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ON THE BOUNDEDNESS OF THE RIESZ POTENTIAL AND ITS COMMUTATOR'S IN THE GLOBAL MORREY TYPE SPACES WITH VARIABLE EXPONENTS

The paper considers the global Morrey-type spaces $GM_{p(.),\theta(.),w(.)}(\Omega)$ with variable exponents $p(.)$, $\theta(.)$, where $\Omega \subset R^n$ is an unbounded domain. The questions of boundedness of the Riesz potential and its commutator in these spaces are investigated. We give the conditions for variable exponents $(p_1(.), p_2(.))$, $(\theta_1(.), \theta_2(.))$ and on the functions $(w_1(.), w_2(.))$ under which the Riesz potential I^{α} , will be bounded from $GM_{p_1(.),\theta_1(.),w_1(.)}(\Omega)$ to $GM_{p_2(.),\theta_2(.),w_2(.)}(\Omega)$. The same conditions are obtained for the boundedness of the commutator of the Riesz potential in these spaces. In the case when the exponents p, θ constant numbers, the questions of boundedness of the Riesz potential and its commutator in global Morrey spaces were previously studied by other authors. There are also well-known results on the boundedness of the Riesz potential in global Morrey-type spaces with variable exponents, when the domain $\Omega \subset R^n$ is bounded.

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Көрсеткіштері айнымалы глобальді Морри типтес кеңістіктердегі Рисс потенциалы және оның коммутаторының шенелгендігі туралы

Бұл жұмыста $p(.)$, $\theta(.)$ көрсеткіштері айнымалы глобальді Морри типтес кеңістіктер $GM_{p(.),\theta(.),w(.)}(\Omega)$ қарастырылады, мұндағы $\Omega \subset R^n$-шенелмеген облыс. Көрсетілген кеңістіктердегі Рисс потенциалы және оның коммутаторының шенелгендігі туралы сұрақтар зерттеледі. $(p_1(.), p_2(.))$, $(\theta_1(.), \theta_2(.))$ көрсеткіштері және $(w_1(.), w_2(.))$ функцияларына I^{α} Рисс потенциалы $GM_{p_1(.),\theta_1(.),w_1(.)}(\Omega)$ кеңістігінен $GM_{p_2(.),\theta_2(.),w_2(.)}(\Omega)$ кеңістігіне шенелген болуының шарттары алынды. Рисс потенциалының коммутаторына да көрсетілген кеңістіктерде дәл осы сияқты шенелгендігік шарттары алынды. p, θ көрсеткіштері тұрақты болатын жағдайда Морри типтес кеңістіктердегі Рисс потенциалы және оның коммутаторының шенелгендігі туралы сұрақтарын басқа авторлар бұрын зерттеген. $\Omega \subset R^n$ шенелген облыс жағдайындағы көрсеткіштері айнымалы глобальді Морри типтес кеңістіктердегі Рисс потенциалының шенелгендік шарттары туралы да белгілі.
Түйін сөздер: глобальді Морри типтес кеңістіктер, айнымалы көрсеткіш, Рисс потенциалы, Рисс потенциалының коммутаторы, оператордың шенелгендігі.

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Об ограниченности потенциала Рисса и его коммутатора в глобальных пространствах типа Морри с переменным показателем

В работе рассматриваются глобальные пространства типа Морри $GM_{p(.),\theta(.),w(.)}(\Omega)$ с переменными показателями $p(.)$, $\theta(.)$, где $\Omega \subset R^n$ - неограниченная область. Исследуются вопросы ограниченности потенциала Рисса и его коммутатора в указанных пространствах. Получены условия на переменные показатели $(p_1(.), p_2(.))$ и $(\theta_1(.), \theta_2(.))$ и на функции $(w_1(.), w_2(.))$ при которых потенциал Рисса I^{α}, будет ограничен из $GM_{p_1(.),\theta_1(.),w_1(.)}(\Omega)$ в $GM_{p_2(.),\theta_2(.),w_2(.)}(\Omega)$.

Такие же условия получены для ограниченности коммутатора потенциала Рисса в рассматриваемых пространствах. В случае, когда показатели p, θ постоянные числа, вопросы ограниченности потенциала Рисса и его коммутатора в глобальных пространствах Морри ранее были исследованы другими авторами. Так же известны результаты об ограниченности потенциала Рисса в глобальных пространствах типа Морри с переменными показателями, когда область $\Omega \subset \mathbb{R}^n$ ограниченная.
Ключевые слова: Глобальные пространства типа Морри, переменный показатель, потенциал Рисса, коммутатор потенциала Рисса, ограниченность операторов.

1 Introduction

1.1 Review of studies by other authors

The Morrey space $M_{p,\lambda}$ was introduced in [1] in connection with the study solutions of differential equations with partial derivatives. The boundedness of integral classical operators of harmonic analysis in global Morrey-type spaces $GM_{p,\theta,w}$ with constant exponents p, θ was well studied ([2]-[5]). The boundedness of classical integral operators in the Lebesgue spaces wih variable exponent was studied in [6]-[7]).

The Morrey-type space $\mathcal{M}_{p(.),\lambda(.)}$ with variable exponents is also well studied in [8]. The generalized Morrey-type space $M_{p(.),w(.)}(\Omega)$ with variable exponent in the case of a bounded domain $\Omega \subset \mathbb{R}^n$ were introduced and studied in [9] and [10], in the case of an unbounded domain $\Omega \subset \mathbb{R}^n$ were studied in [11].

The Riesz potential I^α with exponent α is defined by :

$$I^\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, 0 < \alpha < n.$$

The boundedness of the Riesz potential in generalized Morrey-type spaces with variable exponent was studied in [9] and [10] in the case of a bounded domain $\Omega \subset \mathbb{R}^n$ and in [11] in the case of an unbounded domain $\Omega \subset \mathbb{R}^n$.

Here and below, we denote by $B(x,r)$ the ball with center $x \in \mathbb{R}^n$ and radius $r > 0$, $\tilde{B}(x,r) = B(x,r) \cap \Omega$, $\Omega \subset \mathbb{R}^n$.

The space $BMO(\Omega)$ is defined as the space of all integrable functions f with finite norm

$$\|f\|_{BMO} = \|f\|_* = \sup_{x\in\Omega, r>0} |B(x,r)|^{-1} \int_{\tilde{B}(x,r)} |f(y) - f_{\tilde{B}(x,r)}| dy,$$

where $f_{\tilde{B}(x,r)} = |\tilde{B}(x,r)|^{-1} \int_{\tilde{B}(x,r)} f(y) dy$.

Let $b \in BMO(\Omega)$. The commutator of the Riesz potential is defined by

$$[b, I^\alpha] f = I^\alpha(bf) - b(I^\alpha f) = \int_{R^n} \frac{(b(y) - b(x))}{|x-y|^{n-\alpha}} f(y) dy, 0 < \alpha < n.$$

The boundedness of the commutator of the Riesz potential in weighted Lebesgue spaces with variable exponent was studied in [12].

1.2 Basic definitions. Preliminary results.

Let $p(x)$ be a measurable function on $\Omega\subset\mathbb{R}^n$ with values on $(1,\infty)$. Assume that

$$1<p_-\le p(x)\le p_+<\infty \tag{1}$$

where

$$p_-=p_-(\Omega)=\underset{x\in\Omega}{esinf}\,p(x),$$
$$p_+=p_+(\Omega)=\underset{x\in\Omega}{essup}\,p\left(x\right).$$

We denote by $L_{p(.)}(\Omega)$ the space of all functions $f(x)$ measurable on Ω such that

$$J_{p(.)}(f)=\int_\Omega |f(x)|^{p(x)}\,dx<\infty,$$

where the norm is defined as follows

$$||f||_{p(.)}=\inf\left\{\eta>0,J_{p(.)}\left(\frac{f}{\eta}\right)\le 1\right\}.$$

For details on the Lebesgue space with variable exponent, see [6]. $\mathcal{P}(\Omega)$ is the set of measurable functions $p(x)$ for which $p:\Omega\to[1,\infty)$, $\mathcal{P}^{\log}(\Omega)$ is the set of all measurable functions $p(x)$ satisfying the local logarithmic condition:

$$|p(x)-p(y)|\le\frac{A_p}{-ln\,|x-y|},|x-y|\le\frac{1}{2},x,y\in\Omega,$$

where the constant number A_p does not depend on x and y. $\mathbb{P}^{\log}(\Omega)$ is the set of all measurable functions $p(x)$ satisfying (1) and local logarithmic condition. In the case where Ω is an unbounded set, we denote by $\mathbb{P}^{log}_\infty(\Omega)$ a subset of the set $\mathbb{P}^{\log}(\Omega)$ satisfying the logarithmic conditions at infinity:

$$|p(x)-p(\infty)|\le A_\infty ln(2+|x|),x\in\mathbb{R}^n.$$

Let Ω be a bounded open set, $p\in\mathbb{P}^{\log}(\Omega)$, and $\lambda(x)$ a function measurable on Ω with values on $[0,n]$. Morrey spaces $\mathcal{L}_{p(.),\lambda(.)}(\Omega)$ with variable exponents $p(.)$, $\lambda(.)$ were introduced [8] with the norm

$$\|f\|_{\mathcal{L}_{p(.),\lambda(.)}(\Omega)}=\underset{x\in\Omega,t>0}{sup}\,t^{-\frac{\lambda(x)}{p(x)}}\|f\|_{L_{p(.)}(\tilde{B}(x,t))}.$$

Let $w(x,r)$ be a positive measurable function on $\Omega\times(0,l)$, where $\Omega\subset\mathbb{R}^n$ is a bounded domain, $l=diam\Omega$.The generalized Morrey space $M_{p(.),w(.)}(\Omega)$ with variable exponents on a bounded domain $\Omega\subset\mathbb{R}^n$ were defined in [9] with norm

$$||f||_{M_{p(.),w(.)}(\Omega)}=\underset{x\in\Omega,r>0}{sup}\frac{r^{-\frac{n}{p(x)}}}{w(x,r)}||f||_{L_{p(.)}(\tilde{B}(x,r))}.$$

Let $w(x,r)$ be a measurable function $:\Omega\times(0,l)\to[0,\infty)$, where $\Omega\subset\mathbb{R}^n$ bounded domain, $l=diam\Omega$, measurable function $\theta(r):(0,l)\to[1,\infty]$. Morrey type spaces $M_{p(.),\theta(.),w(.)}(\Omega)$ with variable exponent on a bounded domain $\Omega\subset\mathbb{R}^n$ were defined in [10] with the norm

$$||f||_{M_{p(.),\theta(.),w(.)}(\Omega)}=\underset{x\in\Omega}{sup}\left|\left|w(x,r)r^{-\frac{n}{p(x)}}||f||_{L_{p(.)}(\tilde{B}(x,r))}\right|\right|_{L_{\theta(.)}(0,\delta)}.$$

Let $w(x,r)$ be a positive measurable function on an unbounded domain $\Omega \subset \mathbb{R}^n$.The generalized Morrey space $M_{p(.),w(.)}(\Omega)$ with variable exponent was defined in [11] with the norm

$$||f||_{M_{p(.),w(.)}(\Omega)} = \sup_{x\in\Omega, r>0} \frac{||f||_{L_{p(.)}(\tilde{B}(x,r))}}{w(x,r)}.$$

We introduce global Morrey-type spaces with variable exponents on unbounded domains.

Let's put

$$\eta_p(x,r) = \begin{cases} \frac{n}{p(x)}, & \text{if } r \leq 1; \\ \frac{n}{p(\infty)}, & \text{if } r > 1. \end{cases}$$

Let $p \in P_\infty^{log}(\Omega)$, $w(x,r)$ be a positive measurable function on $\Omega \times [0,\infty]$, where $\Omega \in R^n$, the measurable function $\theta(r)$:$(0,\infty) \to [1,\infty)$. Global Morrey space with variable exponents $GM_{p(.),\theta(.),w(.)}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ unbounded domain, defined as the set of functions $f \in L_{p(.)}^{loc}(\Omega)$ with finite norm

$$||f||_{GM_{p(.),\theta(.),w(.)}(\Omega)} = \sup_{x\in\Omega} \big|\big|w(x,r)r^{-\eta_p(x,r)}||f||_{L_{p(.)}(\tilde{B}(x,r))}\big|\big|_{L_{\theta(.)}(0,\infty)},$$

for $1 \leq \theta(r) < \infty$, with finite norm

$$||f||_{GM_{p(.),\infty,w(.)}(\Omega)} = ||f||_{M_{p(.),w_1(.)}(\Omega)} = \sup_{x\in\Omega, r>0} w(x,r)r^{-\eta_p(x,r)}||f||_{L_{p(.)}(\tilde{B}(x,r))},$$

for $\theta(r) = \infty$.

Note that the space $GM_{p(.),\infty,w(.)}(\Omega)$ coincides with the generalized Morrey-type space $M_{p(.),w_1(.)}(\Omega)$ with variable exponent, where $w_1(x,r) = \frac{r^{\eta_p(x,r)}}{w(x,r)}$.

In the case of $w(x,r) = r^{-\frac{\lambda(x)}{p(x)}+\eta_p(x,r)}$ we denote the indicated space by via $GM_{p(.),\theta(.)}^{\lambda(.)}$:

$$GM_{p(.),\theta(.)}^{\lambda(.)}(\Omega) = GM_{p(.),w(.),\theta}\big|_{w(x,r)=r^{-\frac{\lambda(x)}{p(x)}+\eta_p(x,r)}},$$

$$||f||_{GM_{p(.),\theta(.)}^{\lambda(.)}(\Omega)} = \sup_{x\in\Omega} \big|\big|r^{-\frac{\lambda(x)}{p(x)}}||f||_{L_{p(.)}(\tilde{B}(x,r))}\big|\big|_{L_{\theta(.)}(0,\infty)}.$$

If $p(.) = p = const$, $\theta(x) = \theta = const$, then the space $GM_{p(.),\theta(.),w(.)}(\Omega)$ coincides with the well-known global Morrey space $GM_{p,\theta,w}(\Omega)$ (see, for example, [4]). The following lemma gives a sufficient condition under which the space $GM_{p(.),\theta(.),w(.)}(\Omega)$ is not trivial.

Lemma 2.1. Let

$$\sup_{x\in\Omega} ||w(x,r)||_{L_{\theta(.)}(0,\infty)} < \infty.$$

Then the space $GM_{p(.),\theta(.),w(.)}(\Omega)$ is not empty.

Proof. It suffices to show that the space contains bounded functions. Let $|f(x)| < C$, using the well-known inequality $||1||_{L_{p(.)}(B(x,r))} \ll r^{\eta_p(x,r)}$(see, for example, [11]), we obtain

$$||f||_{GM_{p(.),\theta(.),w(.)}(\Omega)} = \sup_{x\in\Omega} \big|\big|w(x,r)r^{-\eta_p(x,r)}||f||_{L_{p(.)}(\tilde{B}(x,r))}\big|\big|_{L_{\theta(.)}(0,\infty)} <$$

$$< \sup_{x\in\Omega} \big|\big|w(x,r)r^{-\eta_p(x,r)}||C||_{L_{p(.)}(\tilde{B}(x,r))}\big|\big|_{L_{\theta(.)}(0,\infty)} < C \sup_{x\in\Omega} ||w(x,r)||_{L_{\theta(.)}(0,\infty)} < \infty,$$

this means that $f \in GM_{p(.),\theta(.),w(.)}(\Omega)$.

Lemma 2.1 is proved.

The following theorem was proved in [11].

Theorem 1.1 Let $p \in \mathbb{P}^{log}_{\infty}(\Omega)$, $0 < \alpha < n$, $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}$ and positive measurable functions w_1 and w_2 satisfy the condition

$$\int_r^{\infty} \frac{essinf_{t \le s < \infty} w_1(x,s)}{t^{1+\eta_p(x,t)}} dt \le C \frac{w_2(x,r)}{r^{\eta_q(x,r)}},$$

where C does not depend on x and r. Then the operator I^{α} is bounded from $M_{p(.),w_1(.)}(\Omega)$ to $M_{q(.),w_2(.)}(\Omega)$.

Let $\Omega \subset R^n$ be a bounded domain, $l = diam\Omega$. Denote by $\mathcal{W}(\delta, l)$ the set of pairs of measurable functions (θ, w) for which there exists $\delta \in (0, l)$ such that $inf_{x\in\Omega} \|w(x,.)\|_{L_{\theta(.)}(\delta,l)} > 0$.

The following theorem gives a sufficient condition for the boundedness of the Riesz Potential in Morrey-type spaces with variable exponents $p(.), \theta(.), w(.)$ over bounded domains [10].

Theorem 1.2. Assume that $p, \alpha \in \mathcal{P}^{\log}(\Omega)$ and $\alpha > 0$, $(\alpha p(.))_+ = sup_{x\in\Omega} \alpha p(x) < n$, $\frac{1}{p_2(x)} = \frac{1}{p_1(x)} - \frac{\alpha}{n}$, $1 < \theta_1^- \le \theta_1(t) \le \theta_1^+ < \infty$, $1 < \theta_2^- \le \theta_2(t) \le \theta_2^+ < \infty$ for any $0 < t < l$. Suppose there exists $\delta > 0$ such that $\theta_1(t) \le \theta_2(t)$, $t \in (0,\delta)$, $(\theta_1, w_1) \in \mathcal{W}(\delta, l)$. Denote $\tilde{\theta}_1(\xi) = inf_{s\in(\xi,l)\theta_1(s)}$. If

$$\sup_{x\in\Omega, 0<t<\delta} \int_0^t (w_2(x,\xi))^{\theta_2(\xi)} \left(\int_t^{\delta} \left(\frac{r^{\alpha(x)-1}}{w_1(x,r)}\right)^{[\tilde{\theta}_1(\xi)]'} dr\right)^{\frac{\theta_2(\xi)}{[\tilde{\theta}_1(\xi)]'}} d\xi < \infty,$$

then the operator I^{α} is bounded from $M_{p_1(.),\theta_1(.),w_1(.)}(\Omega)$ to $M_{p_2(.),\theta_2(.),w_2(.)}(\Omega)$.

We will need the following theorems on estimating the norm of the Riesz potential and its commutator over the ball, which were proved in [11], [12] respectively.

Theorem 1.3. Let $p \in \mathbb{P}^{log}_{\infty}(\Omega)$ and α satisfy the condition $0 < \alpha < n$, $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}$. Then the following estimate holds

$$||I^{\alpha} f||_{L_{q(.)}(\tilde{B}(x,t))} \le C t^{\eta_q(x,t)} \int_t^{\infty} r^{-\eta_q(x,r)-1} ||f||_{L_{p(.)}(\tilde{B}(x,r))} dr, \tag{2}$$

where C does not depend on $x \in \Omega$ and $t > 0$.

Theorem 1.4. Let $\Omega \subset R^n$ be an unbounded domain, $0 < \alpha < n$, $p \in \mathbb{P}^{log}_{\infty}(\Omega)$, $p_+ < \frac{n}{\alpha}$, $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{alpha}{n}$, $b \in BMO(\Omega)$.Then

$$||[b, I^{\alpha} f]||_{L_{q(.)}(\tilde{B}(x,t))} \le C\|b\|_* t^{\eta_q(x,t)} \int_t^{\infty} (1 + ln\frac{r}{t}) r^{-\eta_q(x,r)-1} ||f||_{L_{p(.)}(\tilde{B}(x,r))} dr, \tag{3}$$

where C does not depend on $x \in \Omega$ and $t > 0$.

Let u and v be positive measurable functions on R_+. The conjugate Hardy operator is defined by

$$\tilde{H}_{v,u} f(r) = v(x) \int_r^{\infty} f(t) u(t) dt, x \in R_+,$$

where $R_+ = (0, +\infty)$. Suppose a is a fixed positive number. Let $\theta_{1,a}(r) = essinf_{y\in[r,a)}\,\theta_1(y)$,

$$\tilde{\theta}_1(r) = \begin{cases} \theta_{1,a}(r) & \text{if } r \in [0,a]; \\ \overline{\theta}_1 = const & \text{if } r \in [a,\infty); \end{cases},$$

$\theta_1 = essinf_{r\in R_+}\,\theta_1(r)$, $\Theta_2 = essup_{r\in R_+}\,\theta_2(r)$.

The following theorem was proved in [13].

Theorem 1.5. Let $\theta_1(r)$ and $\theta_2(r)$ be positive measurable functions on R_+ and there exists a positive number a such that that $\theta_1(r) = \overline{\theta}_1 = const$, $\theta_2(r) = \overline{\theta_2} = const$ for all $r > a$, inequalities $1 < \theta_1 \le \tilde{\theta}_1(r) \le \theta_2(r) \le \Theta_2 < \infty$ hold almost everywhere on R_+. If

$$G = \sup_{t>0} \int_0^t [v(r)]^{\theta_2(r)} (\int_t^\infty u^{\tilde{\theta}_1'(r)}(\tau)d\tau)^{\frac{\theta_2(r)}{(\tilde{\theta}_1)'(r)}} dr < \infty, \tag{4}$$

hen the operator $\tilde{H}_{v,u}$ is bounded from $L_{\theta_1(.)}(R^+)$ to $L_{\theta_2(.)}(R^+)$.

2 The main results

Theorem 2.1. Let $p(.) \in \mathbb{P}_\infty^{log}(\Omega)$ and a constant number α satisfy the conditions $\alpha > 0$, $(\alpha p(.))_+ = sup_{x\in\Omega}\,\alpha p(x) < n$, $\theta_1(r)$ and $\theta_2(r)$ are positive measurable functions on R_+ and there exists a positive number a such that $\theta_1(r) = \overline{\theta}_1 = const$, $\theta_2(r) = \overline{\theta_2} = const$ for all $r > a$, inequality $1 < \theta_1 \le \tilde{\theta}_1(r) \le \theta_2(r) \le \Theta_2 < \infty$ are executed almost everywhere. Suppose that the functions $p_1(x)$ and $p_2(x)$ satisfy the equality $\frac{1}{p_2(x)} = \frac{1}{p_1(x)} - \frac{\alpha}{n}$, positive measurable functions w_1 and w_2 satisfy the condition

$$T = \sup_{x\in\Omega, t>0} \int_0^t (w_2(x,r))^{\theta_2(r)} (\int_t^\infty (\frac{s^{\alpha-1}}{w_1(x,s)})^{[\tilde{\theta}_1(r)]'} ds)^{\frac{\theta_2(r)}{[\tilde{\theta}_1(r)]'}} dr < \infty. \tag{5}$$

Then the operator I^α is bounded from $GM_{p_1(.),\theta_1(.),w_1(.)}(\Omega)$ to $GM_{p_2(.),\theta_2(.),w_2(.)}(\Omega)$.

Proof of Theorem 2.1. Using Theorem 1.3, we have

$$||I^\alpha||_{GM_{p_2(.),\theta_2(.),w_2(.)}(\Omega)} = \sup_{x\in\Omega} \left\| w_2(x,r) r^{-\eta_{p_2}(x,r)} \| I_\alpha f \|_{L_{p_2(.)}(B(x,r))} \right\|_{L_{\theta_2(.)}(0,\infty)} \le$$

$$\le C \sup_{x\in\Omega} ||w_2(x,r) \int_r^\infty t^{-\eta_{p_2}(x,t)-1} ||f||_{L_{p_1(.)}(B(x,t))} dt||_{L_{\theta_2(.)}(0,\infty)}.$$

Denote

$$\tilde{H}_{v,u} f(r) = v(r) \int_r^\infty g(t)u(t)dt,$$

where

$$v(r) = w_2(x,r),$$

$$g(t) = \frac{w_1(x,t)}{t^{\eta_{p_1}(x,t)}} ||f||_{L_{p_1(.)}(B(x,t))},$$

$$u(t) = \frac{t^{\eta_{p_1}(x,t)-\eta_{p_2}(x,t)-1}}{w_1(x,t)} = \frac{t^{\alpha-1}}{w_1(x,t)},$$

for every fixed $x \in \Omega$. Then condition (4) has the form (5), which, according to Theorem 1.5, implies that the operator $\tilde{H}_{v,u}f(r)$ is bounded from $L_{\theta_1(.)}(0,\infty)$ to $L_{\theta_2(.)}(0,\infty)$. Finally, we have

$$\|I^{\alpha}f\|_{GM_{p_2(.),\theta_2(.),w_2(.)}(\Omega)} \leq CT \cdot \sup_{x\in\Omega} \|w_1(x,t)t^{-\eta_{p_1}(x,t)}\|f\|_{L_{p_1(.)}(B(x,t))}\|_{L_{\theta_1(.)}(0,\infty)} =$$

$$= CT \cdot \|f\|_{GM_{p_1(.),\theta_1(.),w_1(.)}(\Omega)},$$

this means that the operator I^{α} is bounded from $GM_{p_1(.),\theta_1(.),w_1(.)}(\Omega)$ to $GM_{p_2(.),\theta_2(.),w_2(.)}(\Omega)$. Theorem 2.1 is proved.

Theorem 2.2. Let $p(.) \in \mathbb{P}^{log}_{\infty}(\Omega)$ and a constant number α satisfy the conditions $\alpha > 0$, $(\alpha p(.))_+ = sup_{x\in\Omega}\, \alpha p(x) < n$, $\theta_1(r)$ and $\theta_2(r)$ are positive measurable functions on R_+ and there exists a positive number a such that $\theta_1(r) = \overline{\theta}_1 = const$, $\theta_2(r) = \overline{\theta_2} = const$ for all $r > a$, inequality $1 < \theta_1 \leq \tilde{\theta}_1(r) \leq \theta_2(r) \leq \Theta_2 < \infty$ are executed almost everywhere. Suppose that the functions $p_1(x)$ and $p_2(x)$ satisfy the equality $\frac{1}{p_2(x)} = \frac{1}{p_1(x)} - \frac{\alpha}{n}$, positive measurable functions w_1 and w_2 satisfy the condition

$$B = \sup_{x\in\Omega, t>0} \int_0^t \left(\frac{w_2(x,r)}{r}\right)^{\theta_2(r)} \left(\int_t^{\infty} \left(\frac{s^{\alpha}}{w_1(x,s)}\right)^{[\tilde{\theta}_1(r)]'} ds\right)^{\frac{\theta_2(r)}{[\tilde{\theta}_1(r)]'}} dr < \infty. \tag{6}$$

Then the commutator $[b, I^{\alpha}]$ is bounded from $GM_{p_1(.),\theta_1(.),w_1(.)}(\Omega)$ to $GM_{p_2(.),\theta_2(.),w_2(.)}(\Omega)$. Proof of Theorem 2.2. According to Theorem 1.4, we have

$$||[b, I^{\alpha}]f||_{GM_{p_2(.),\theta_2(.),w_2(.)}(\Omega)} = \sup_{x\in\Omega} \|w_2(x,r)r^{-\eta_{p_2}(x,r)}\|[b, I_{\alpha}]f\|_{L_{p_2(.)}(B(x,r))}\|_{L_{\theta_2(.)}(0,\infty)} \leq$$

$$\leq C \sup_{x\in\Omega} ||\frac{w_2(x,r)}{r} \int_r^{\infty} t^{-\eta_{p_2}(x,t)}||f||_{L_{p_1(.)}(B(x,t))} dt||_{L_{\theta_2(.)}(0,\infty)},$$

here we use the inequality $1 + ln\frac{t}{r} < \frac{t}{r}$ for $t > r > 0$. Denote

$$\tilde{H}_{v,u}f(r) = v(r) \int_r^{\infty} g(t)u(t)dt,$$

where

$$v(r) = \frac{w_2(x,r)}{r},$$

$$g(t) = \frac{w_1(x,t)}{t^{\eta_{p_1}(x,t)}}||f||_{L_{p_1(.)}(B(x,t))},$$

$$u(t) = \frac{t^{\alpha}}{w_1(x,t)},$$

for every fixed $x \in \Omega$. Then condition (4) takes the form (6), from which, according to Theorem 1.5, it follows that the operator $\tilde{H}_{v,u}f(r)$ is bounded from $L_{\theta_1(.)}(0,\infty)$ to $L_{\theta_2(.)}(0,\infty)$. Finally, we have

$$\|[b, I^{\alpha}]f\|_{GM_{p_2(.),\theta_2(.),w_2(.)}(\Omega)} \leq CB \cdot \sup_{x\in\Omega} \|w_1(x,t)t^{-\eta_{p_1}(x,t)}\|f\|_{L_{p_1(.)}(B(x,t))}\|_{L_{\theta_1(.)}(0,\infty)} =$$

$$= CB \cdot \|f\|_{GM_{p_1(.),\theta_1(.),w_1(.)}(\Omega)},$$

which means that the commutator $[b, I^{\alpha}]$ is bounded from $GM_{p_1(.),\theta_1(.),w_1(.)}(\Omega)$ to $GM_{p_2(.),\theta_2(.),w_2(.)}(\Omega)$.

Theorem 2.2 is proved.

3 Conclusion

We have obtained the sufficient conditions for the boundedness Riesz potential and its commutator the global Morrey-type spaces with variable exponents.

We gave the conditions for variable exponents $(p_1(.), p_2(.))$, $(\theta_1(.), \theta_2(.))$ and on the functions $(w_1(.), w_2(.))$ under which the Riesz potential I^{α} , would be bounded from $GM_{p_1(.),\theta_1(.),w_1(.)}(\Omega)$ to $GM_{p_2(.),\theta_2(.),w_2(.)}(\Omega)$.

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