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MULTI-TERM TIME-FRACTIONAL DERIVATIVE HEAT EQUATION FOR ONE-DIMENSIONAL DUNKL OPERATOR

In this paper, we investigate the well-posedness for Cauchy problem for multi-term time-fractional heat equation associated with Dunkl operator. The equation under consideration includes a linear combination of Caputo derivatives in time with decreasing orders in $(0,1)$ and positive constant coefficients and one-dimensional Dunkl operator. To show solvability of this problem we use several important properties of multinomial Mittag-Leffler functions and Dunkl transforms, since various estimates follow from the explicit solutions in form of these special functions and transforms. Then we prove the uniqueness and existence results. To achieve our goals, we use methods corresponding to the different areas of mathematics such as the theory of partial differential equations, mathematical physics, hypoelliptic operators theory and functional analysis. In particular, we use the direct and inverse Dunkl transform to establish the existence and uniqueness of solutions to this problem on the Sobolev space. The generalized solutions of this problem are studied.
Key words: Dunkl operator, heat equation, Cauchy problem, Caputo fractional derivative.

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Бір өлшемді Данкл операторы үшін уақыт бойынша көпмүшелік бөлшек туындысы бар жылу өткізгіштік теңдеуі

Бұл мақалада біз Данкл операторымен байланысты уақыт бойынша көпмүшелік бөлшек туындысы бар жылу өткізгіштік теңдеуі үшін Коши есебінің қисынды екенін зерттейміз. Қарастырылып отырған теңдеу уақыт бойынша Капуто туындыларының сызықтық комбинациясы $(0,1)$ оң коэффициенттерімен және де бір өлшемді Данкл операторынан туындаған. Бұл есептің шешімділігін көрсету үшін біз Миттаг-Леффлер көпмүшелік арнайы функциялары және Данкл түрлендіруінің маңызды қасиеттерін қолданамыз, өйткені әртүрлі бағалаулар осы арнайы функциялармен түрлендірулер түріндегі нақты шешімдерден туындайды. Содан кейін біз осы есептің шешімі бар және жалғыз екенін дәлелдейміз. Осы айтылғанды дәлелдеу үшін біз математиканың әртүрлі салаларына сәйкес келетін әдістерді қолданамыз, атап айтқанда дербес туындылы дифференциалдық теңдеулер теориясы, математикалық физика теңдеулері, гипоэллиптикалық операторлар теориясы және функционалдық талдау. Қарастырып отырған есептің шешімі бар және жалғыз болатынын Соболев кеңістігінде дәлелдейміз, ол үшін біз негізгі әдіс ретінде тура және кері Данкл түрлендіруін қолданамыз. Бұл есепте жалпылама шешім қарастырылады.
Түйін сөздер: Данкл операторы, жылу өткізгіштік теңдеуі, Коши есебі, Капуто бөлшек туындысы.

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Уравнение теплопроводности с многочленной дробной производной по времени для одномерного оператора Данкля

В этой статье мы исследуем корректность задачи Коши для уравнения теплопроводности с многочленным дробным производным по временем связанного с оператором Данкла. Рассматриваемое уравнение включает линейную комбинацию производных Капуто по времени с убывающими порядками в $(0, 1)$ и положительными коэффициентами и одномерным оператором Данкла. Чтобы показать разрешимость этой задачи, мы используем несколько важных свойств многочленных функций Миттага-Леффлера и преобразований Данкла, поскольку из явных решений в виде этих специальных функций и преобразований вытекают различные оценки. Затем мы докажем единственность и существования решения этой задачи. Для достижения наших цель мы используем методы, соответствующие различным областям математики, таким как теория дифференциальных уравнений в частных производных, математическая физика, теория гипоэллиптических операторов и функциональный анализ. В частности, мы используем прямое и обратное преобразование Данкла, чтобы установить существование и единственность решений этой задачи в Соболевском пространстве. Изучаются обобщенные решения этой задачи.

Ключевые слова: Оператор Данкла, уравнение теплопроводности, задача Коши, дробная производная Капуто.

1 Introduction

Let γ be $0 < \gamma < 1$. For a fixed positive integer m, $a_j \in \mathbb{R}$ and γ_j $(j = 1, ..., m)$ be constants such that $1 > \gamma > \gamma_1 >, ..., > \gamma_m > 0$. We consider the following equation

$$\partial_t^{\gamma} u(t, x) - \sum_{j=1}^{m} a_j \partial_t^{\gamma_j} u(t, x) - \Lambda_{\alpha,x}^2 u(t, x) = f(t, x) \tag{1}$$

in the domain $(t, x) \in Q_T$, under the initial condition

$$u(0, x) = g(x), \quad x \in \mathbb{R}, \tag{2}$$

where f and g are sufficiently smooth functions.

Here $\partial_t^{\alpha_j}$ denotes the Caputo derivative defined by

$$\partial_t^{\alpha_j} u(t) := \frac{1}{\Gamma(1 - \alpha_j)} \int_0^t \frac{u'(s)}{(t - s)^{\alpha_j}} ds,$$

where $\Gamma(\cdot)$ is a usual Gamma function. For various properties of the Caputo derivative, we refer to Kilbas et al. [9], Podlubny [10].

The operator Λ_α is called the Dunkl operator which was introduced in 1989 by C. Dunkl [2], where $\alpha \geq 1/2$. The Dunkl operator is associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. The Dunkl operators are very important in pure mathematics and physics. Solution of the spectral problem generated by the Dunkl operator is called the Dunkl kernel $E_\alpha(ix\lambda)$ which is used to define the Dunkl transform $\mathcal{F}_\alpha$ [4]. Main properties of the Dunkl transform is given by M.F.E. de Jeu in 1993 [5]. For more information about harmonic analysis associated with the operator Λ_α, we refer the readers to the papers [1, 3, 5, 6].

A general solution of problem (1)–(2) is the function $u \in C^\alpha([0, T], L^2(\mathbb{R}, \mu_\alpha)) \cap C([0, T], W_\alpha^{2,2}(\mathbb{R}, \mu_\alpha))$ satisfying the equation (1). Let us denote that by $\mathcal{D}_t^\gamma := \partial_t^\gamma - \sum_{j=1}^m a_j \partial_t^{\gamma_j}$.

2 Auxiliary materials

In this section we introduce the Dunkl operator and it's necessary properties to our research.

2.1 The Dunkl operator and the Dunkl transform

The first-order singular differential-difference operator $\Lambda_\alpha, \alpha \geq -1/2$, given by

$$\Lambda_\alpha y(x) = \frac{d}{dx}y(x) + \frac{2\alpha+1}{x}\left(\frac{y(x)-y(-x)}{2}\right), \quad y \in C^1(\mathbb{R})$$

called the Dunkl operator, associated with the reflexion group $\mathbb{Z}_2$ on $\mathbb{R}$. If $\alpha = -1/2$, the Dunkl operator turns into the ordinary differential operator $\Lambda_{-1/2} = \frac{d}{dx}$.

For $\alpha \geq -1/2$ and $\lambda \in \mathbb{R}$ the spectral problem associated with Dunkl operator

$$\begin{cases} \Lambda_\alpha y(x) - (i\lambda)y(x) = 0, \\ y(0) = 1. \end{cases}$$

has a unique solution $y(x) = D_\alpha(ix\lambda)$ called Dunkl kernel given by

$$D_\alpha(ix\lambda) = j_\alpha(ix\lambda) + \frac{ix\lambda}{2(\alpha+1)}j_{\alpha+1}(ix\lambda), \quad x \in \mathbb{R},$$

where

$$j_\alpha(ix\lambda) = \Gamma(\alpha+1)\sum_{k=0}^{+\infty}\frac{1}{k!}\frac{(ix\lambda/2)^{2k}}{\Gamma(k+\alpha+1)}$$

is the normalized Bessel function of order α.

Замечание 1 *For $\alpha = -\frac{1}{2}$, we have*

$$\begin{cases} \frac{d}{dx}y(x) - (i\lambda)y(x) = 0, \\ y(0) = 1. \end{cases}$$

The solution of this problem is

$$D_{-1/2}(ix\lambda) = e^{ix\lambda}.$$

Definition 1 *We denote by $L^p(\mathbb{R}, \mu_\alpha), 1 \leq p \leq +\infty$, the space of measurable functions h on $\mathbb{R}$ such that*

$$\|h\|_{p,\alpha} = \left(\int_{\mathbb{R}} |h(x)|^p d\mu_\alpha(x)\right)^{\frac{1}{p}} < +\infty, \quad 1 \leq p < +\infty,$$

$$\|h\|_\infty = \sup_{x\in\mathbb{R}} |h(x)| < +\infty.$$

Here μ_α is the measure defined on $\mathbb{R}$ by

$$d\mu_\alpha(x) = \frac{|x|^{2\alpha+1}}{2^{\alpha+1}\Gamma(\alpha+1)}dx, \quad \alpha \geq -1/2.$$

For $h \in L^1(\mathbb{R}, \mu_\alpha)$ the Dunkl transform is defined by

$$\mathcal{F}_\alpha(h)(\lambda) = \widehat{h}(\lambda) := \int_{\mathbb{R}} h(x) D_\alpha(-ix\lambda) d\mu_\alpha(x), \quad \lambda \in \mathbb{R}. \tag{3}$$

This transform has the following properties ([5]):
i) For all $h \in \mathcal{S}(\mathbb{R})$, we have

$$\mathcal{F}_\alpha(\Lambda_\alpha h)(\lambda) = i\lambda \mathcal{F}_\alpha(h)(\lambda), \quad \lambda \in \mathbb{R}. \tag{4}$$

ii) For all $h \in L^1(\mathbb{R}, \mu_\alpha)$, the Dunkl transform $\mathcal{F}_\alpha$ is a continuous function on $\mathbb{R}$ satisfying

$$\|\mathcal{F}_\alpha(h)\|_\infty \leq \|h\|_{1,\alpha}.$$

iii) (L^1-inversion) For all $h \in L^1(\mathbb{R}, \mu_\alpha)$ with $\mathcal{F}_\alpha(h) \in L^1(\mathbb{R}, \mu_\alpha)$, we have

$$h(x) = \int_{\mathbb{R}} \mathcal{F}_\alpha(h)(\lambda) D_\alpha(ix\lambda) d\mu_\alpha(\lambda). \tag{5}$$

iv) $\mathcal{F}_\alpha$ is a topological isomorphism on $\mathcal{S}(\mathbb{R})$ which extends to a topological isomorphism on $\mathcal{S}'(\mathbb{R})$.
v) (Plancherel theorem) The Dunkl transform $\mathcal{F}_\alpha$ is an isometric isomorphism of $L^2(\mathbb{R}, \mu_\alpha)$. In particular,

$$\|\mathcal{F}_\alpha(h)\|_{2,\alpha} = \|h\|_{2,\alpha}. \tag{6}$$

Notation. ([6, p. 22]) For $s \in \mathbb{R}$ we denote by

$$W_\alpha^{s,2}(\mathbb{R}, \mu_\alpha) := \{h \in \mathcal{S}'(\mathbb{R}) : \quad \|h\|^2_{W_\alpha^{s,2}(\mathbb{R},\mu_\alpha)} = \int_{\mathbb{R}} (1+\lambda^2)^s |\mathcal{F}_\alpha(h)(\lambda)|^2 d\mu_\alpha(\lambda) < \infty\}$$

the usual Sobolev space on $\mathbb{R}$.

3 Main results and methods

Theorem 1 *Let $g \in W_\alpha^{2,2}(\mathbb{R}, \mu_\alpha)$, $f \in C([0,T], W_\alpha^{2,2}(\mathbb{R}, \mu_\alpha))$ and $0 < \gamma < 1$. Then there exists a unique solution of problem (1)–(2). Moreover, it is given by the expression*

$$u(t,x) = \int_{\mathbb{R}} \int_{\mathbb{R}} g(y) \left(1 - \lambda^2 t^\gamma E_{\gamma+1}(t)\right) D_\alpha(ix\lambda) D_\alpha(-iy\lambda) d\mu_\alpha(y) d\mu_\alpha(\lambda)$$
$$+ \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^t \frac{d}{d\tau} \{\tau^\gamma E_{1+\gamma}(\tau)\} f(t-\tau, y) D_\alpha(ix\lambda) D_\alpha(-iy\lambda) d\tau d\mu_\alpha(y) d\mu_\alpha(\lambda),$$

where

$$E_{\gamma+1}(t) = E_{(\gamma-\gamma_1,\ldots,\gamma-\gamma_m,\gamma),1+\gamma}(a_1 t^{\gamma-\gamma_1}, \ldots, a_m t^{\gamma-\gamma_m}, -\lambda^2 t^\gamma).$$

Here

$$E_{(\gamma_1,\ldots,\gamma_{m+1}),\beta(z_1,\ldots,z_{m+1})} = \sum_{k=0}^{\infty} \sum_{\substack{l_1+l_2+\cdots+l_{m+1}=k,\\ l_1\geq 0,\ldots,l_{m+1}\geq 0}} \frac{k!}{l_1!\ldots l_{m+1}!} \frac{\prod_{j=1}^{m+1} z_j^{l_j}}{\Gamma\left(\beta + \sum_{j=1}^{m+1} \gamma_j l_j\right)} \tag{7}$$

is the multivariate Mittag-Leffler function [7].

Existence of the solution. Now to show that there is a generalised solution to problem (1)–(2), we apply the Dunkl transform $\mathcal{F}_\alpha$ to the equation (1) and the initial condition (2). It gives us

$$\partial_t^\gamma \widehat{u}(t,\lambda) - \sum_{j=1}^m a_j \partial_t^{\gamma_j} \widehat{u}(t,\lambda) + \lambda^2 \widehat{u}(t,\lambda) = \widehat{f}(t,\lambda), \tag{8}$$

and

$$\widehat{u}(0,\lambda) = \widehat{g}(\lambda), \tag{9}$$

for all $\lambda \in \mathbb{R}$, where $\widehat{u}(\cdot,\lambda)$ is an unknown function. Then by solving the equation (14) under the initial condition (15) (see [7]), we get

$$\begin{aligned}\widehat{u}(t,\lambda) &= \widehat{g}(\lambda)\left(1 - \lambda^2 t^\gamma E_{\gamma+1}(t)\right) \\ &\quad + \int_0^t \tau^{\gamma-1} E_\gamma(\tau)\, \widehat{f}(t-\tau,\lambda) d\tau.\end{aligned} \tag{10}$$

Lemma 1 *[8] Let* $0 < \gamma < 1$. *Then*

$$\frac{d}{dt}\{t^\gamma E_{1+\gamma}(t)\} = t^{\gamma-1} E_\gamma(t),\ t > 0.$$

Using Lemma 1 we can rewrite the formula (10) in the form:

$$\begin{aligned}\widehat{u}(t,\lambda) &= \widehat{g}(\lambda)\left(1 - \lambda^2 t^\gamma E_{\gamma+1}(t)\right) \\ &\quad + \int_0^t \frac{d}{d\tau}\{\tau^\gamma E_{1+\gamma}(\tau)\} \widehat{f}(t-\tau,\lambda) d\tau.\end{aligned} \tag{11}$$

Consequently, by using the inverse Dunkl transform (5) to (11), one obtains the solution of problem (1)–(2)

$$\begin{aligned}u(t,x) &= \int_{\mathbb{R}}\int_{\mathbb{R}} g(y)\left(1 - \lambda^2 t^\gamma E_{\gamma+1}(t)\right) D_\alpha(ix\lambda) D_\alpha(-iy\lambda) d\mu_\alpha(y) d\mu_\alpha(\lambda) \\ &+ \int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^t \frac{d}{d\tau}\{\tau^\gamma E_{1+\gamma}(\tau)\} f(t-\tau,y) D_\alpha(ix\lambda) D_\alpha(-iy\lambda) d\tau d\mu_\alpha(y) d\mu_\alpha(\lambda).\end{aligned} \tag{12}$$

Here, we prove convergence of the obtained solution (12) and it's derivatives $\mathcal{D}_t^\gamma u(t,x)$, $\Lambda_\alpha^2 u(t,x)$. To prove the convergence of these, we use the estimate for the multivariate Mittag–Leffler function (7), obtained in [8], of the form

$$|E_{(\gamma-\gamma_1,\ldots,\gamma-\gamma_m,\gamma),1+\gamma}(a_1 t^{\gamma-\gamma_1}, \ldots, a_m t^{\gamma-\gamma_m}, -\lambda^2 t^\gamma)| \leq \frac{C}{1+\lambda^2 t^\gamma}.$$

Let us to show absolute convergence the first term of (11):

$$\widehat{g}(\lambda)\left(1 - \lambda^2 t^\gamma E_{\gamma+1}(t)\right) \leq C|\widehat{g}(\lambda)|.$$

Let us to show the convergence of the integral term of (11):

$$\begin{aligned} &\left|\int_0^t \frac{d}{d\tau}\{\tau^\gamma E_{1+\gamma}(\tau)\}\widehat{f}(t-\tau,\lambda)d\tau\right| \leq \\ &\leq \max_{0\leq\tau\leq t}|\widehat{f}(t-\tau,\lambda)|t^{\gamma-1}E_\gamma(t) \leq \\ &\leq \max_{0\leq t\leq T}|\widehat{f}(t,\lambda)|T^{\gamma-1}E_\gamma(T) \leq C\|\widehat{f}(\cdot,\lambda)\|_{C([0,T])}. \end{aligned} \tag{13}$$

Now using obtained above inequalities and in view of Plancherel theorem we have

$$\begin{aligned} \|u(t,\cdot)\|^2_{2,\alpha} = \|\widehat{u}(t,\cdot)\|^2_{2,\alpha} &= \int_{\mathbb{R}} |\widehat{u}(t,\lambda)|^2 d\mu_\alpha(\lambda) \leq \\ &\leq C\|g\|^2_{2,\alpha} + C\|f\|^2_{C([0,T];L^2(\mathbb{R},\mu_\alpha))}. \end{aligned}$$

This implies $u(t,x) \in C([0,T];L^2(\mathbb{R},\mu_\alpha))$. Similarly we can obtain

$$\begin{aligned} \|\Lambda^2_\alpha u(t,\cdot)\|^2_{2,\alpha} = \|\lambda^2\widehat{u}(t,\cdot)\|^2_{2,\alpha} &= \int_{\mathbb{R}} |\lambda^2\widehat{u}(t,\lambda)|^2 d\mu_\alpha(\lambda) \\ &\leq C\|g\|^2_{W^{2,2}_\alpha(\mathbb{R},\mu_\alpha)} + C\|f\|^2_{C([0,T];W^{2,2}_\alpha(\mathbb{R},\mu_\alpha))}. \end{aligned}$$

This implies immediately $\Lambda^2_\alpha u(t,x) \in C([0,T];L^2(\mathbb{R},\mu_\alpha))$. Finally, we have

$$\begin{aligned} \|\mathcal{D}^\gamma_t u(t,\cdot)\|^2_{2,\alpha} = \|\mathcal{D}^\gamma_t \widehat{u}(t,\cdot)\|^2_{2,\alpha} &= \|\widehat{f}(t,\cdot) - \lambda^2\widehat{u}(t,\cdot)\|^2_{2,\alpha} \\ &\leq C\|\widehat{f}(t,\cdot)\|^2_{2,\alpha} + C\|\lambda^2\widehat{u}(t,\cdot)\|^2_{2,\alpha} \\ &\leq C\|g\|^2_{W^{2,2}_\alpha(\mathbb{R},\mu_\alpha)} + C\|f\|^2_{C([0,T];W^{2,2}_\alpha(\mathbb{R},\mu_\alpha))}. \end{aligned}$$

This is $\mathcal{D}^\gamma_t u(t,x) \in C([0,T];L^2(\mathbb{R},\mu_\alpha))$. So we finally proved existence of the generalized solution of the problem (1)–(2) and it belongs to the class $u(t,x) \in C^\alpha([0,T],L^2(\mathbb{R},\mu_\alpha)) \cap C([0,T],W^{2,2}_\alpha(\mathbb{R},\mu_\alpha))$.

Now, we are in a position to show the uniqueness of the solutions. Suppose that there are two solutions u_1 and u_2 of the problem (1)–(2). Denote

$$u(t,x) = u_1(t,x) - u_2(t,x).$$

Then the function u satisfies the equation

$$\partial^\gamma_t u(t,x) - \sum_{j=1}^m a_j \partial^{\gamma_j}_t u(t,x) - \Lambda^2_{\alpha,x} u(t,x) = 0, \tag{14}$$

with homogeneous condition

$$u(0,x) = 0. \tag{15}$$

Then by applying the Dunkl transform $\mathcal{F}_\alpha$ to the problem (14)–(15) one obtains

$$\mathcal{D}^\gamma_{0+,t}\widehat{u}(t,\lambda) + \lambda^2\widehat{u}(t,\lambda) = \widehat{f}(t,\lambda), \quad \widehat{u}(0,\lambda) = 0.$$

Then we have the trivial solution, i.e. $\widehat{u}(t,\lambda) \equiv 0$. Then by acting inverse Dunkl transform to this trivial solution we see that the solution u of the problem (14)–(15) is equal to zero. This means $u_1 \equiv u_2$. It contradicts to the our assumption, so the solution of the problem (1)–(2) is unique.

4 Conclusion

In this paper we showed existence and uniqueness of the solution to the (1)–(2) by using direct and inverse Dunkl transforms and using property of the multivariate Mittag-Leffler function. Further investigation problems will be application this technique for other type of equations such as Multi-term time-fractional derivative wave equation for Dunkl operator.

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