the second kind, which are satisfied by an optimal boundary control that minimizes a strictly convex target functional on a convex set. Along the way, the linear part of the increment of the target functional is highlighted. Necessary and sufficient conditions for the minimum of a smooth convex functional on a convex set are established. The difference between the results of this work and the available ones is that in the proposed work, the temperature field is given by the heat conduction equation with a loaded term. As a result, the conjugate problem has a slightly different domain of definition than the domain of the conjugate problem in the case of no load.

Key words: initial-boundary value problem, heat equation, boundary control, Green's function, Fredholm integral equation of the second kind, spectral properties, eigenfunction, eigenvalues.

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Таңдалған нүктесі бар өзекшенің температуралық өрісін шекаралық басқару

Бұл жұмыста таңдалған нүктесі бар өзекшенің температуралық өрісін шекаралық басқару мәселесі зерттеледі. Жұмыстың негізгі мақсаты – температуралық өрістің бастапқы күйден соңғы күйге өтуін қамтамасыз ететін шекаралық басқарудың бар болуы шарттарын анықтау. Шекаралық басқаруларды бастапқы және финалдық күйлермен, сондай-ақ сыртқы температура өрісімен байланыстыратын қатынастар табылды. Мұндай шекаралық басқарулар, жалпы айтқанда, шексіз жиынды құрайды. Шекаралық басқарудың бірегей таңдалуы үшін қатаң дөңес мақсат функционал таңдалады. Таңдалған мақсат функционалды минимумдаушы шекаралық басқару ізделеді. Ол үшін жұмыста алдымен бастапқы-шекаралық есеп пен түйіндес есептің шешімдерінің бар болуы мен жалғыздығын зерттейміз. Сондай-ақ дөңес жиында қатаң дөңес мақсат функционалын минимумдаушы тиімді шекаралық басқарумен қанағаттандырылатын Фредгольмның екінші текті сызықты интегралдық теңдеулер жүйесінің алынуы келтірілген. Осы орайда мақсат функционалдың өсімшесінің сызықтық бөлігі айқындалған. Дөңес жиында тегіс дөңес функционал минимумының қажетті және жеткілікті шарттары анықталған. Жұмыстың нәтижесінің белгілі жұмыстардан айырмашылығы температура өрісі жүктелген мүшесі бар жылуөткізгіштік теңдеуі арқылы берілгендігі.

Нәтижесінде, түйіндес есептің жүктемесі болмаған жағдайда түйіндес есептің облысынан біршама өзгеше анықталу облысы болады.

Түйін сөздер: бастапқы-шекаралық есеп, жылуөткізгіштік теңдеуі, шекаралық басқару, Грин функциясы, екінші текті Фредгольм интегралдық теңдеуі, спектрлік қасиеттер, меншікті функция, меншікті мәндер.

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Граничное управление температурным полем стержня с выделенной точкой

В данной работе изучается вопрос о граничном управлении температурным полем стержня с выделенной точкой. Основная цель работы – выяснение условий существования граничного управления, обеспечивающего переход температурного поля из начального состояния в конечное состояние. Найдены соотношения, связывающие граничные управления с начальным и финальным состояниями, а также внешним температурным полем. Такие граничные управления, вообще говоря, составляют бесконечное множество. Для однозначного выбора граничного управления выбран строго выпуклый целевой функционал. Ищется граничное управление, которое минимизирует выбранный целевой функционал. Для этого в работе сначала исследуются существование и единственность решений начально-граничной задачи и сопряженной задачи. А также, дан вывод системы линейных интегральных уравнений Фредгольма второго рода, которым удовлетворяет оптимальное граничное управление, которое минимизирует строго выпуклый целевой функционал на выпуклом множестве. По пути выделена линейная часть приращения целевого функционала. Установлены необходимые и достаточные условия минимума гладкого выпуклого функционала на выпуклом множестве. Отличие результатов данной работы от имеющихся заключается в том, что в предлагаемой работе температурное поле задается уравнением теплопроводности с нагруженным членом. Вследствие чего сопряженная задача имеет несколько отличительную область определения, чем область определения сопряженной задачи в случае отсутствия нагрузки.

Ключевые слова: начально-граничная задача, уравнение теплопроводности, граничное управление, функция Грина, интегральное уравнение Фредгольма второго рода, спектральные свойства, собственная функция, собственные значения.

1 Introduction

In this paper, we study the issue of boundary control of rod temperature field with a selected point x_0.

$$u_t(x,t) - u_{xx}(x,t) + \alpha u(x_0,t) = f(x,t), \quad (x,t) \in Q, \tag{1}$$

where $Q = \{(x,t):\ 0 < x < b,\ 0 < t < T < +\infty\}$.

It is assumed that at the initial moment $t = 0$ the temperature along the rod of length b is given by law $u(x,0) = u_0(x)$, $0 < x < b$, where $u_0(x)$ is a twice continuously differentiable function. At the moment of time $t = T$ the temperature of the rod is equal to $u(x,T) = \gamma(x)$, $0 < x < b$, where $\gamma(x)$ is also a twice continuously differentiable function. The main purpose of the work is to clarify the conditions for the existence of the boundary control $u(0,t) = \mu(t)$, $u(b,t) = \eta(t)$, which ensures the transition of the temperature field from the state $\{u(x,0) = u_0(x)\}$ to the state $\{u(x,T) = \gamma(x)\}$. Similar problems were considered in $[1,2]$.

According to the optimization method, we choose the following functional

$$\mathcal{J}[\mu,\eta] = \|u(\cdot,T;\mu,\eta) - \gamma(\cdot)\|^2_{W_2^1(0,b)} + \beta_1 \int_0^T |\mu(t)|^2 dt + \beta_2 \int_0^T |\eta(t)|^2 dt,$$

where β_1, β_2 are positive numbers, γ is a given function from class $W_2^1(0,b)$.

The boundary control problem is as follows: it is required to find boundary controls $(\mu(t), \eta(t))$ and the corresponding solution $u(x,t)$, that satisfies equation (1) with initial boundary controls

$$u(0,t) = \mu(t), \quad u(b,t) = \eta(t), \quad 0 < t < T, \tag{2}$$

$$u(x,0) = u_0(x), \quad 0 < x < b, \tag{3}$$

and minimizes functional $\mathcal{J}[\mu,\eta]$.

Many natural and fundamental physical phenomena can be modeled by partial differential equations (PDEs), such as heat conduction, sound, electrostatics, electrodynamics, fluid flow and quantum mechanics in which states depend on not only time but also space, for example, see [3–5]. In particular, heat diffusion phenomena are extended mainly in describing fluid, thermal, and chemical dynamics, including the wide applications of sea ice melting and freezing [6], lithium-ion batteries [7], etc. The work [8] is concerned with the problem of boundary observer-based finite-time output feedback control for a heat system with Neumann boundary condition and Dirichlet boundary actuator. Finite-time stabilization, which is a key feature in the sliding mode control theory, is investigated in the work [9]. More specifically, finite-time control for the heat equation with Dirichlet boundary condition and the Dirichlet control is investigated in [10]. In work [11] the heat equation with prescribed lateral and final data is studied in half-plane and the uniqueness of the bounded solution is proved. In work [12] the solvability problems of an nonhomogeneous boundary value problem in the first quadrant for a fractionally loaded heat equation are studied. For parabolic equations in a bounded domain, various aspects of inverse source problems has been studied in [13–16], etc.

The paper presents a derivation of a system of linear Fredholm integral equations of the second kind, which optimal boundary control is satisfied. In the proposed work, for the first time, the conjugate problem to a mixed boundary value problem for the heat conduction equation with a loaded term is explicitly written out. As a result, it was possible to obtain more precise information about the solutions of the conjugate problem. We note that in [1], the solution of the mixed boundary value problem for the heat conduction equation is decomposed by the eigenfunctions of a periodic problem with a specially selected period. In [2], the method of work [1] is extended to the heat conduction equation with a loaded term. In this paper, the expansion of the solution to the mixed problem for the heat equation with a loaded term is carried out in terms of the eigenfunctions of the corresponding spectral problem. At the same time, it is necessary to select a period and continue the solution in a wider area. Moreover, the solution of the conjugate problem is carried out similarly to the solution of the mixed problem for the heat conduction equation with a load.

2 Existence and uniqueness of the solutions to the initial-boundary value problem and the conjugate problem

Before studying the boundary control problem, it is necessary to investigate the question of the existence and uniqueness of the solution to problem (1)–(3). To do this, select a function $w(x,t)$ from class $L_2((0,T);W_2^1(0,b))$ such that

$$w(x,t) = \mu(t) + \frac{x}{b}(\eta(t) - \mu(t)).$$

Then, instead of studying problem (1)–(3) it is enough to study the following problem:

$$v_t(x,t) - v_{xx}(x,t) + \alpha v(x_0,t) = F(x,t), \quad (x,t) \in Q, \tag{4}$$

$$v(0,t) = 0, \quad v(b,t) = 0, \tag{5}$$

$$v(x,0) = v_0(x), \quad 0 < x < b, \tag{6}$$

where

$$F(x,t) = f(x,t) - w_t(x,t) - \alpha w(x_0,t), \quad v_0(x) = u_0(x) - \mu(0) - \frac{x}{b}(\eta(0) - \mu(0)).$$

The solution to problem (4)–(6) is sought in the form

$$v(x,t) = \sum_{k\geqslant 1} d_k(t)\varphi_k(t).$$

Here $\{\varphi_k\}$ is the system of root functions of the following eigenvalue problem

$$-\varphi_{xx}(x) + \alpha\varphi(x_0) = \lambda\varphi(x), \quad 0 < x < b, \tag{7}$$

$$\varphi(0) = 0, \quad \varphi(b) = 0. \tag{8}$$

In this case $\varphi_k(x) \equiv \varphi_k(x,\lambda_k)$, where $\{\lambda_k\}$ is a sequence of eigenvalues of (7)–(8). The eigenfunctions $\varphi_k(x) \equiv \varphi(x,\lambda_k)$ and the biorthogonal system of functions $\left\{\psi_k(x) = \frac{\overline{\psi(x,\lambda_k)}}{\langle\varphi(\cdot,\lambda_k),\psi(\cdot,\lambda_k)\rangle}\right\}$ are defined by formulas:

$$\varphi(x,\lambda) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} + \alpha\frac{\dfrac{\sin\sqrt{\lambda}x_0}{\sqrt{\lambda}}}{\lambda - \alpha(1-\cos\sqrt{\lambda}x_0)}(1-\cos\sqrt{\lambda}x), \quad 0 < x < b,$$

$$\psi(x,\lambda) = \frac{\sin\sqrt{\lambda}(b-x)}{\sqrt{\lambda}}, \quad x_0 < x < b,$$

$$\begin{aligned}\psi(x,\lambda) &= \frac{\sin\sqrt{\lambda}(b-x_0)}{\sqrt{\lambda}}\cos\sqrt{\lambda}(x_0-x) \\ &+ \frac{\sin\sqrt{\lambda}(x_0-x)}{\sqrt{\lambda}}\left(\cos\sqrt{\alpha}\,(b-x_0) - \overline{\alpha}\frac{1-\cos\sqrt{\lambda}(b-x_0)}{\lambda}\right) - \overline{\alpha}\,\frac{\sin\sqrt{\lambda}(x_0-x)}{\sqrt{\lambda}} \\ &\times \frac{\cos\sqrt{\lambda}(b-x_0) - \cos\sqrt{\lambda}b - \overline{\alpha}\,\dfrac{(1-\cos\sqrt{\lambda}(b-x_0))(1-\cos\sqrt{\lambda}x_0)}{\lambda}}{\lambda + \overline{\alpha}\,(1-\cos\sqrt{\lambda}x_0)}, \ 0 < x < x_0,\end{aligned}$$

The acceptable values of parameter λ are selected according to the conditions $\lambda - \alpha(1 - \cos\sqrt{\lambda}x_0) \neq 0$, $\lambda + \overline{\alpha}(1 - \cos\sqrt{\lambda}x_0) \neq 0$. Each function $f(x)$ from $L_2(0,b)$ is decomposed into a Fourier series by the system $\{\varphi_k\}$:

$$f(x) = \sum_{k\geqslant 1} C_k(f)\varphi_k(x),$$

where $C_k(f) = \langle f, \psi_k\rangle$, $k \geqslant 1$.

In this case, the Fourier coefficients $\{d_k(t),\ k \geqslant 1\}$ in terms of system $\{\varphi_k(x)\}$ of the solution $v(x,t)$ satisfy equations

$$d_{tk}(t) + \lambda_k d_k(t) = D_k(t), \quad t > 0 \tag{9}$$

and initial conditions

$$d_k(0) = d_k^{(0)}. \tag{10}$$

Here $\{D_k(t)\}$ and $\{d_k^{(0)}\}$ are sequences of Fourier coefficients in terms of system $\{\varphi_k\}$ of functions $F(x,t)$ and $v_0(x)$. Relations (9)–(10) imply the following representation

$$d_k(t) = d_k^{(0)} e^{-\lambda_k t} + \int_0^t e^{-\lambda_k(t-\tau)} D_k(\tau)d\tau, \quad t > 0. \tag{11}$$

Thus, problem (4)–(6) has a solution $v(x,t)$, representable in the form

$$v(x,t) = \sum_{k\geqslant 1} d_k(t)\varphi_k(x), \tag{12}$$

and the coefficients $d_k(t)$ are calculated by formulas (11). Thus, we can formulate the following statement.

Theorem 1 *Let $v_0(x)$ be a twice continuously differentiable function on a finite segment $[0,b]$, and the matching conditions $v_0(0) = v_0(b) = 0$ are satisfied. Suppose also that $F(x,t) = \mathrm{L}^2((0,T); L_2(0,b))$. Then there is a solution $v(x,t)$ of problem* (4)–(6)*, which can be represented as a Fourier series* (12)*, the coefficients $\{d_k(t)\}$ of which are found by formulas* (11).

Remark 1 *Note that $v_0(x)$ is decomposed into a Fourier series by the system $\{\varphi_k\}$ and the corresponding Fourier series on $[0,b]$ converges uniformly. This follows from the fact that $v_0(x)$ belongs to the domain of operator B. The monograph [17] contains theorems on the uniform convergence of spectral decompositions in such cases.*

We denote by $G(x,\xi,t) = \sum\limits_{k\geqslant 1} e^{-\lambda_k t}\varphi_k(x)\overline{\psi_k(\xi)}$, the function that represents the Green function of the Dirichlet problem for the heat equation with the selected point [18]. Then the statement follows.

Corollary 1 *Let the conditions of Theorem 1 be satisfied. Then there exists a solution $u(x,t)$ of problem* (1)–(3)*, which can be represented in the form*

$$u(x,t) = \int_0^b u_0(\xi)G(x,\xi,t)d\xi + \int_0^t d\tau \int_0^b f(\xi,\tau)G(x,\xi,t-\tau)d\xi - u_0(0)\int_0^b G(x,\xi,t)d\xi$$
$$- \frac{u_0(b)-u_0(0)}{b}\int_0^b \xi G(x,\xi,t)d\xi - \alpha \int_0^t \mu(\tau)d\tau \int_0^b G(x,\xi,t-\tau)d\xi$$
$$- \frac{\alpha}{b}\int_0^t (\eta(\tau)-\mu(\tau))\,d\tau \int_0^b \xi G(x,\xi,t-\tau)d\xi.$$

We now formulate and prove a uniqueness theorem for a solution.

Theorem 2 *Let the conditions of Theorem 1 be satisfied. Then problem* (4)–(6) *has a unique solution.*

Proof 1 *The idea of this proof is borrowed from the work of V.A. Il'in [19]. Let $r(x)$ be one of the eigenfunctions of operator B^*. We denote by $\Phi(x,t)$ any of the functions of the form*

$$\Phi(x,t) = r(x)f(t),$$

where $f(t)$ is a function that is continuously differentiable on the entire numerical axis, which is equal to zero for all $t > t_0$, where t_0 is some number less than T.

Let $v(x,t)$ be a solution to problem (4)–(6) *for $F \equiv 0$, $v_0 \equiv 0$. Let consider integral*

$$0 = \int_0^b \int_0^T (v_t(x,t) - v_{xx}(x,t) + \alpha v(x_0,t))\,\Phi(x,t)dtdx = \int_0^b r(x)dx \int_0^T v_t(x,t)f(t)dt$$
$$+ \int_0^T f(t)dt \int_0^b Bv(x,t)r(x)dx = \int_0^b r(x)dx\left(\varphi(x,t)f(t)\Big|_0^T - \int_0^T v(x,t)f_t(t)dt\right)$$
$$+ \int_0^T f(t)dt \int_0^b v(x,t)B^*r(x)dx = -\int_0^b\int_0^T v(x,t)r(x)f_t(t)dtdx$$
$$+ \overline{\lambda}\int_0^b\int_0^T v(x,t)r(x)f(t)dtdx, \quad (13)$$

where $\overline{\lambda}$ is the eigenvalue of operator B^* corresponding to eigenfunction $r(x)$.

Let us continue $v(x,t)$ on domain $t < 0$, by setting it equal to zero there. Then, taking into account that $f(t) = 0$ for $t > t_0$, relation (13) *can be rewritten in the form*

$$\int_0^b \int_{-\infty}^{\infty} v(x,t)r(x)\left(-f_t(t) + \overline{\lambda}f(t)\right)dtdx = 0. \quad (14)$$

We fix any $\xi \geqslant 0$. Then function $f(\xi+t)$ is a priori equal to zero for $t \geqslant t_0$. In equality (14) *we substitute $f(\xi+t)$ instead of $f(t)$. Then for all $\xi \geqslant 0$ we have equality*

$$\int_0^b \int_{-\infty}^{\infty} v(x,t)r(x)\left(-f_\xi(\xi+t) + \overline{\lambda}f(\xi+t)\right)dtdx = 0. \quad (15)$$

From (15) *it follows*

$$\int_0^b \int_{-\infty}^{\infty} v(x,t)r(x)f(\xi+t)dtdx = c\cdot e^{\overline{\lambda}\xi}, \quad \xi \geqslant 0. \tag{16}$$

However, for $\xi \geqslant t_0$ *and* $t>0$ *the function* $f(\xi+t)\equiv 0$. *Therefore, it follows from relation* (16) *that* $c=0$. *Therefore, we have the equality*

$$\int_0^b \int_{-\infty}^{\infty} v(x,t)r(x)f(\xi+t)dtdx = 0, \quad \xi \geqslant 0. \tag{17}$$

Since the system of eigenfunctions $\{\overline{\psi(x,\lambda_k)},\ k\geqslant 1\}$ *of operator* B^* *is complete in space* $L_2(0,b)$, *equalities* (17) *imply*

$$\int_0^{\infty} v(x,t)f(\xi+t)dt \equiv 0 \text{ in space } L_2(0,b).$$

In particular, for $\xi=0$ *we find that*

$$\int_0^T v(x,t)f(t)dt = 0.$$

The latter equality holds for any function $f(t)$, *that has the properties described above. Therefore* $v(x,t)\equiv 0$ *for* $0<x<b$, $0<t<T$. *Theorem 2 is completely proved.*

Therefore, the conjugate problem to problem (4)–(6) takes the form

$$-\Psi_t(x,t) - \Psi_{xx}(x,t) = E(x,t), \quad (x,t)\in Q, \tag{18}$$

$$\Psi(0,t)=0, \quad \Psi(b,t)=0, \quad t>0, \tag{19}$$

$$\begin{cases} \Psi(x_0+0,t)=\Psi(x_0-0,t), \quad t>0, \\ \Psi_x(x_0+0,t)=\Psi_x(x_0-0,t)+\overline{\alpha}\int_0^b \psi(x,t)dx, \quad t>0, \end{cases} \tag{20}$$

$$\Psi(x,T)=\Psi_T(x), \quad 0<x<b. \tag{21}$$

Thus, we can formulate the following statement.

Theorem 3 *Let* $\Psi_T(x)$ *be a twice continuously differentiable function on a finite segment* $[0,b]$, *moreover, for* $\Psi_T(x)$ *the matching conditions* (19)–(20) *are satisfied. Suppose also that* $E(x,t)\in \mathrm{L}^2((0,T);L_2(0,b))$. *Then there is a solution* $\Psi(x,t)$ *to problem* (18)–(21), *which can be represented as a Fourier series dual to series* (12).

Theorem 3 implies the following statement.

Corollary 2 *Let the conditions of Theorem 3 be satisfied. Then there exists a solution* $\Psi(x,t)$ *to problem* (18)–(21), *which can be represented in the form*

$$\Psi(x,t) = \int_0^b \Psi_T(\xi)\overline{G(\xi,x,T-t)}d\xi + \int_t^T d\tau \int_0^b E(\xi,\tau)\overline{G(\xi,x,T-\tau)}d\xi.$$

Now we formulate and prove the uniqueness theorem.

Theorem 4 *Let the conditions of Theorem 3 be satisfied. Then problem* (18)–(21) *has a unique solution.*

The proof of Theorem 4 repeats the proof of Theorem 2. Let $r(x)$ be one of the eigenfunctions of operator B. We denote by $\Phi(x,t)$ any function of the form

$$\Phi(x,t) = r(f)f(t),$$

where $f(t)$ is a continuously differentiable function on the entire numerical axis, which is equal to zero for all $t < t_0$, where t_0 is some positive number. Further, the reasoning from the proof of Theorem 2 is repeated.

3 Necessary conditions for maintaining the final temperature regime

In this section, we study the boundary control problem I:

$$W_t(x,t) - W_{xx}(x,t) + \alpha W(x_0,t) = f(x,t), \quad (x,t) \in Q. \tag{22}$$

$$W(x,T) = u_T(x), \quad 0 < x < b, \tag{23}$$

Statement of the boundary control Problem I:

Let $W(x,t;f,u_T)$ be an arbitrary solution of problem (22)–(23). We denote the boundary controls corresponding to $W(x,t;f,u_T)$, by $\mu(t) = W(0,t;f,u_T)$ and $\eta(t) = W(b,t;f,u_T)$, as well as by $u_0(x) = W(x,0;f,u_T)$ the initial temperature regime.

What necessary conditions do $\mu(t)$, $\eta(t)$, $u_0(x)$, satisfy if $W(x,t;f,u_T)$ satisfies (22)–(23)?

This boundary control Problem I corresponds to a given final temperature regime $u_T(x)$. To answer the question posed, it is convenient to introduce solutions $\Psi(x,t) = \Psi(x,t;\Psi_T)$ to conjugate equation

$$-\Psi_t(x,t) - \Psi_{xx}(x,t) = 0, \quad (x,t) \in Q, \quad x \neq x_0, \tag{24}$$

with conditions

$$\Psi(0,t) = 0, \quad \Psi(b,t) = 0, \quad t > 0, \tag{25}$$

$$\begin{cases} \Psi(x_0+0,t) = \Psi(x_0-0,t), \\ \Psi'(x_0+0,t) = \Psi'(x_0-0,t) + \overline{\alpha}\int_0^b \Psi(\xi,t)d\xi, \quad t > 0, \end{cases} \tag{26}$$

and the final temperature distribution

$$\Psi(x,T) = \Psi_T(x), \quad 0 < x < b \tag{27}$$

for an arbitrary function $\Psi_T(x)$ from class $W_2^2[0,b]$.

Lemma 1 *For an arbitrary solution* $u(x,t) \equiv u(x,t;f,\mu,\eta,u_0)$ *of equation*

$$u_t(x,t) - u_{xx}(x,t) + \alpha u(x_0,t) = f(x,t), \quad (x,t) \in Q, \tag{28}$$

with boundary conditions

$$u(0,t) = \mu(t), \quad u(b,t) = \eta(t), \quad T > t > 0, \tag{29}$$

and with the initial temperature distribution

$$u(x,0) = u_0(x), 0 < x < b, \tag{30}$$

the following integral relation is valid

$$\int_0^T \int_0^b f(x,t)\overline{\Psi(x,t)}dxdt = \int_0^b \Big(u(x,T)\overline{\Psi_T(x)} - u_0(x)\overline{\Psi(x,0)}\Big)\, dx$$
$$- \int_0^T \mu(t)\overline{\Psi_x(0,t)}dt + \int_0^T \eta(t)\overline{\Psi_x(b,t)}dt,$$

where $\Psi(x,t) \equiv \Psi(x,t;\Psi_T)$ *is the solution to conjugate problem* (24)–(27) *for an arbitrary* $\Psi_T(x) \in W_2^2[0,b]$.

Let us formulate another useful lemma.

Lemma 2 *For an arbitrary solution* $v(x,t) \equiv v(x,t;f,\mu_1,\eta_1,v_T)$ *of equation*

$$v_t(x,t) - v_{xx}(x,t) + \alpha v(x_0,t) = f(x,t), \quad (x,t) \in Q, \tag{31}$$

with boundary conditions

$$v(0,t) = \mu_1(t), \quad v(b,t) = \eta_1(t), \quad T > t > 0, \tag{32}$$

and with the final temperature distribution

$$v(x,T) = v_T(x), \quad 0 < x < b, \tag{33}$$

the following integral relation is valid

$$\int_0^T \int_0^b f(x,t)\overline{\Psi(x,t)}dxdt = \int_0^b \Big(v_T(x)\overline{\Psi_T(x)} - v(x,0)\overline{\Psi(x,0)}\Big)\, dx$$
$$- \int_0^T \mu_1(t)\overline{\Psi_x(0,t)}dt + \int_0^T \eta_1(t)\overline{\Psi_x(b,t)}dt,$$

where $\Psi(x,t) \equiv \Psi(x,t;\Psi_T)$ *is the solution to conjugate problem* (24)–(27) *for an arbitrary* $\Psi_T(x) \in W_2^2[0,b]$.

We now formulate an important statement.

Theorem 5 *For the solution* $u(x,t) \equiv u(x,t;f,\mu,\eta,u_0)$ *to problem* (28)–(30) *and for the solution* $v(x,t) \equiv v(x,t;f,\mu_1,\eta_1,v_T)$ *to problem* (31)–(33) *the following integral identity is valid*

$$\int_0^T (\eta_1(t) - \eta(t))\, G_x(\xi,b,T-t)dt - \int_0^T (\mu_1(t) - \mu(t))\, G_x(\xi,0,T-t)dt$$
$$\equiv \int_0^b (v(x,0) - u_0(x))\, G(\xi,x,T)dx, \quad \forall\, \xi \in (0,b), \tag{34}$$

where $G(x,\xi,t) = \sum\limits_{k\geqslant 1} e^{-\lambda_k t}\varphi_k(x)\overline{\psi_k(\xi)}$ *is a Green's function.*

Proof 2 *Lemmas 1 and 2 imply the integral identity*

$$0 = -\int_0^b (v(x,0) - u_0(x))\,\overline{\Psi(x,0)}dx - \int_0^T (\mu_1(t) - \mu(t))\,\overline{\Psi_x(0,t)}dt + \int_0^T (\eta_1(t) - \eta(t))\,\overline{\Psi_x(b,t)}dt, \tag{35}$$

for all $\Psi(x,t)$ *at any* $\Psi_T(x)$*. Corollary 2 implies that*

$$\Psi(x,t) = \int_0^b \Psi_T(\xi)\overline{G(\xi,x,T-t)}d\xi.$$

Therefore, relation (35) *takes the form*

$$-\int_0^b (v(x,0) - u_0(x))\,dx \int_0^b \overline{\Psi_T(\xi)}G(\xi,x,T)d\xi - \int_0^T (\mu_1(t) - \mu(t))\,dt \int_0^b \overline{\Psi_T(\xi)}G_x(\xi,0,T-t)d\xi + \int_0^T (\eta_1(t) - \eta(t))\,dt \int_0^b \overline{\Psi_T(\xi)}G_x(\xi,b,T-t)d\xi.$$

Rearranging the order of the integrals, we obtain the equality

$$\int_0^b \overline{\Psi_T(\xi)}\Bigg\{\int_0^T (\eta_1(t) - \eta(t))\,G_x(\xi,b,T-t)dt - \int_0^T (\mu_1(t) - \mu(t))\,G_x(\xi,0,T-t)dt - \int_0^b (v(x,0) - u_0(x))\,G(\xi,x,T)dx\Bigg\}d\xi = 0.$$

Since $\Psi_T(\xi)$ *is an arbitrary function from* $W_2^2[0,b]$*, then relation* (34) *follows from the last equality. Theorem 5 is completely proved.*

This implies the following statement.

Corollary 3 *Let* $u(x,t) \equiv u(x,t;f,\mu,\eta,u_0)$ *and* $v(x,t) \equiv v(x,t;f,\mu_1,\eta_1,v_T)$ *are solutions to problems* (28)–(30) *and* (31)–(33)*, respectively. If* $u_0 = v(x,0)$*,* $x \in (0,b)$*, then the following identity is valid*

$$\int_0^T (\eta_1(t) - \eta(t))\,G_x(\xi,b,T-t)dt - \int_0^T (\mu_1(t) - \mu(t))\,G_x(\xi,0,T-t)dt \equiv 0, \quad \forall\,\xi \in (0,b).$$

4 Optimality criteria

In this section, the target functional is investigated.

$$\mathcal{J}[\mu,\eta] = \int_0^b |u(x,T;\mu,\eta) - \gamma(x)|^2 dx + \int_0^b |u'(x,T;\mu,\eta) - \gamma'(x)|^2 dx + \beta_1 \int_0^T \mu^2(t)dt + \beta_2 \int_0^T \eta^2(t)dt.$$

Let us take arbitrary controls $(\mu(t), \eta(t))$ and $(\mu(t) + h(y), \eta(t) + q(t))$, where $h(0) = 0$, $q(0) = 0$. We denote the corresponding solutions of problem (1)–(3) by $u(x, t; \mu, \eta)$ and $u(x, t; \mu + h, \eta + q)$. Let us introduce the notation

$$\Delta u(x, t) = u(x, t; \mu + h, \eta + q) - u(x, t; \mu, \eta).$$

Then from (1)–(3) it follows

$$\frac{\partial}{\partial t}\Delta u - \frac{\partial^2}{\partial x^2}\Delta u + \alpha \Delta u(x_0, t) = 0, \quad (x, t) \in Q, \tag{36}$$

$$\Delta u\Big|_{x=0} = h(t), \quad \Delta u\Big|_{x=b} = q(t), \quad t > 0, \tag{37}$$

$$\Delta u\Big|_{t=0} = 0, \quad 0 < x < b. \tag{38}$$

Arguing as in the proof of Theorem 1, we obtain the representation

$$\Delta u(x, t) = \frac{1}{b}\int_0^t (q(\tau) - h(\tau))\frac{\partial}{\partial \tau}K_1(x, t - \tau)d\tau - \frac{\alpha x_0}{b}\int_0^t q(\tau)K_0(x, t - \tau)d\tau$$
$$- \alpha\left(1 - \frac{x_0}{b}\right)\int_0^t h(\tau)K_0(x, t - \tau)d\tau + \int_0^t h(\tau)\frac{\partial}{\partial \tau}K_0(x, t - \tau)d\tau,$$

where $K_0(x, t) = \sum\limits_{k\geqslant 1} \beta_k^{(0)} e^{-\lambda_k t} \cdot \varphi_k(x)$, $K_1(x, t) = \sum\limits_{k\geqslant 1} \beta_k^{(1)} e^{-\lambda_k t} \cdot \varphi_k(x)$.

Consider the increment of the target functional $\mathcal{J}[\mu, \eta]$.

$$\Delta\mathcal{J}[\mu, \eta] = 2\int_0^b \mathrm{Re}\left(\left(\overline{u(x, T; \mu, \eta)} - \overline{\gamma(x)}\right)\Delta u(x, T)\right)dx$$
$$+ 2\int_0^b \mathrm{Re}\left(\left(\overline{u_x(x, T; \mu, \eta)} - \overline{\gamma_x(x)}\right)\frac{\partial}{\partial x}\Delta u(x, T)\right)dx + 2\beta_1\int_0^T \mathrm{Re}\left(\overline{\mu(t)}h(t)\right)dt$$
$$+ 2\beta_2\int_0^T \mathrm{Re}\left(\overline{\eta(t)}q(t)\right)dt + \bar{\bar{o}}\left(\sqrt{\int_0^T (|h(t)|^2 + |q(t)|^2)dt}\right)$$
$$= 2\int_0^b \mathrm{Re}\left(\left(\overline{u(x, T; \mu, \eta)} - \overline{\gamma(x)}\right)\Delta u(x, T)\right)dx$$
$$+ 2\mathrm{Re}\left(\left(\overline{u_x(x, T; \mu, \eta)} - \overline{\gamma_x(x)}\right)\Delta u(x, T)\right)\Big|_{x=0}^{x=b}$$
$$- 2\int_0^b \mathrm{Re}\left(\left(\overline{u_{xx}(x, T; \mu, \eta)} - \overline{\gamma_{xx}(x)}\right)\frac{\partial}{\partial x}\Delta u(x, T)\right)dx$$
$$+ 2\beta_1\int_0^T \mathrm{Re}\left(\overline{\mu(t)}h(t)\right)dt + 2\beta_2\int_0^T \mathrm{Re}\left(\overline{\eta(t)}q(t)\right)dt + \bar{\bar{o}}\left(\sqrt{\int_0^T (|h(t)|^2 + |q(t)|^2)dt}\right)$$
$$= 2\mathrm{Re}\left(\left(\overline{u_x(b, T; \mu, \eta)} - \overline{\gamma_x(b)}\right)q(T)\right) - 2\mathrm{Re}\left(\left(\overline{u_x(0, T; \mu, \eta)} - \overline{\gamma_x(0)}\right)h(T)\right)$$
$$+ 2\int_0^b \mathrm{Re}\left(\Delta u(x, T)\left[-\left(\overline{u_{xx}(x, T; \mu, \eta)} - \overline{\gamma_{xx}(x)}\right) + \left(\overline{u(x, T; \mu, \eta)} - \overline{\gamma(x)}\right)\right]\right)dx$$

$$+2\beta_1\int_0^T \mathrm{Re}\left(\overline{\mu(t)}h(t)\right)dt+2\beta_2\int_0^T \mathrm{Re}\left(\overline{\eta(t)}q(t)\right)dt+\overline{\overline{o}}\left(\sqrt{\int_0^T(|h(t)|^2+|q(t)|^2)dt}\right), \tag{39}$$

where

$$\lim_{q,h\to 0}\frac{\overline{\overline{o}}\left(\sqrt{\int_0^T(|h(t)|^2+|q(t)|^2)dt}\right)}{\int_0^T(|h(t)|^2+|q(t)|^2)dt}=0.$$

Let us introduce the solution $\Psi(x,t;\mu,\eta)$ of the following conjugate problem (18)–(21) at $E(x,t)\equiv 0$, $\Psi_T(x)=\left(-\frac{d^2}{dx^2}+I\right)(u(x,T;\mu,\eta)-\gamma(x))$. For further purposes, we transform the integral

$$\begin{aligned}
&\int_0^b \Delta u(x,T)\left[-\left(u_{xx}(x,T;\mu,\eta)-\gamma_{xx}(x)\right)+\left(u(x,T;\mu,\eta)-\gamma(x)\right)\right]dx\\
&\quad=\int_0^T\frac{\partial}{\partial t}\left(\int_0^b \Delta u(x,t)\overline{\Psi(x,t;\mu,\eta)}dx\right)dt\\
&\quad=\int_0^T dt\int_0^{x_0}\left[\frac{\partial^2}{\partial x^2}\Delta u(x,t)-\alpha\Delta u(x_0,t)\right]\overline{\Psi(x,t;\mu,\eta)}dx\\
&\quad+\int_0^T dt\int_{x_0}^b\left[\frac{\partial^2}{\partial x^2}\Delta u(x,t)-\alpha\Delta u(x_0,t)\right]\overline{\Psi(x,t;\mu,\eta)}dx\\
&\quad+\int_0^T dt\int_0^b \Delta u(x,t)\frac{\partial}{\partial t}\overline{\Psi(x,t;\mu,\eta)}dx\\
&\quad=\int_0^T dt\left\{\left[\frac{\partial}{\partial x}\Delta u(x,t)\overline{\Psi(x,t;\mu,\eta)}-\Delta u(x,t)\frac{\partial}{\partial x}\overline{\Psi(x,t;\mu,\eta)}\right]_{x=0}^{x=x_0-0}\right.\\
&\quad+\left.\left[\frac{\partial}{\partial x}\Delta u(x,t)\overline{\Psi(x,t;\mu,\eta)}-\Delta u(x,t)\frac{\partial}{\partial x}\overline{\Psi(x,t;\mu,\eta)}\right]_{x=x_0+0}^{x=b}\right\}\\
&\quad+\alpha\int_0^T\Delta u(x_0,t)\overline{\Psi(x,t;\mu,\eta)}dt\\
&\quad+\int_0^T dt\int_{x_0}^b\Delta u(x,t)\left(\frac{\partial^2\overline{\Psi(x,t;\mu,\eta)}}{\partial x^2}+\frac{\partial\overline{\Psi(x,t;\mu,\eta)}}{\partial t}\right)dx\\
&\quad=\int_0^T h(t)\frac{\partial}{\partial x}\overline{\Psi(0,t;\mu,\eta)}dt-\int_0^T q(t)\frac{\partial}{\partial x}\overline{\Psi(b,t;\mu,\eta)}dt.
\end{aligned}$$

Thus, the following relation is true

$$\begin{aligned}
&2\int_0^b \mathrm{Re}\left(\Delta u(x,T)\left[-\frac{d^2}{dx^2}+I\right](u(x,T;\mu,\eta)-\gamma(x))\right)dx\\
&=2\int_0^T \mathrm{Re}\left(h(t)\frac{\partial}{\partial x}\overline{\Psi(0,t;\mu,\eta)}\right)dt-2\int_0^T \mathrm{Re}\left(q(t)\frac{\partial}{\partial x}\overline{\Psi(b,t;\mu,\eta)}\right)dt.
\end{aligned}$$

From the last relation and equality (39) it follows that the increment of the target functional will take the form

$$\Delta\mathcal{J}[\mu,\eta] = 2\mathrm{Re}\left(\left[\overline{u_x(b,T;\mu,\eta)} - \overline{\gamma_x(b)}\right] q(T)\right) - 2\mathrm{Re}\left(\left[\overline{u_x(0,T;\mu,\eta)} - \overline{\gamma_x(0)}\right] h(T)\right)$$
$$+ 2\int_0^T \mathrm{Re}\left(h(t)\frac{\partial}{\partial x}\overline{\Psi(0,t;\mu,\eta)}\right) dt - 2\int_0^T \mathrm{Re}\left(q(t)\frac{\partial}{\partial x}\overline{\Psi(b,t;\mu,\eta)}\right) dt$$
$$+ 2\beta_1 \int_0^T \mathrm{Re}\left(\overline{\mu(t)}h(t)\right) dt + 2\beta_2 \int_0^T \mathrm{Re}\left(\overline{\eta(t)}q(t)\right) dt + \overline{\overline{o}}\left(\sqrt{\int_0^T (|h(t)|^2 + |q(t)|^2) dt}\right).$$

Thus, we were able to isolate the linear part of the increment of the target functional $\Delta\mathcal{J}[\mu,\eta]$. Necessary and sufficient conditions for the minimum of a smooth convex functional $\mathcal{J}[\mu,\eta]$ on a convex set $U = \{\mu(t),\eta(t) : \mu,\eta \in W_2^1[0,T]\}$ [20] are given in the following statement.

Theorem 6 *Let $(\mu^*(t),\eta^*(t)) \in U$ and gives a minimum to functional $\mathcal{J}[\mu,\eta]$. Then the following inequality must be*

$$2\mathrm{Re}\left(\left[\overline{u_x(b,T;\mu^*,\eta^*)} - \overline{\gamma_x(b)}\right] q(T)\right) - 2\mathrm{Re}\left(\left[\overline{u_x(0,T;\mu^*,\eta^*)} - \overline{\gamma_x(0)}\right] h(T)\right)$$
$$+ 2\int_0^T \mathrm{Re}\left(h(t)\frac{\partial}{\partial x}\overline{\Psi(0,t;\mu^*,\eta^*)}\right) dt - 2\int_0^T \mathrm{Re}\left(q(t)\frac{\partial}{\partial x}\overline{\Psi(b,t;\mu^*,\eta^*)}\right) dt$$
$$+ 2\beta_1 \int_0^T \mathrm{Re}\left(\overline{\mu^*(t)}h(t)\right) dt + 2\beta_2 \int_0^T \mathrm{Re}\left(\overline{\eta^*(t)}q(t)\right) dt = 0.$$

for all $h,q \in W_2^1[0,T]$ with conditions $h(0) = q(0) = 0$. Moreover, since $\mathcal{J}[\mu,\eta]$ is convex on U, the above necessary condition is also sufficient for $(\mu^*(t),\eta^*(t)) \in U$.

5 System of linear integral equations for optimal boundary control

In this section, we derive a system of linear Fredholm integral equations of the second kind, which are satisfied by the optimal boundary control

$$(\mu^*(t),\eta^*(t)) \in U.$$

Since in Theorem 6 the functions $h(t)$ and $q(x)$ are arbitrary from $W_2^1[0,T]$ and independent of each other, we can write the system of relations

$$\begin{cases} \dfrac{\partial}{\partial x}\Psi(0,t;\mu^*,\eta^*) + \beta_1\mu^*(t) = 0, \quad T > t > 0, \\ -\dfrac{\partial}{\partial x}\Psi(b,t;\mu^*,\eta^*) + \beta_2\eta^*(t) = 0, \quad T > t > 0, \end{cases} \tag{40}$$

$$\begin{cases} u_x\Psi(b,t;\mu^*,\eta^*) = \gamma_x(b), \\ u_x\Psi(0,t;\mu^*,\eta^*) = \gamma_x(0). \end{cases} \tag{41}$$

Now, according to Corollary 2, we have the representation

$$u(x,T;\mu^*,\eta^*) = \int_0^T \mathrm{H}_1(x,T-\tau)\mu^*(\tau)d\tau + \int_0^T \mathrm{H}_2(x,T-\tau)\eta^*(\tau)d\tau + \mathrm{H}_3(x),$$

where

$$\mathrm{H}_1(x,t-\tau) = -\alpha\int_0^b \left(1-\frac{\xi}{b}\right) G(x,\xi,t-\tau)d\xi, \quad \mathrm{H}_2(x,t-\tau) = -\frac{\alpha}{b}\int_0^b G(x,\xi,t-\tau)d\xi,$$

$$\mathrm{H}_3(x,t) = \int_0^b u_0(\xi)G(x,\xi,t)d\xi + \int_0^T d\tau \int_0^b f(\xi,\tau)G(x,\xi,t-\tau)d\xi$$
$$-u_0(0)\int_0^b G(x,\xi,T)d\xi - \frac{u_0(b)-u_0(0)}{b}\int_0^b \xi G(x,\xi,T)\xi.$$

Further, Corollary 3 implies the representation

$$\begin{aligned}
\Psi(x,t) &= \int_0^b \left(-\frac{d^2}{d\xi^2}+I\right)(u(\xi,T;\mu^*,\eta^*)-\gamma(\xi))\,\overline{G(\xi,x,T-t)}d\xi \\
&= \Big(-\frac{d}{d\xi}\,(u(\xi,T;\mu^*,\eta^*)-\gamma(\xi))\,\overline{G(\xi,x,T-t)} \\
&+ (u(\xi,T;\mu^*,\eta^*)-\gamma(\xi))\,\frac{d}{d\xi}\overline{G(\xi,x,T-t)}\Big)\Big|_{\xi=0}^{\xi=b} \\
&+ \int_0^b (u(\xi,T;\mu^*,\eta^*)-\gamma(\xi))\left(-\frac{\partial^2}{\partial\xi^2}+I\right)\overline{G(\xi,x,T-t)}d\xi \\
&= \int_0^T Q_1(T-\tau,x,T-t)\mu^*(\tau)d\tau + \int_0^T Q_2(T-\tau,x,T-t)\eta^*(\tau)d\tau + Q_3(x,t),
\end{aligned}$$

where

$$Q_1(T-\tau,x,T-t) = \int_0^b \left(-\frac{\partial^2}{\partial\xi^2}+I\right)\overline{G(\xi,x,T-t)}\mathrm{H}_1(\xi,T-\tau)d\xi,$$
$$Q_2(T-\tau,x,T-t) = \int_0^b \left(-\frac{\partial^2}{\partial\xi^2}+I\right)\overline{G(\xi,x,T-t)}\mathrm{H}_2(\xi,T-\tau)d\xi,$$
$$Q_3(x,t) = \int_0^b \left(-\frac{\partial^2}{\partial\xi^2}+I\right)\overline{G(\xi,x,T-t)}\mathrm{H}_3(\xi)d\xi - \int_0^b \gamma(\xi)\left(-\frac{\partial^2}{\partial\xi^2}+I\right)\overline{G(\xi,x,T-t)}\xi.$$

Thus, relations (40) and (41) imply the required system of linear integral equations with respect $(\mu^*(t),\eta^*(t))$.

$$\begin{cases} \beta_1\mu^*(t) + \int_0^T P_1^0(T-\tau,T-t)\mu^*(\tau)d\tau + \int_0^T P_2^0(T-\tau,T-t)\eta^*(\tau)d\tau = P_3^0(t), \\ \beta_2\eta^*(t) + \int_0^T P_1^b(T-\tau,T-t)\mu^*(\tau)d\tau + \int_0^T P_2^b(T-\tau,T-t)\eta^*(\tau)d\tau = P_3^b(t), \end{cases}$$

where

$$P_1^a(T-\tau,T-t) = \frac{\partial}{\partial x}Q_1(T-\tau,x,T-t)\Big|_{x=a},$$
$$P_2^a(T-\tau,T-t) = \frac{\partial}{\partial x}Q_2(T-\tau,x,T-t)\Big|_{x=a}, \quad P_3^a(t) = \frac{\partial}{\partial x}Q_3(x,t)\Big|_{x=a},$$

6 Conclusion

The results of the work, the boundary control of the temperature field of the bar with a selected point can be useful in solving the problem of stabilization a loaded parabolic equation using boundary control, which can be used in problems of mathematical modeling using controlled loaded differential equations.

Acknowledgements

This research first author has is funded by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP08855372).

This research second author has is funded by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP08855402).

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