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ON THE SCHWARZ PROBLEM FOR THE MOISIL–TEODORESCU SYSTEM IN A SPHERICAL LAYER AND IN THE INTERIOR OF A TORUS

Key words: Cauchy–Riemann system, Moisil–Teodorescu system, Schwartz problem, spherical layer, torus interior, solvability of the problem.

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Сфералық қабаттағы және тордың ішкі бөлігіндегі Моисил–Теодореско жүйесі үшін Шварц есебі туралы

Аналитикалық функциялар теориясы жазықтықтағы эллиптикалық теңдеулер мен аралас типтес теңдеулерді зерттеуде классикалық бағыт болып табылады. Үшөлшемді шенелген $\Omega \subseteq \mathbb{R}^3$ обылысында келесі эллиптикалық жүйесін қарастырамыз

$$M(\partial/\partial x)u(x) \equiv \begin{pmatrix} 0 & \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ \partial/\partial x_1 & 0 & -\partial/\partial x_3 & \partial/\partial x_2 \\ \partial/\partial x_2 & \partial/\partial x_3 & 0 & -\partial/\partial x_1 \\ \partial/\partial x_3 & -\partial/\partial x_2 & \partial/\partial x_1 & 0 \end{pmatrix} u(x) = 0,$$

мұндағы $u(x) = (u_0, u_1, u_2, u_3)$ вектор- функциясы $C^1(\Omega)$ класынан. Мұндай жүйе Моисил-Теодореско жүйесі деп аталады. Бұл жүйенің шешімдері үшін жазықтықтағы аналитикалық функциялар теориясының негізгі фактілері, соның ішінде Кошидің интегралдық теоремасы мен формуласы, Морер теоремасы және басқалар. Екі байланысты облыстар сұйықтықтар механикасында маңызды рөл атқарады. Мысалы, өз осімен нормаль бағытта қозғалатын ұзын тұтас цилиндрден жасалған ағын дәл екі байланысты облыста жүзеге асады. Бұл жұмыста $M(\partial/\partial x)$ дифференциалдық операторы үшін $\mathbb{R}^3$ кеңістігінде іргелі шешімі жазылған және сфералық қабаттағы және тордың ішкі бөлігіндегі Моисил–Теодореско жүйесі үшін тиянақты есептер келтірілген. Моисил–Теодореско жүйесі эллиптикалық Коши–Риман жүйесінің жалпыланған мысалы болып табылады. Бұл жұмыстың нәтижелерінен сфералық қабаттағы тиянақты қойылған есеп пен тордың ішкі бөлігінде қойылған есептің арасында айтарлықтай айырмашылықты көреміз.

Түйін сөздер: Коши–Риман жүйесі, Моисил–Теодореско жүйесі, Шварц есебі, сфералық қабат, тордың ішкі бөлігі, есептің шешімділігі.

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О задаче Шварца для системы Моисила–Теодореску в шаровом слое и во внутренности тора

Теория аналитических функции является классическим направлением в изучении эллиптических уравнений и уравнений смешанного типа на плоскости. В трехмерной ограниченной области $\Omega \subseteq \mathbb{R}^3$ рассматривается эллиптическая система

$$M(\partial/\partial x)u(x) \equiv \begin{pmatrix} 0 & \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ \partial/\partial x_1 & 0 & -\partial/\partial x_3 & \partial/\partial x_2 \\ \partial/\partial x_2 & \partial/\partial x_3 & 0 & -\partial/\partial x_1 \\ \partial/\partial x_3 & -\partial/\partial x_2 & \partial/\partial x_1 & 0 \end{pmatrix} u(x) = 0,$$

где $u(x) = (u_0, u_1, u_2, u_3)$ – искомая вектор- функция $u \in C^1(\Omega)$. Такая система называется системой Моисила – Теодореску. Для решений этой системы справедливы основные факты теории аналитических функций на плоскости, включая интегральную теорему и формулу Коши, теорему Морера и другие. Двусвязные области играют значительную роль в механике жидкости. К примеру течение, создаваемое длинным твердым цилиндром, движущегося в направлении нормали к своей оси, происходит именно в двусвязной области. В данной работе выписан фундаментальное решение дифференциального оператора $M(\partial/\partial x)$ в пространстве $\mathbb{R}^3$ и приведены корректные задачи для системы Моисила–Теодореску в случае шарового слоя и внутренности тора. Из результатов данной работы видно существенное отличие корректной задачи в шаровом слое от аналогичной задачи в торе.

Ключевые слова: система Коши–Римана, система Моисила–Теодореску, задача Шварца, шаровой слой, внутренность тора, разрешимость задачи.

1 Introduction

Complex analysis methods constitute a classical direction in the study of elliptic equations and equations of mixed type on the plane. At present, active research is being carried out in this direction in many mathematical centers of the world.

Multiply connected (in particular, doubly-connected) domains play an important role in fluid mechanics. For example [1] the flow created by a long solid cylinder moving in the direction of the normal to its axis, occurs precisely in a two-connected domain. From the fact that certain closed curves in such a domain are non-contractible to a point, it follows that the presence of lifting power. Another example [1] is the motion of a smoke ring in the outside of the torus. Thus, it makes sense to study the well-posed formulation of the problems for elliptic systems in multiply connected domains. Plane multiply connected domains are usually described by the number of connected components of the boundary of the domain. Spatially multiply connected domains already require a large number of topological characteristics. For spatial multiply connected domains along with the number of connected components of the boundary of the domain, it is convenient to consider also so-called the order of connectedness of the domain [2, 3].

In this paper, we denote the number of connected components of the boundary of the domain by n, and the order of connectedness of the domain is denoted by m. For example, for a spherical layer in a three-dimensional space $n = 2, m = 1$, and for the interior of a torus in the same space $n = 1, m = 2$.

It is noted that the formulation of well-posed problems for first order elliptic systems depend on the numbers n, m in [4–6].

This paper presents the well-posed problems for the Moisil–Theodorescu system in the case of the spherical layer and the interior of the torus. The results of this work show a significant difference between well-posed problem in the spherical layer and the similar problem in the torus. A more general investigation of the Fredholm property of boundary value problems of first order elliptic systems in multiply connected domains can be found in the papers of A.P. Soldatov [7–9]. Moreover the index of the studied problems is calculated in [7].

Materials and methods

2 Cauchy–Riemann and Moisil–Teodorescu systems

In a flat bounded domain $\Omega \subseteq \mathbb{R}^2$, we consider the elliptic system

$$M(\partial/\partial x)u(x) \equiv \begin{pmatrix} \partial/\partial x_1 & -\partial/\partial x_2 \\ \partial/\partial x_2 & -\partial/\partial x_1 \end{pmatrix} u(x) = 0,$$

where $u(x) = (u_1, u_2)$ is the desired vector function $u \in C^1(\Omega)$. Such a system is called a Cauchy-Riemann system.

In a three-dimensional bounded domain $\Omega \subseteq \mathbb{R}^3$, we consider the elliptic system

$$M(\partial/\partial x)u(x) \equiv \begin{pmatrix} 0 & \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ \partial/\partial x_1 & 0 & -\partial/\partial x_3 & \partial/\partial x_2 \\ \partial/\partial x_2 & \partial/\partial x_3 & 0 & -\partial/\partial x_1 \\ \partial/\partial x_3 & -\partial/\partial x_2 & \partial/\partial x_1 & 0 \end{pmatrix} u(x) = 0, \quad (1)$$

where $u(x) = (u_0, u_1, u_2, u_3)$ is the desired vector function $u \in C^1(\Omega)$. Such a system is called the Moisil-Teodorescu system.

For the solutions of this system, the basic facts of the theory of analytic functions on the plane are valid, including the integral theorem and the Cauchy formula, the Morera theorem, etc. The foundations of this theory were laid in the works of G.K. Moisil and N. Teodorescu [10]. It is easy to show that all components u_j of the solution $u = (u_0, u_1, u_2, u_3)$ of system (1) are harmonic functions. In this sense, it is an example of a multidimensional generalized Cauchy-Riemann system [11].

This theory was further developed in the works of A.V. Bitsadze [12, 13]. In particular, he introduced the concept of a Cauchy-type integral for system (1) and pointed out its various applications.

The fundamental solution of the differential operator $M(\partial/\partial x)$ in the space $\mathbb{R}^3$ is the matrix function $M^\top(x)/|x|^3$, where $\top$ – matrix transposition symbol. In these notations, the integral

$$(I\varphi)(x) = \frac{1}{2\pi}\int_\Gamma \frac{M^\top(y-x)}{|y-x|^3} M[n(y)]\varphi(y)d_2y, \quad x \notin \Gamma, \tag{2}$$

where d_2y is the area element on the surface $\Gamma = \partial D$ and $n(y)$ is the unit normal, determines the solution of system (1). The choice of density in the form $M[n(y)]\varphi(y)$ is dictated by the fact that it ensures the validity of the analogue of the Sokhotsky-Plemelja formulas.

Namely, if the function φ satisfies the Holder condition and the surface Γ is a Lyapunov surface, then there exist limit values

$$u^\pm(y_0) = \lim_{x\to y_0, x\in D^\pm} u(x), \quad y_0 \in \Gamma,$$

for which the analogue of the Sokhotsky-Plemelya formulas is valid

$$u^\pm = \pm\varphi + u^*. \tag{3}$$

Here $D^+ = D,\ D^- = \mathbb{R}^3 \setminus \overline{D}$, the normal n is assumed to be external to D and the function $u^* = I^*\varphi$ is defined by the singular integral

$$(I^*\varphi)(y_0) = \frac{1}{2\pi}\int_\Gamma \frac{M^\top(y-y_0)}{|y-y_0|^3} M[n(y)]\varphi(y)d_2y, \tag{4}$$

which is understood as the limit at $\varepsilon \to 0$ of integrals over $\Gamma \cap \{|y-y_0| \geq \varepsilon\}$. These formulas were first obtained by A.V. Bitsadze [12]. From the point of view of the minimum requirements for surface smoothness, this result was refined in [14]: if Γ belongs to the class $C^{1,\nu},\ 0<\nu<1$, then the operator I is bounded $C^\mu(\Gamma) \to C^\mu(\overline{D}),\ 0<\mu<\nu$. Here and below, by $C^\mu(G)$ we mean the Banach Holder space defined by the usual norm

$$|\varphi|_{\mu,G} = |\varphi|_{0,G} + [\varphi]_{\mu,G} \quad [\varphi]_{\mu,G} = \sup_{x\neq y, x,y\in G} \frac{|\varphi(x)-\varphi(y)|}{|x-y|^\mu},$$

where $|\varphi|_{0,G}$ means sup –norm. Similar meaning has the space $C^{1,\mu}(\overline{D})$ continuously - differentiable functions and the class of $C^{1,\mu}$ surfaces.

In terms of the integral (2), the Cauchy integral formula for solutions $u \in C^\mu(\overline{D})$ of system (1) in a finite domain D can be written as

$$u(x) = (Iu^+)(x) \quad x \in D. \tag{5}$$

In this case, the Cauchy theorem gives the equality

$$(Iu^+)(x) = 0, \quad x \in D^-. \tag{6}$$

If the domain D is infinite, then under the additional assumption

$$u(x) = o(|x|^{-1}) \tag{7}$$

for $|x| \to \infty$ these formulas remain valid.

Let the function $u(x)$ be given and be a solution to (1) in each component of the complement to Γ, satisfies the Holder condition in their closure and condition (7) at infinity. Then from the formulas (5) and (6) applied in these components are completely similarly to the case of analytic functions, we derive the representation

$$u = I(u^+ - u^-) \tag{8}$$

general solution in the form of a Cauchy-type integral. Taking into account the Sokhotsky-Plemelya formulas (3), this representation allows the problem of linear conjugation

$$u^+ - Gu^- = f$$

with a given invertible $(4 \times 4)-$ matrix $G \in C^\mu(\Gamma)$ reduce to an equivalent two-dimensional singular integral equation

$$(\varphi + I^*\varphi) + G(\varphi - I^*\varphi) = f.$$

Results and discussion

3 The Schwartz problem for the Moisil–Theodorescu system in the spherical layer

Let $\Omega = \{x \in \mathbb{R}^3 : 0 < r_1 < |x| < r_2\}$, where r_1, r_2 are some positive numbers. We denote by Γ the boundary of the domain Ω, i.e. $\Gamma = \{x \in \mathbb{R}^3 : |x| = r_1\} \cup \{x \in \mathbb{R}^3 : |x| = r_2\}$. It is required to find the vector-function $u = (u_0, u_1, u_2, u_3) = (u_0, \widetilde{u})$ that satisfies the Moisil–Theodorescu system

$$\begin{cases} div\, \tilde{u} = 0, \quad x \in \Omega, \\ grad\, u_0 + rot\, \tilde{u} = 0, \quad x \in \Omega, \end{cases} \tag{9}$$

and Schwartz conditions

$$\begin{cases} u_0^+(y) = f_1(y), \quad y \in \Gamma, \\ \tilde{u}^+(y)\, n(y) = f_2(y), \quad y \in \Gamma, \end{cases} \tag{10}$$

where $\vec{n}(y)$ is the outer normal to the boundary Γ at the point y.

Here, in what follows, we will use the operations of a vector field, which for the vector function $u = (u_1, u_2, u_3) \in C^1(\Omega)$ and the scalar function $w \in C^1(\Omega)$ are defined by the equalities

$$div\, \tilde{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}, grad w = \left(\frac{\partial w}{\partial x_1}, \frac{\partial w}{\partial x_2}, \frac{\partial w}{\partial x_3}\right),$$

and

$$rot\, \tilde{u} = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right).$$

It is directly verified that the system (9) is an elliptic system. The boundary conditions (10) satisfy the complementarity condition [5-7,11]. Therefore the problem (9),(10) has a Fredholm property. Necessary and sufficient conditions for the solvability of problem (9),(10) are noted in [7]. To describe the condition of solvability of (9),(10), we need the following constructions.

We choose an open covering $S_k, k = 1, 2, 3, 4$ of the surface Γ and the unit tangent vectors $\vec{p}_k(y), \vec{q}_k(y)$ to Γ from the class $C^\mu(S_k)$ so that at each point $y \in S_k$ the vectors $\vec{p}_k(y), \vec{q}_k(y)$ and $\vec{n}(y)$ were pairwise orthogonal.

Since Γ is the union of two spheres, it follows that such a choice is possible. By the results of [7], we introduce the conjugate problem to (9),(10)

$$\begin{cases} div\, \tilde{v} = 0, \quad x \in \Omega, \\ grad\, v_0 + rot\, \tilde{v} = 0, \quad x \in \Omega, \end{cases} \tag{11}$$

$$\begin{cases} \tilde{v}^+(y)\, p_k(y) = 0, \quad y \in \Gamma, \\ \tilde{v}^+(y)\, q_k(y) = 0, \quad y \in S_k, \quad k = 1, 2, 3, 4. \end{cases} \tag{12}$$

Proposition 1 *[7] The nonhomogeneous problem (9),(10) is solvable in the class $C^\mu(\overline{\Omega})$ if and only if the orthogonality condition*

$$\begin{cases} \int_{|y|=r_1} [-y f_1(y)\, \tilde{v}^+(y) + r_1 f_2(y)\, v_0^+(y)] d_2 y = 0, \\ \\ \int_{|y|=r_2} [y f_1(y)\, \tilde{v}^+(y) + r_2 f_2(y)\, v_0^+(y)] d_2 y = 0, \end{cases} \tag{13}$$

holds for all $(v_0, \tilde{v})$ representing the solutions of the homogeneous problem (11),(12).

Further we assume that the orthogonality requirements (13) for the data f_1, f_2 are satisfied. So, problem (9),(10) is solvable (can be ambiguously solvable). One of the possible solutions of the problem (9),(10) is denoted by $(w_0, \tilde{w})$, $x \in \Omega$.

We formulate the following statement that is useful for further investigation.

Lemma 1 *The first component $u_0(x)$ of the vector-function $u = (u_0, u_1, u_2, u_3)$ represents the solution of the Dirichlet problem for the Laplace equation*

$$\begin{cases} \Delta u_0 = 0, \quad x \in \Omega, \\ u_0^+(y) = f_1(y), \quad y \in \Gamma. \end{cases} \tag{14}$$

Since the Dirichlet problem for the Laplace equation (14) has a unique solution, then previously introduced $w_0(x) \equiv u_0(x)$ for all $x \in \Omega$.

The second equation of system (9) implies that

$$grad\, u_0 + rot\, \tilde{u} \equiv 0, \quad x \in \Omega,$$

$$grad\, w_0 + rot\, \tilde{w} \equiv 0, \quad x \in \Omega.$$

Subtracting one equality from the other, we obtain the following equation

$$rot\, (\tilde{u} - \tilde{w}) = 0, \quad x \in \Omega.$$

By the same way we can write down the boundary condition

$$(\tilde{u}^+ - \tilde{w}^+) \cdot \vec{n}(y) = 0, \quad y \in \Gamma.$$

The difference $\tilde{u}(x) - \tilde{w}(x)$ we denote by $\tilde{\theta}(x)$. Hence, it follows that $\tilde{\theta}(x)$ is the solution of the homogeneous problem

$$\begin{cases} div\, \tilde{\theta} = 0, \quad x \in \Omega, \\ rot\, \tilde{\theta} = 0, \quad x \in \Omega, \\ \tilde{\theta}^+ n = 0, \quad y \in \Gamma. \end{cases} \tag{15}$$

Lemma 2 *To solve the inhomogeneous problem*

$$\begin{cases} div\,\tilde{u} = 0, \quad x \in \Omega, \\ rot\,\tilde{u} = -\theta(x), \quad x \in \Omega, \\ \tilde{u}^{+}n = 0, \quad y \in \Gamma \end{cases} \tag{16}$$

the relation

$$\int_{l'_{y_0,y}} \tilde{u}^{+}(y)\, e(y)d_1y - \int_{l^{-1}_{y_0,y}} \tilde{u}^{+}(y)\, e(y)d_1y = -\int_S \theta(x)\, n(x)d_2x,$$

for any $y \in \Omega$ and for any $l_{y_0,y}$.

Proof 1 *Let's fix a point $y_0 \in D$ and choose an arbitrary $y \in \Omega$. Let $l_{y_0,y}, l'_{y_0,y} \subset D$ be arbitrary paths that connecting the point y_0, y and lying entirely in this region. These paths $l_{y_0,y} \sim l'_{y_0,y}$ are homotopic in Ω,, since the domain Ω is a spherical layer. Let $L = l_{y_0,y} \cup l'_{y_0,y}$ and denote by S the surface that formed by the closed contour L.*
Therefore, the closed-loop integral L by the Stokes formula is equal to

$$\int_L \tilde{u}^{+}(y)e(y)d_1y = \int_S (\mathrm{rot}\,\tilde{u})^{+}(x)n(x)d_2x,$$

where $e(y)$ is a unit tangent vector to the contour $\partial\Gamma_0$, oriented positively with respect to n (i.e. the traversal of this contour, as viewed from the end of the vector n, is carried out counterclockwise). According to the second equation of system (16), we have relation

$$\int_L \tilde{u}^{+}(y)e(y)d_1y = -\int_S \theta(x)n(x)d_2x.$$

Since $L = l'_{y_0,y} \cup l^{-1}_{y_0,y}$, then we rewrite the last relation in the form

$$\int_{l'_{y_0,y}} \tilde{u}^{+}(y)e(y)d_1y - \int_{l^{-1}_{y_0,y}} \tilde{u}^{+}(y)e(y)d_1y = -\int_S \theta(x)n(x)d_2x.$$

This is true for any $y \in D$ and for any $l_{y_0,y}$.

The following statement is proved in [7].

Theorem 1 *[7] The homogeneous problem (15) defined in the spherical layer has a unique solution belonging to the class $C^{\mu}(\overline{\Omega})$.*

Proposition 1 implies that the nonhomogeneous problem (9),(10) has a solution if the requirements (13) hold. Thus, the results of [7] imply the existence of a single well-posed problem for system (9) in the spherical layer.

4 The Schwartz problems for the Moisil–Teodorescu system in the interior of a torus in three-dimensional space

Let $\Omega = \{(x_1, x_2, x_3) : x_1 = rcos\varphi, \quad x_2 = rcos\theta(3 + sin\varphi), \quad x_3 = rsin\theta(3 + sin\varphi), \quad r < 1, 0 \le \varphi \le 2\pi, 0 \le \theta \le 2\pi\}$ presents the interior of the torus in three-dimensional space.

By Γ we denote the boundary of the domain Ω, namely $\Gamma = \{(y_1, y_2, y_3) : y_1 = cos\varphi, y_2 = cos\theta(3+sin\varphi), y_3 = sin\theta(3+sin\varphi), 0 \le \varphi \le 2\pi, 0 \le \theta \le 2\pi\}$. It is required to find the scalar function $u_0(x)$ and the vector-function $\tilde{u} = (u_1, u_2, u_3)$ that satisfy the Moisil–Teodorescu system with Schwartz conditions

$$\begin{cases} div\, \tilde{u} = 0, \quad x \in \Omega, \\ gradu_0 + rot\, \tilde{u} = 0, \quad x \in \Omega, \end{cases} \tag{17}$$

$$\begin{cases} u_0^+(y) = f_1(y), \quad y \in \Gamma, \\ \tilde{u}^+(y)\, n(y) = f_2(y), \quad y \in \Gamma, \end{cases} \tag{18}$$

$$\int_{-\pi}^{\pi} [-u_2(0, 3cos\theta, 3sin\theta)sin\theta + u_3(0, 3cos\theta, 3sin\theta)cos\theta]d\theta = \alpha(u_0^+, \tilde{u}^+ n), \tag{19}$$

where the quantity α represents an arbitrary linear continuous functional in the space $C^1(\Gamma) \times C^1(\Gamma)$.

The Fredholm index of the problem (17),(18) (without condition (19)) is calculated in [7]. By the results of the work [7] the nonhomogeneous problem (17),(18) (without condition (19)) is solvable in $C^\mu(\overline{\Omega})$ if and only if the orthogonality condition

$$\int_\Gamma f_2(y) d_2 y = 0. \tag{20}$$

holds.

Further we assume that the orthogonality condition (20) holds.

Proposition 2 *If the condition (19) holds, then the problem (17),(18) is uniquely solvable.*

Proof 2 *We will prove this proposition by contradiction. Suppose that there exist two solutions of the problem (17),(18). We denote them by $u_0(x)$, $\tilde{u}(x)$ and $w_0(x)$, $\tilde{w}(x)$.*

It is clear that $u_0(x) = w_0(x)$, $x \in \Omega$. The similar statement is proved in section 3. The difference $\tilde{u}(x) - \tilde{w}(x)$ we denote by $\check{\theta}(x)$. Thus, $\check{\theta}(x)$ is a solution of the problem

$$\begin{cases} div\, v(x) = 0, \quad x \in \Omega, \\ rot\, v(x) = 0, \quad x \in \Omega, \end{cases}$$

$$v(y)^+ n(y) = 0, \quad y \in \Gamma,$$

$$\int_{-\pi}^{\pi} [-v_2(0, 3cos\theta, 3sin\theta)sin\theta + v_3(0, 3cos\theta, 3sin\theta)cos\theta]d\theta = 0, \tag{21}$$

By the results of the work [7] there exists a harmonic function $\varphi(x)$ such that $\check{\theta}(x) = grad\, \varphi(x)$. On the boundary Γ the harmonic function $\varphi(x)$ satisfies the following condition

$$\frac{\partial \varphi}{\partial n} = 0, \quad y \in \Gamma.$$

The third condition in (21) means that

$$\lim_{\theta\to\pi}\varphi(0,3cos\theta,3sin\theta)=\lim_{\theta\to-\pi}\varphi(0,3cos\theta,3sin\theta). \tag{22}$$

It was proved in [7] that the difference of the limits

$$\lim_{x_3\to 0^+,x_2<0}\varphi(x_1,x_2,x_3)-\lim_{x_3\to 0^-,x_2<0}\varphi(x_1,x_2,x_3)$$

does not depend on the points $(x_1,x_2,x_3)\in\Omega,\ x_2<0,x_3=0$ *for* $(x_1,x_2,x_3)\in\Omega$.

Then (22) implies that

$$\lim_{x_3\to 0^+,x_2<0}\varphi(x_1,x_2,x_3)-\lim_{x_3\to 0^-,x_2<0}\varphi(x_1,x_2,x_3)=0$$

for $(x_1,x_2,x_3)\in\Omega$, $x_2<0$. *In this case we conclude that* $\varphi(x)=const$ *for all* $x\in\Omega$. *Consequently,* $\breve{\theta}(x)\equiv 0$ *for all* $x\in\Omega$.

We now state the main result of this section.

Theorem 2 *Let* f_1 *and* f_2 *be arbitrary functions in* $C^1(\Gamma)$, *and (20) holds for* f_2. *Then the problem (17),(18) has a unique solution for arbitrary linear continuous functional* $\alpha(f_1,f_2)$ *in* $C^1(\Gamma)\times C^1(\Gamma)$.

Remark 1 *The functional* $\alpha(f_1,f_2)$ *can be defined by the formula*

$$\alpha(f_1,f_2)=\int_\Gamma f_1(y)\mu_1(y)d_2y-\int_\Gamma f_2(y)\mu_2(y)d_2y,$$

where $\mu_1(\cdot),\mu_2(\cdot)$ *are continuous functions on the surface* Γ.

In this case, in the problem (17),(18) condition (19) is a nonlocal boundary condition. Nonlocal boundary value problems for differential equations have been studied by many authors. In particular, in the work [16] systematically studied solutions of nonlocal problems for pseudo-hyperbolic equations.

In [17–19] works questions of the Fredholm solvability of the Neumann problem for a higher order elliptic equation on the plane were studied, and the equivalence of the solvability condition for the generalized Neumann problem with the complementary condition (the Shapiro-Lopatinsky condition) was proved.

Conclusion. Thus, in this paper, we considered the Moisil – Teodorescu elliptic system $M(\partial/\partial x)u(x)=0$ in a three-dimensional bounded domain $\Omega\subseteq\mathbb{R}^3$. For solutions of this system, the basic facts of the theory of analytic functions on the plane are valid, including the integral theorem and Cauchy's formula, Morera's theorem, and others. In this paper, we write out the fundamental solution of the differential operator $M(\partial/\partial x)$ in the space $\mathbb{R}^3$ and present well-posed problems for the Moisil–Teodorescu system in the case of a spherical layer and the interior of a torus. The results of this work show a significant difference between the well-posed problem in a spherical layer and a similar problem in a torus.

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References

[1] Batchelor G.K., An Introduction to Fluid Dynamics, Cambridge University Press. 2000.

[2] Hatcher A., Algebraic Topology, Cambridge University Press. New Yourk. 2002.

[3] Simmons G.F., Introduction to Topology and Modern Analysis, McGraw Hill Book Company. 1968.

[4] Agmon S., Douglis A., Nirenberg L., Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II, Comm. Pure and Appl. Math., 17 (1964), 35—92.

[5] Shevchenko V.I., Some boundary value problem for holomorphic vector, Sborn. Math. Phys. Kiev, 8 (1970), 172—187. (in Russian)

[6] Yao Y., The Riemann boundary value problem for the Moisil-Theodorescu system, In Tournal-Sichuan Normal University. Natural Science Edition, 29 (2006), 188—191.

[7] Soldatov A.P., The Riemann-Hilbert problem for the Moisil-Theodorescu system, Izv. RAN, Ser. Math. (2018) (in Russian).

[8] Polunin V.A., Soldatov A.P., Riemann-Hilbert problem for the Moisil-Teodorescu system in multiply connected domains, Electronic Journal of Differential Equations, 310 (2016), 1–5.

[9] Polunin V.A., Soldatov A.P., On the integral represenation of solutions of the Moisil-Theodorescu system in multiply connected domains, Dokl. RAN, 475:4 (2017), 192–201.

[10] Moisil Gr. C., Theodoresco N., Fonctions holomorphes dans l'espace, Mathematica, 5 (1931), 142-153.

[11] Stein I., Weiss G., Introduction to harmonic analysis on Euclidean spaces, Moscow. Mir. 1974.

[12] Bitsadze A.V., Spatial analogue of the Cauchy-type integral and some of its applications, Izv. USSR Academy of Sciences, ser. math., 17:6 (1953), 525-538.

[13] Bitsadze A.V., Boundary value problems for second order elliptic equations, Moscow. Mir. 1966.

[14] Polunin V.A., Soldatov A.P., Three-dimensional analogue of the Cauchy-type integral, Differ. equations, 47:3 (2011), 366-375.

[15] Nazarov S., Plamenevsky B.A., Elliptic problems in domains with piecewise smooth boundaries. Walter de Gruyter, 2011.

[16] Kozhanov A.I., Koshanov B.D., Sultangazieva Zh.B., New boundary value problems for fourth-order quasi-hyperbolic equations, Siberian Electronic Mathematical Reports, 16 (2019), 1383-1409. http://dx.doi.org/10.33048/semi.2019.16.098

[17] Koshanov B.D., Soldatov A.P., Boundary value problem with normal derivatives for a higher order elliptic equation on the plane, Differential Equations, 52:12 (2016), 1594-1609. https://doi.org/10.1134/S0012266116120077

[18] Koshanov B.D., Soldatov A.P., On the Solvability of the Boundary Value Problems for the Elliptic Equation of High Order on a Plane, Bulletin of the Karaganda University. Mathematics series, 91:3 (2018), 24-31. https://doi.org/10.31489/2018M3/24-30

[19] Koshanov B.D., Equivalence of the Fredholm solvability condition for the Neumann problem to the complementarity condition, Journal of Mathematics, Mechanics and Computer Science, 109:1 (2021), 34-54. https://doi.org/10.26577/JMMCS.2021.v111.i3.04