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B. Rysbaiuly , S.D. Alpar*
International information technology university, Kazakhstan, Almaty
*e-mail: rapla.natlus@gmail.com

SOLUTION OF A NONLINEAR HEAT TRANSFER PROBLEM BASED ON EXPERIMENTAL DATA

The paper develops a method for solving nonlinear equations of heat conduction. Two-layer container complexes have been created, the side faces of which are thermally insulated so that the 1D heat equation can be used. In order not to solve the boundary value problem with a contact discontinuity and lose the accuracy of the solution method, a temperature sensor was placed at the junction of two media, and a mixed boundary value problem is solved in each area (container). To provide initial data for the initial boundary value problem, three temperature sensors were used: two sensors measure air temperatures at the left and right boundaries of the container complex; the third sensor measures the temperature of the soil at the junction of two media. The paper numerically investigates the initial-boundary problem of heat conduction with nonlinear coefficients of heat conduction, heat capacity, heat transfer and density of the material. To solve a nonlinear initial-boundary value problem, the grid method is used. Two types of difference schemes are constructed: linearized and nonlinear. The linearized difference scheme is implemented numerically by the scalar sweep method, and the nonlinear difference problem is solved by the Newton method. On the basis of an a priori estimate of the solution of a nonlinear difference problem, we prove the convergence of the second degree of Newton's method. The numerical calculations carried out show that, for small time intervals, the solutions of the linearized difference problem differ little from the solution of the nonlinear difference problem (1 - 3%). And for long periods of time, tens of days or months, the solutions of the two methods differ significantly, sometimes exceeding 20%.
Key words: thermal conductivity, nonlinearity, difference problem, convergence, inverse problem, differentiation with respect to a parameter.

Б. Рысбайұлы, С.Д. Алпар*
Халықаралық ақпараттық технологиялар университеті, Қазақстан, Алматы қ.
*e-mail: rapla.natlus@gmail.com

Эксперименттік мәліметтер негізінде сызықты емес жылу алмасу есебін шешу

Жұмыста жылуөткізгіштіктің сызыққ емес теңдеулерін шешу әдісі әзірленген. 1D жылу теңдеуін қолдану үшін бүйір беттері жылу оқшауланған екі қабатты контейнерлік кешендер жасалды. Контакті үзілісімен шекаралық есептерді шешпеу және шешу әдісінің дәлдігін жоғалтпау үшін екі ортаның түйіскен жеріне температура датчигі қойылып, әр аймаққа (контейнер) аралас шекаралық есеп шығарылды. Бастапқы шекаралық есептің бастапқы деректерін қамтамасыз ету үшін үш температура датчигі пайдаланылды: екі датчиктер контейнер кешенінің сол және оң жақ шекараларында ауа температурасын өлшейді; үшінші сенсор екі ортаның түйіскен жеріндегі топырақ температурасын өлшейді. Жұмыста жылу өткізгіштіктің сызықты емес коэффициенттері, жылу сыйымдылығы, жылу беру және материалдың тығыздығы бар жылу өткізгіштіктің бастапқы-шекаралық есептері сандық түрде зерттеледі. Сызықты емес бастапқы-шекаралық есептерді шешу үшін тор әдісі қолданылады. Айырмашылық схемалардың екі түрі құрастырылады: сызықтық және сызықтық емес. Сызықтық айырым схемасы скалярлық сыпыру әдісімен сандық түрде жүзеге асырылады, ал сызықтық емес айырмашылық мәселесі Ньютон әдісімен шешіледі. Сызықты емес айырмашылық есебінің шешімін априорлық бағалау негізінде Ньютон әдісінің екінші дәрежелі жинақтылығын дәлелдейміз. Жүргізілген сандық есептеулер шағын уақыт аралықтары үшін сызықтық айырым есебінің шешімдерінің сызықтық емес айырмашылық есебінің шешімінен (1 - 3%) айырмашылығы аз екенін көрсетеді. Ал ұзақ уақыт бойы, ондаған күндер немесе айлар үшін екі әдістің шешімдері айтарлықтай ерекшеленеді, кейде 20% -дан асады.

Түйін сөздер: жылу өткізгіштік, сызықтық емес, айырықша есеп, жинақтылық, кері есеп, параметрге қатысты дифференциалдау.

Б. Рысбайұлы*, С.Д. Алпар
Международый университет информационных технологий, Казахстан, г. Алматы
*e-mail: rapla.natlus@gmail.com

Решение нелинейной задачи теплопередачи, основанной на экспериментальных данных

В работе разрабатывается метод решения нелинейной уравнений теплопроводности. Созданы двухслойные комплексы контейнеров, боковые грани которых теплоизолированные, чтобы можно было воспользоваться 1D уравнением теплопроводности. Чтобы не решать краевую задачу с контактным разрывом и терять точность метода решения, на стыке двух сред поставили датчик температуры, и в каждой области (контейнера) решается смешанная краевая задача. Чтобы обеспечить исходными данными начально граничную задачу, использовали три датчика температуры: два датчика измеряет температуры воздуха на левой и правой границе комплекса контейнеров; третьи датчик измеряет температуру грунта на стыке двух сред. В работе численно исследуется начально-краевая задача теплопроводности с нелинейными коэффициентами теплопроводности, теплоемкости, теплоотдачи и плотности материала. Чтобы решить нелинейную начально-краевую задачу используется метод сеток. Строятся два вида разностных схем: линеаризованная и нелинейная. Линеаризованная разностная схема реализуется численно методом скалярной прогонки, а нелинейная разностная задача решается методом Ньютона. На основе априорной оценки решения нелинейной разностной задачи доказывается сходимость второй степени метода Ньютона. Проведенные численные расчеты показывают, что при небольших промежутках времени решения линеаризованной разностной задачи мало отличаются от решения нелинейной разностной задачи (1 - 3%). А при больших промежутках времени, десятки дней или месяцы, решения двух методов значительно отличаются, порой переваливая за 20%.

Ключевые слова: теплопроводность, нелинейность, разностная задача, сходимость, обратная задача, дифференцирование по параметру.

1 Introduction

It is well known that the main source of information about thermophysical properties is the performance of a physical experiment [1,4]. The law of conservation of energy is used for the theoretical basis of the method for finding the thermophysical characteristics of the medium, the consequence of which is the nonlinear differential equation of heat conduction [1,5–7]. Where κ is the coefficient of thermal conductivity, ρ is the density, c is the specific heat, h is the heat transfer coefficient, all coefficients depend on the temperature of the material and determine the process of heat transfer in the medium (solid or liquid). Temperature is one of the main factors affecting the thermal conductivity of the soil. It has been experimentally established that the nature of the influence of temperature on the thermophysical parameters of the soil is non-linear [8–10]. In this regard, there is an urgent need to solve the inverse problem of the nonlinear heat equation.

Therefore, the main purpose of our study is the conduction of a thermophysical experiment and on the basis of the obtained data, numerical calculation of the nonlinear inverse problem of heat conduction to find the thermophysical coefficients [2,3,5,11–13]. At the beginning, the linearized direct problem of heat transfer is solved with respect to the input parameters (temperature values at the boundaries). Then, using the solution of the linearized scheme for the initial data in the Newton method, the nonlinear heat equation is solved. Differentiation of the nonlinear difference problem with respect to the parameter and experimental temperature values at the accessible soil-ground boundary make it possible to find thermophysical characteristics in inverse coefficient problems of heat transfer and heat fluxes in inverse boundary problems [14–18].

The article is structured as follows: the 2 section shows a demonstration of a mathematical model for describing the physical phenomenon of heat conduction. Additionally one can find there the discretization of the computational domain and the mathematical model. Section 3 presents a numerical method for solving the direct problem of a nonlinear differential heat equation, which is the Newton's method. And the main goal of this section is proof of the Newton's method convergence. The 4 section provides us scheme and an overview of the experimental setup. The 5 section illustrates the solution of a numerical method for solving the problem of a nonlinear differential heat equation by differentiating a nonlinear difference problem with respect to the required parameter.

2 Mathematical model

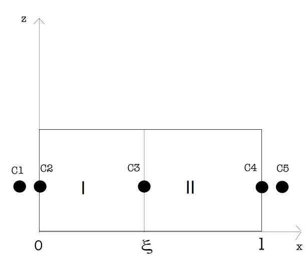

Figure 1: Two-chamber container

Problem formulation. Fig.1 schematically shows a two-chamber container, which has the thermally insulated side faces, and the end faces bound with the environment (air).

Taking into account these limitations, instead of using the three-dimensional heat equation, we consider the one-dimensional non-stationary heat equation (1).

$$c(u)\,\rho(u)\,\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(u)\,\frac{\partial u}{\partial x}\right), \; x \in (0,\xi) \times (\xi, l), \; t \in (0, 4t_{max}). \tag{1}$$

where $u(x,t)$ is the temperature distribution inside of the container chambers; x is the coordinate of the complex along the Ox axis; t is the current time. At the initial time of observation, the temperature distribution of the both chambers is: $t = 0$, $u(x,0) = u_0(x), x \in (0,l)$. The ambient temperature at the left boundary of the region at $x = 0$ will be denoted by $u_{ins}(t)$, and at the right boundary at $x = l$ we will denote by $u_{out}(t)$. In engineering calculations, the parameters c, ρ and κ are usually considered like constants [1]. However, many scientists [10,12,13,16,17] come to the conclusion that the study of nonlinear processes is of great practical interest. Since the vast majority of processes occurring in nature are non-linear. Taking into account the nonlinearity - greatly complicates the solution of the mathematical model.

The boundary conditions that determine the features of the process on the wall surface are given as follows: the left and right boundaries of the region $\Omega = (0,\xi) \times (\xi, l)$ are in contact with the gaseous medium (air), therefore, at these boundaries, it is advisable to formulate the Robin boundary condition - the relationship between the heat flux due to thermal conductivity from the solid wall and the heat flux from the gaseous medium. Thus, the boundary conditions on the left and right boundaries are written as follows:

$$x = 0: \; k_1(u)\frac{\partial u}{\partial x} = h_{ins}(u)\left(u - u_{ins}(t)\right), \tag{2}$$

$$x = l: \; k_2(u)\frac{\partial u}{\partial x} = -h_{out}(u)\left(u - u_{out}(t)\right), \tag{3}$$

here $u_{ins}(t)$, $u_{out}(t)$ is the ambient temperature; $h_{ins}(u)$, $h_{out}(u)$ are heat transfer coefficients; $k_1(u), k_2(u)$ are thermal conductivity coefficients of the «I» and «II» mediums (Fig.1).

Usually, boundary conditions are set on the contact surface of the layers $x = \xi$, which determine the equality of temperatures and heat fluxes at the junction of materials:

$$u_1(\xi, t) = u_2(\xi,\ t), \tag{4}$$

$$k_1(u)\frac{\partial u_1}{\partial x}(\xi, t) = k_2(u)\frac{\partial u_2}{\partial x}(\xi, t). \tag{5}$$

Here $u_1(x,t)$ and $u_2(x,t)$ are the temperatures of the material layers in contact. When solving problems with contact conditions of the form (4) - (5), the rate of convergence of the homogeneous difference scheme becomes very low [2]. Therefore, in order to avoid this problem, as well as to solve the inverse problem, we placed a separate sensor at the point $x = \xi$, which measures the change in soil temperature at the contact point of two media. Due to this, the original task is split into two tasks, i.e. using the measured data in each container, solves its own inverse problem of nonlinear thermal conductivity. Further, for brevity, we will describe the method for solving the inverse problem only on the left container (I), shown in Fig.1.

In addition to $u_{ins}(t)$, $u_{out}(t)$, the initial temperature values are measured - $T_{ins}(t), T_\xi(t), T_{out}(t)$, $t \in [0, 4t_{max}]$ where T_{ins}, T_ξ, T_{out} - respectively measured temperatures of materials at points $x = 0$, $x = \xi$ and $x = l$. For convenience of notation, we introduce the notation $h_{ins}(u) = h_1(u)$

Problem. Using the measured values $u_{ins}(t), T_{ins}(t), T_\xi(t), t \in [0, 4t_{max}]$, it is required to develop a method for finding the environment parameters $\rho_1(u)$, $c_1(u)$, $k_1(u)$, $h_1(u)$.

Since we consider only the left container, for simplicity we omit the indexes of the coefficients and use $k(u), c(u), \rho(u), h(u)$. We introduce the notation

Based on (1) - (5), the inverse problem is formulated as follows: in the region $Q_1 = (0,\xi) \times (0, 4t_{max})$ the following system is being studied

$$c(u)\rho(u)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right) \tag{6}$$

$$u(x, 0) = u_0(x) \tag{7}$$

$$k_1(u)\frac{\partial u}{\partial x} = h_1(u)\left(u - u_{ins}(t)\right), \quad x = 0 \tag{8}$$

$$u(\xi, t) = T_\xi(t) \tag{9}$$

where the following dependences on temperature are used for thermophysical characteristics: $c\left(u\right)=c_{0}+c_{1}u,\ \rho\left(u\right)=\rho_{0}+\rho_{1}u,\ k\left(u\right)=\ k_{0}+k_{1}u+k_{2}u^{2}+k_{3}u^{3},\ h\left(u\right)=\ h_{0}+h_{1}u+h_{2}u^{2}.$

And the experimentally measured value on the left boundary of the region

$$T_{ins}\left(t\right),t\in\left[0,4t_{max}\right]. \tag{10}$$

It is required to develop a method for finding thermophysical parameters $\rho\left(u\right),\ c\left(u\right),\ k\left(u\right),\ h\left(u\right)$.

Grid method. The segment $(0,\xi)$ is divided into I equals with a step $\Delta x=\xi/I$. Then $\xi=I\Delta x$, where I is the node number of the contact point $x=\xi$. And the segment $(0,t_{max})$ is divided into m equal parts with the step $\Delta t=\frac{t_{max}}{m}$.

As a result of this action, we get a grid

$\omega_1=\{x_i=i\Delta x,\ t_j=j\Delta t,\ i=0,1,...,I;\ j=0,1,...,4m\}$

Further, the problem (6) - (10) is solved by splitting in the grid domain

$\omega_{1s}=\{x_i=i\Delta x,\ t_j=j\Delta t,\ i=0,1,...,I,\ j=(s-1)m,...,s\cdot m;\ s=1,\ 2,\ 3,\ 4\}.$

The difference scheme L_{11} is studied in the grid domain:

$$\begin{gathered}
\rho\left(u_i^{j+1}\right)c\left(u_i^{j+1}\right)\frac{u_i^{j+1}-u_i^j}{\Delta t}=\\
=\frac{1}{\Delta x}\left(k\left(u_{i+1/2}^{j+1}\right)\frac{u_{i+1}^{j+1}-u_i^{j+1}}{\Delta x}-k\left(u_{i-1/2}^{j+1}\right)\frac{u_i^{j+1}-u_{i-1}^{j+1}}{\Delta x}\right),\\
i=1,2,...,I-1;\ j=0,1,...,m-1,\\
u_i^0=u_0\left(x_i\right),\ i=0,1,...,I,\\
u_I^{j+1}=T_\xi\left(t_{j+1}\right),\ j=0,1,...,m-1,\\
k\left(u_{1/2}^{j+1}\right)\frac{u_1^{j+1}-u_0^{j+1}}{\Delta x}=h\left(u_0^{j+1}\right)\left(u_0^{j+1}-u_{ins}^{j+1}\right),\ j=0,1,...,m-1,
\end{gathered} \tag{11}$$

where $u_{i+\frac{1}{2}}=\frac{u_{i+1}^{j+1}+u_i^{j+1}}{2},\ i=0,1,...,I-1.$

3 An iterative method for solving a nonlinear difference problem

For simplicity of presentation, consider the nonlinear difference problem (11), when $c=const$, $h=const$. Let's rewrite (11) as

$$\begin{gathered}
C\cdot\frac{v_i^{j+1}-v_i^j}{\Delta x}=\frac{1}{\Delta x}\left[k\left(\frac{v_{i+1}^{j+1}+v_i^{j+1}}{2}+T_\xi^{j+1}\right)\frac{v_{i+1}^{j+1}-v_i^{j+1}}{\Delta x}-\right.\\
\left.-k\left(\frac{v_i^{j+1}+v_{i-1}^{j+1}}{2}+T_\xi^{j+1}\right)\frac{v_i^{j+1}-v_{i-1}^{j+1}}{\Delta x}\right]-C\cdot\frac{T_\xi^{j+1}-T_\xi^j}{\Delta t},\\
i=1,2,\ldots,N-1,\ j=0,1,\ldots,\ m-1,
\end{gathered} \tag{12}$$

$$v_i^0=N_i^0,\quad i=0,1,\ldots,\ I,\quad N_i^0=u_0(x_i)-T_\xi^0,$$

$$v_I^{j+1}=0,\quad j=0,1,\ldots,\ m-1,$$

$$k\left(\frac{v_1^{j+1}+v_0^{j+1}}{2}+T_\xi^{j+1}\right)\cdot\frac{v_1^{j+1}-v_0^{j+1}}{\Delta x}=h\cdot v_0^{j+1}+h\left(T_\xi^{j+1}-u_{ins}^{j+1}\right),$$

$$j=0,1,\ldots,\ m-1.$$

Замечание 1 *The solution to the problem* (11) *and* (12) *are related by the equality*

$$U_i^{j+1}=V_i^{j+1}+T_\xi^{j+1}, i=0,1,\ldots,I;\quad j=0,1,\ldots,m-1.$$

The following statements are proved in the work:

Lemma 1 *If* $u_0(x)\in l_2(\Omega=(0,\xi)), T_\xi(t), u_{ins}(t)\in l_2[0,t_{max}]$, *then the solution of the problem* (12) *satisfies the estimate*

$$C\|V^{j+1}\|^2+2k\sum_{j=0}^{J}\|V_{\bar{x}}^{j+1}\|^2\Delta t+h\sum_{j=0}^{J}(V_0^{j+1})^2\Delta t\le C_3,$$

where C_3 is the limited value.

Lemma 2 *If the original functions have the property* $u_0(x)\in l_\infty(0,\xi), T_\xi(t)-u_{ins}(t)\in l_\infty(0,t_{max}), T_{\xi,\bar{t}}^{j+1}\in l_\infty(0,t_{max})$, *then the solution to the difference scheme* (12) *satisfies the estimate*

$$\max_{0\le i\le i-1}|V_i^{j+1}|\le\max_j|N_i^0|+\max_j|T_\xi^{j+1}-u_{ins}^{j+1}|+\max_j T_{\xi,\bar{t}}^{j+1}|=C_4,$$

where C_4 *is the limited value.*

Lemma 3 *Let* $|\sum_{i=1}^{I-1}V_{i,\bar{t}}^{j+1}\Delta t|$ *be limited value, then*

$$\max_i|V_{i,\bar{x}}^{j+1}|\le C_5<\infty,$$

where C_5 *is limited value, for every* j *from 0 to* $m-1$.

Lemma 4 *If there is a condition*

$$u_0(x)\in l_\infty(0,\xi), T'_\xi(t)-u'_{ins}(t)\in l_2(0,t_{max}), T_{\xi,\bar{t}t}^{j+1}\in l_2(0,t_{max})$$

and inequality

$$C\left|\sum_{i=1}^{I}V_{i,t}^{j}\right|\Delta t<\infty,\quad i=1,2,\ldots,I; j=0,1,\ldots,m-1,$$

then the solution to the difference scheme (12) *satisfies the estimate*

$$\|V_t^j\|^2+\sum_{j=0}^{J}\|V_{\bar{x},t}^j\|^2\Delta t+\sum_{j=0}^{J}\left(V_{(0,t)}^j\right)^2\Delta t\le C_6<\infty.$$

where C_6 *is the limited value.*

The system of nonlinear algebraic equations (12) is solved by an iterative method for each

$$j = 0, 1, \ldots, m-1.$$

We fix the variable j and look for the desired value v_i^{j+1} in the form v_i^s, where s is the number of iterations. The initial approximation of the problem (12) when $s = 0$ is determined from the solution of the linearized problem for a fixed j :

$$\begin{aligned}
&C \cdot \frac{v_i^0 - v_i^j}{\Delta t} = \frac{1}{\Delta x}\left[k\left(\frac{v_{i+1}^j + v_i^{j+1}}{2} + T_\xi^{j+1}\right)\frac{v_{i+1}^0 - v_i^0}{\Delta x} - \right.\\
&\left. - k\left(\frac{v_i^j + v_{i-1}^j}{2} + T_\xi^{j+1}\right)\frac{v_i^0 - v_{i-1}^0}{\Delta x}\right] - C \cdot \frac{T_x i^{j+1} - T_\xi^j}{\Delta t},\\
&k\left(\frac{v_1^j + v_0^j}{2} + T_\xi^{j+1}\right) \cdot \frac{v_1^1 - v_0^0}{\Delta x} = h \cdot v_0^0 + h\left(T_\xi^{j+1} - u_{ins}^{j+1}\right),\\
&i = 1, 2, \ldots, N-1, \quad j = 0, 1, \ldots, m-1,\\
&v_i^0 = N_i^0, \quad i = 0, 1, \ldots, N,\\
&v_N^0 = 0.
\end{aligned} \tag{13}$$

Before proceeding to solve the (12) problem by the iterative method, let's estimate the error of the initial approximation described by the (13) system. To do this, subtract the system (13) from (12). We introduce the notation

$$\Delta v_i^0 = v_i^{j+1} - v_i^0, \qquad i = 0, 1, \ldots, I.$$

Then

$$\begin{aligned}
&C \cdot \frac{\Delta v_i^0}{\Delta t} = \left[k'\left(\theta v_{i+\frac{1}{2}}^{j+1} + (1-\theta)\, v_{i+\frac{1}{2}}^j + T_\xi^{j+1}\right)\frac{1}{2}\left(v_{i+1}^{j+1} + v_i^{j+1}\right)_{\bar{t}} \cdot \right.\\
&\left. \cdot \Delta t \cdot v_{i,x}^{j+1} - k\left(\frac{v_{i+1}^j + v_i^j}{2} + T_\xi^{j+1}\right) \cdot \Delta v_{i,x}^0\right]_{\bar{x}},\\
&i = 1, 2, \ldots, N-1, \quad j = 0, 1, \ldots, m-1,
\end{aligned} \tag{14}$$

$$\Delta v_i^0 = 0, \quad i = 0, 1, \ldots, N,$$
$$\Delta v_N^0 = 0, \quad j = 0, 1, \ldots, m-1,$$

$$\begin{aligned}
&k'\left(\theta v_{\frac{1}{2}}^{j+1} + (1-\theta)\, v_{\frac{1}{2}}^j + T_\xi^{j+1}\right) \cdot \frac{1}{2}\left(v_1^{j+1} + v_0^{j+1}\right)_{\bar{t}} \cdot v_{1,x}^{j+1} + \\
&+ k\left(\frac{v_1^j + v_0^j}{2} + T_\xi^{j+1}\right) \cdot \Delta v_{i,x}^0 = h_1 \Delta v_0^0,\\
&j = 0, 1, \ldots,\ m-1.
\end{aligned} \tag{15}$$

Замечание 2 *It can be proved that the previous lemmas hold for* (14).

To obtain an estimate for the solution of the (14) problem, we multiply the first equation of the (14) system by $\Delta v_i^0 \Delta t\ \Delta x$ and sum over i from 1 to $I-1$. We apply the summation by parts formula over the variable i, and taking into account the boundary conditions of the problem (14) we have the equality

$$C\|\Delta v^0\| + h|\Delta v_0^0|\Delta t + \sum_{i=1}^{I} k\left(\frac{v_{i+1}^j + v_i^j}{2} + T_\xi^{j+1}\right)\cdot\left(\Delta v_{i,\bar{x}}^0\right)^2 \Delta t\Delta x =$$

$$= -\Delta t\sum_{i=1}^{I} k'\left(\theta v_{i+\frac{1}{2}}^{j+1} + (1-\theta)\, v_{i+\frac{1}{2}}^{j} + T_\xi^{j+1}\right)\cdot v_{i+\frac{1}{2},\bar{t}}\cdot v_{i,\bar{x}}^{j+1}\cdot \Delta v_{i,\bar{x}}^0 \Delta x\Delta t.$$

The third lemma says that

$$\max_i |v_{i,\bar{x}}^{j+1}| < C_7, \quad j = 0, 1, \ldots,\ m-1.$$

Taking into account this information, applying the Cauchy inequality, we have the inequality

$$C\|\Delta v^0\|^2 + h|\Delta v_0^0| + k\|\Delta v_{\bar{x}}^0\|^2\Delta t \le C_7\Delta t\cdot\|v_{i+\frac{1}{2},\bar{t}}^{j+1}\|\cdot\|\Delta v_{\bar{x}}^0\|.$$

But $v_{i-\frac{1}{2},\bar{t}} = \frac{1}{2}\left(v_{i-1,\bar{t}}^{j+1} + v_{i,\bar{t}}^{j+1}\right)$, therefore

$$\sum_{i=1}^{N}\left(v_{i-\frac{1}{2},\bar{t}}\right)^2\Delta x = \sum_{i=1}^{N}\left(v_{i,\bar{t}}\right)^2\Delta x = \|v_{\bar{t}}^{j+1}\|^2$$

.

Applying ε - Cauchy's inequality, we deduce the estimate

$$C_1\|\Delta v^0\|^2 + h_1|\Delta v_0^0|\Delta t + \left(k - \frac{\varepsilon}{2}\right)\|\Delta v_{\bar{x}}^0\|^2\Delta t \le \frac{C_8}{2\varepsilon}\|v_{\bar{t}}^{j+1}\|^2\Delta t.$$

Let $\varepsilon = \frac{k}{C_2}$, then the inequality

$$C\|\Delta v^0\|^2 + h|\Delta v_0^0| + \frac{k}{2}\|\Delta v_{\bar{x}}^0\|^2\Delta t \le \frac{C_8^2}{2k}\|v_{\bar{t}}^{j+1}\|^2\left(\Delta t\right)^2.$$

In the fourth lemma, we proved that $\|v_{\bar{t}}^{j+1}\|^2 < \infty$, so we finally get the estimate

$$q_0 = C\|v_i^{j+1} - v_i^0\|^2 + h_1|v_0^{j+1} - v_0^0|\Delta t + \frac{k}{2}\|v_{\bar{x}}^{j+1} - v_{\bar{x}}^0\|^2\Delta t \le C_9\left(\Delta t\right)^2. \tag{16}$$

Theorem 1 *If the initial approximation of Newton's method is taken from the solution of the linearized problem* (13)*, then the solution of the system of nonlinear algebraic equations in the iterative Newton scheme* (12) *converges, and the convergence rate estimate takes place*

$$q^{s+1} \le q_0^{2^{s+1}}, \quad s = 0, 1, \ldots,$$

$$q^s = C\frac{\|\Delta v^s\|^2}{\sqrt{\Delta t}} + k\|\Delta v_{\bar{x}}^S\|^2\sqrt{\Delta t} + h\left(\Delta v_0^S\right)^2\sqrt{\Delta t}, \quad \Delta v_i^s = v_i^{j+1} - v_i^s.$$

Figure 2: Soil containers

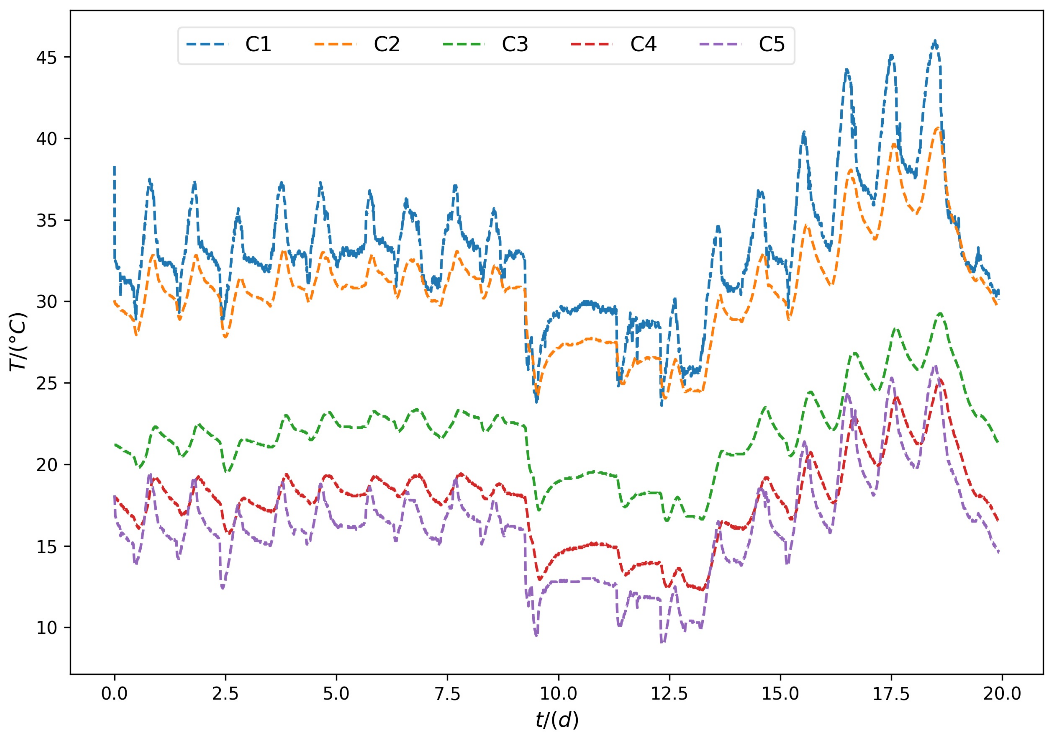

Figure 3: Measured temperature at five points of the container.

4 Experimental setup

During the experiment, the data of the problem of one-dimensional heat and mass transfer for various soils were obtained.

Photos of containers are shown in Fig.4. The side edges of the chambers consist of 2 cm thermally insulated material, and the end edges are in bound with the environment (air). In each compartment of the container, 15 cm long, there are various soils. One end side is heated with lamps. The second outer side is affected by the ambient temperature.

3 sensors (C2, C3, C4) are evenly distributed inside the material as shown in Fig.1. They produce temperature measurements with an error of 0.3 degrees Celsius according to the technical passport of the sensor. In addition to these sensors, there are 2 more sensors (C1, C5) close to the ends to measure the ambient temperature. The errors of these sensors are the same as those of the previous sensors. The temperature data measurement is taken at intervals of 10 minutes.

For calculations, a two-chamber container was considered and, accordingly, with two materials: sand and black soil. The data were measured over the course of three months, and the physical length of the entire container was determined through the interval $x \in (0, l)$, where $l = 30$ cm. The boundary of the two media is at a distance of $x = 15$ cm and there is also a temperature measurement sensor. Since there is an exchange with the environment at the end boundaries, Robin boundary conditions were considered for the numerical solution. Measurements at the points $x = 0$ cm and $x = 30$ cm determine the temperature at the end boundaries. The temperature values of the measured data can be seen in Fig.3.

5 Results

The measured temperature data were used to solve a numerical problem to find all thermophysical coefficients (thermal conductivity coefficient, specific heat capacity, specific density and heat transfer coefficient). As can be seen from Fig.4 and 5, thanks to the steepest descent method, the functionals converge fairly quickly and reach a minimum in 6 and 7 iterations. The minimization of the functional continued until the relative error between the nonlinear solution and the experimental data reached 4.3% for chernozem and 3.12% for sand, which in turn shows a fairly good accuracy of the solution. If we look at the absolute errors in two environments - 6.3% and 5.3%, we see that they also meet our expectations. Due to the fact that it is difficult for the reader to evaluate each parameter separately, in Fig.5, Fig.6 and Fig.7 functions of thermophysical coefficients are given depending on temperature for each medium. For clarity, the interval from $10°C$ to $50°C$ was used. The graphs show that with increasing temperature, the values of thermophysical parameters increase. In this case, it is possible to evaluate the behavior of the thermal conductivity and heat capacity coefficients. It can be seen that they show a stronger dependence on temperature values than the density coefficient, which in turn is confirmed by the theoretical basis. It is also worth noting that the data obtained are in good agreement with the tabular, experimental values [6]. And in the case of the heat transfer coefficient, it can be seen that it practically does not change with increasing temperature. This is explained by the fact that the heat transfer is affected by the difference between the ambient temperature and the temperature at the end boundaries of the soil-soil. In our case, according to the experimental data, it can be seen that the difference is not large.

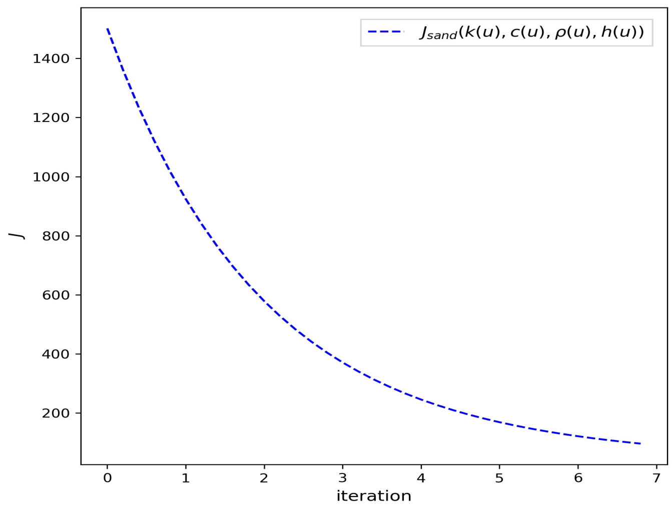

Figure 4: Convergence graph of functionals for two media: sand

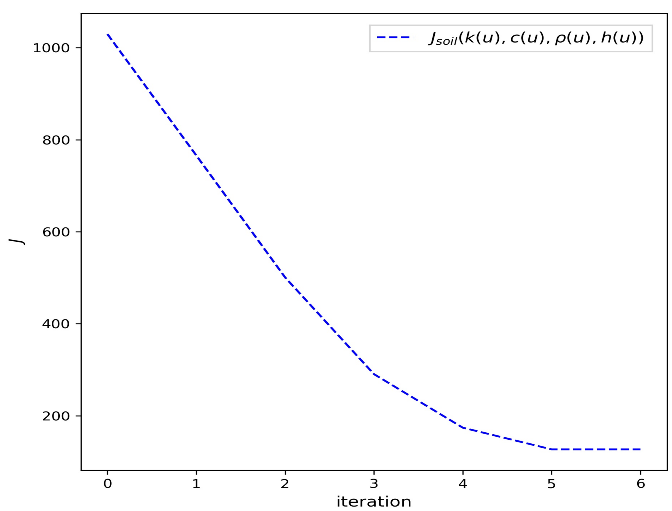

Figure 5: Convergence graph of functionals for two media: soil

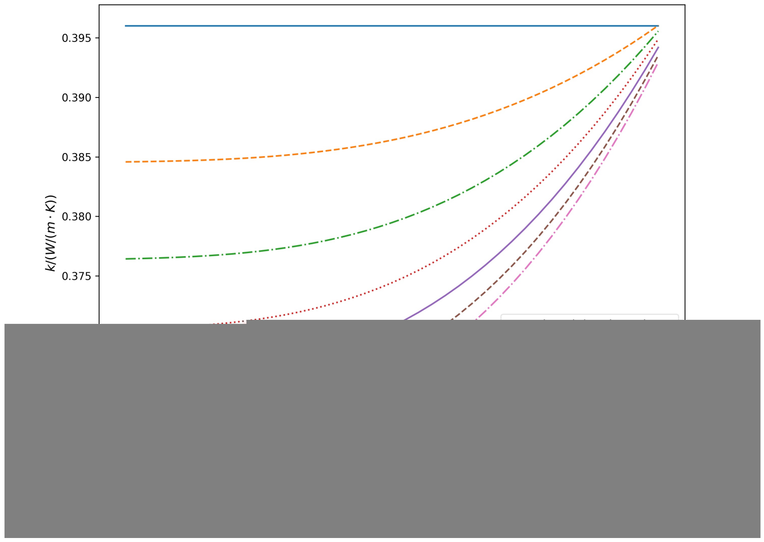

Figure 6: Changing the heat conduction function at each iteration of the calculation from the initial data. Sand

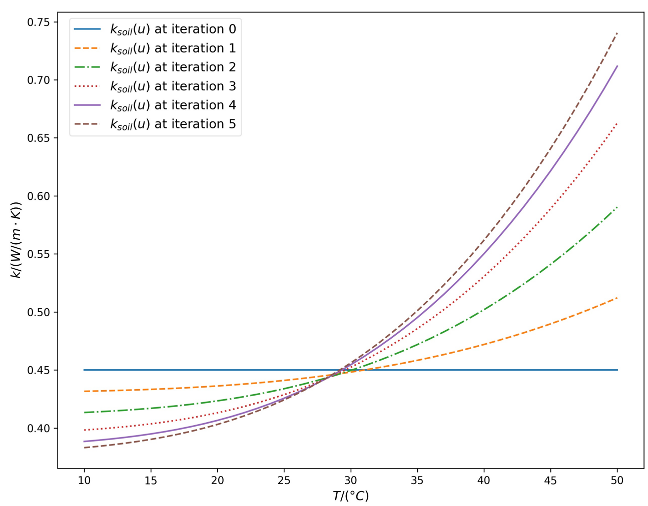

Figure 7: Changing the heat conduction function at each iteration of the calculation from the initial data. The soil

6 Conclusion

Thus, the constructed numerical method allows obtaining an approximate solution to the problem of fluid flow in a fractured porous medium with the second order in both time and spatial variable. The results of computational experiments carried out for various orders of fractional derivatives and grid configurations fully confirm the results of theoretical analysis. The methods used and the conclusions drawn, described in the work, can be used to solve other classes of fractional differential equations.

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References

[1] Desta T. Z., Langmans J., Roels S., "Experimental data set for validation of heat, air and moisture transport models of building envelopes " , *Building and Environment,* (2011).

[2] Rysbaiuly B., Adamov A.A., "Matematicheskoe modelirovanie tepla i massoobmennogo processa v mnogosloynom grunte " [Monographiya] *Izdatelskyi dom "Kazak Universitety* (2020): 211.

[3] Luikov A.V., "Heat and Mass Transfer in Capillary-Porous Bodies" , (1964).

[4] Berger J., Dutykh D., Mendes N., Rysbaiuly B., "SA new model for simulating heat, air and moisture transport in porous building material" , *International Journal of Heat and Mass Transfer,* 134 (2019): 1041-1060.

[5] Tien-Mo Shih, Chao-Ho Sung, Bao Yang, "A Numerical Method for Solving Nonlinear Heat Transfer Equations" , *Numerical Heat Transfer, Part B: Fundamentals: An International Journal of Computation and Methodology,* (2008).

[6] Mazumder S., "Numerical Methods for Partial Differential Equations: Finite Difference and Finite Volume Methods" , *Academic Press* (2016).

[7] Lopushansky A., Lopushansky O., Sharyn S., "Nonlinear inverse problem of control diffusivity parameter determination for a space-time fractional diffusion equation" , *Applied Mathematics and Computation* (2021).

[8] Moore T., Jones M., "Solving nonlinear heat transfer problems using variation of parameters" , *International Journal of Thermal Sciences* (2015).

[9] Battaglia J-L., Maachou A., Malti R., Melchior P., "Nonlinear heat diffusion simulation using Volterra series expansion" , *International Journal of Thermal Sciences* (2013).

[10] Nguyen Huy Tuana, Pham Hoang Quanc, "Some extended results on a nonlinear ill-posed heat equation and remarks on a general case of nonlinear terms" , *Nonlinear Analysis: Real World Applications* (2011).

[11] Huntul M., Lesnic D., "Determination of the time-dependent convection coefficient in two-dimensional free boundary problems" , *Engineering Computations* (2020).

[12] Jumabekova A., Berger J., Dutykh D., Le Meur H., "An efficient numerical model for liquid water uptake in porous material and its parameter estimation" , *Numerical Heat Transfer, Part A: Applications* (2019).

[13] Hasanov A., "Lipschitz continuity of the Fr?chet gradient in an inverse coefficient problem for a parabolic equation with Dirichlet measured output" , *Journal of Inverse and Ill-Posed Problems* (2018).

[14] Cao K., Lesnic D., Ismailov M., "Determination of the time-dependent thermal grooving coefficient" , *Journal of Inverse and Ill-Posed Problems* (2021).

[15] Lesnic D., Hussein M.S., Kamynin V., Kostina B., "Direct and inverse source problems for degenerate parabolic equations" , *Journal of Inverse and Ill-Posed Problems* (2020).

[16] Kabanikhin S. I., Shishlenin M.A., "Theory and numerical methods for solving inverse and ill-posed problems" , *Journal of Inverse and Ill-Posed Problems* (2019).

[17] Lesnic D., "Inverse Problems with Applications in Science and Engineering" , *JRC Press* (Abingdon, UK, 2021): 349.

[18] Rysbaiuly B., Rysbaeva N., "The method of solving nonlinear heat transfer model in freezing soil" , *Eurasian Journal of Mathematical and Computer Applications (EJMCA).* V.8, No 4 (2020): 83-96.

[19] Rysbaiuly B., "Obratnye zadachi nelineynoy teploperedachi" , *Kazakh Universitety* (2020): 369.