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ON A BOUNDARY VALUE PROBLEM FOR A BOUSSINESQ-TYPE EQUATION IN A TRIANGLE

Earlier, we considered an initial-boundary value problem for a one-dimensional Boussinesq-type equation in a domain that is a trapezoid, in which the theorems on its unique weak solvability in Sobolev classes were established by the methods of the theory of monotone operators. In this article, we continue research in this direction and study the issues of correct formulation of the boundary value problem for a one-dimensional Boussinesq-type equation in a degenerate domain, which is a triangle. A scalar product is proposed with the help of which the monotonicity of the main operators is shown, and uniform a priori estimates are obtained. Further, using the methods of the theory of monotone operators and a priori estimates, theorems on its unique weak solvability in Sobolev classes are established. A theorem on increasing the smoothness of a weak solution is established. In proving the smoothness enhancement theorem, we use a generalization of the classical result on compactness in Banach spaces proved by Yu.I. Dubinsky ("Weak convergence in nonlinear elliptic and parabolic equations Sbornik: Mathematics, 67 (109): 4 (1965)) in the presence of a bounded set from a semi-normed space instead of a normed one. It is also shown that the solution may have a singularity at the point of degeneracy of the domain. The order of this feature is determined, and the corresponding theorem is proved.
Key words: Boussinesq equation, degenerating domain, a priori estimates, Sobolev space.

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Үшбұрыштағы Буссинеск типтес теңдеуіне қойылған шекаралық есеп

Осыған дейін біз трапециялы облыстағы бірөлшемді Буссинеск типтес теңдеуі үшін қойылған бастапқы-шекаралық есепті қарастырдық. Есептің соболев кеңістіктеріндегі бірмәнді әлсіз шешімділігі туралы теоремалар монотонды операторлар теориясы әдісімен дәлелденді. Осы мақалада біз осы бағыттағы зерттеулерді жалғастырып, азғындалатын үшбұрышты облыстағы бірөлшемді Буссинеск типтес теңдеуі үшін қойылған шекаралық есептің қисынды қойылуын қарастырамыз. Негізгі операторлардың монотондылығын көрсету кезінде қолданылған скалярлы көбейтінді ұсынылып, бірқалыпты априорлы бағалаулар алынды. Әрі қарай монотонды операторлар теориясы және априорлы бағалаулар көмегімен есептің соболев кластарындағы бірмәнді әлсіз шешімділігі туралы теоремалар дәлелденді. Әлсіз шешімнің дифференциалдық қасиеттерін жақсартатын теорема дәлелденді. Шешімнің дифференциалдық қасиеттерін жақсартатын теореманы дәлелдеу кезінде біз банах кеңістіктеріндегі компактылық туралы классикалық нәтиженің нормаланған кеңістіктің орнына полунормаланған кеңістіктегі шенелген жиын бар болу жағдайының Ю.И. Дубинский ("Weak convergence in nonlinear elliptic and parabolic equations Sbornik: Mathematics, 67 (109): 4 (1965)) дәлелдеген жалпылауын пайдаландық. Бұған қоса облыстың азғындалу нүктесінде шешімнің ерекшігі бар екендігі көрсетілген. Осы ерекшеліктің реті анықталып, сәйкес теорема дәлелденді.
Түйін сөздер: Буссинеск теңдеуі, азғындалатын облыс, априорлы бағалаулар, Соболев кеңістігі.

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Граничная задача для уравнения типа Буссинеска в треугольнике

Ранее нами была рассмотрена начально-граничная задача для одномерного уравнения типа Буссинеска в области, представляющей собой трапецию, в которой методами теории монотонных операторов установлены теоремы об её однозначной слабой разрешимости в соболевских классах. В этой статье мы продолжаем исследования в данном направлении и изучаем вопросы корректной постановки граничной задачи для одномерного уравнения типа Буссинеска в вырождающейся области, представляющей собой треугольник. Предложено скалярное произведение с помощью которого показана монотонность основных операторов, и получены равномерные априорные оценки. Далее методами теории монотонных операторов и априорных оценок установлены теоремы об её однозначной слабой разрешимости в соболевских классах. Установлена теорема о повышении гладкости слабого решения. При доказательстве теоремы о повышении гладкости мы используем обобщение классического результата о компактности в банаховых пространствах, доказанного Ю.И. Дубинским ("Weak convergence in nonlinear elliptic and parabolic equations Sbornik: Mathematics, 67 (109): 4 (1965)) при наличии ограниченного множества из полунормированного пространства вместо нормированного. Также показано, что решение может иметь особенность в точке вырождения области. Порядок данной особенности определен, и доказана соответствующая теорема.

Ключевые слова: уравнение Буссинеска, вырождающаяся область, априорные оценки, пространство Соболева.

Introduction

The theory of Boussinesq equations and its modifications always attracts the attention of both mathematicians and applied scientists. The Boussinesq equation, as well as their modifications, occupy an important place in describing the motion of liquid and gas, including in the theory of non-stationary filtration in porous media [1]– [13]. Additionally, here we note only the works [14]– [19]. In recent years, boundary value problems for these equations have been actively studied, since they model processes in porous media. These problems acquire particular importance for deep understanding and comprehension in the tasks of exploration and effective development of oil and gas fields.

In this paper, we study questions of the correct formulation of boundary value problems for a one-dimensional Boussinesq-type equation in a degenerating domain. The domain is represented by a triangle. Using the method of monotone operators, we prove theorems on the unique weak solvability of the considered boundary value problems, and also establish a theorem on improving the smoothness of a weak solution.

1 Statement of the boundary value problem and the main result

Let $\Omega_t = \{0 < x < t\}$ and $\partial\Omega_t$ be the boundary of the Ω_t, $\;0 < t < T < \infty$. In the domain $Q_{xt} = \{x, t | \, x \in \Omega_t, \; t \in (0, T)\}$, which is a triangle, we consider the following boundary value problem for a Boussinesq-type equation

$$\partial_t u - \partial_x \left(|u| \partial_x u \right) = f, \;\; \{x, t\} \in Q_{xt}, \tag{1}$$

with boundary conditions

$$u = 0, \ \ \{x, t\} \in \Sigma_{xt} = \partial\Omega_t \times (0, T), \tag{2}$$

where $f(x, t)$ is a given function.

It can be directly shown that the nonlinear operator $A_0(v) = -\partial_x\left(|v|\partial_x v\right)$ of boundary value problem (1)–(2) has the following properties:

$$A_0(v) : L_3(\Omega_t) \to L_{3/2}(\Omega_t) \text{ is a hemicontinuous operator,} \tag{3}$$

$$\|A_0(v)\|_{L_{3/2}(\Omega_t)} \leq c\|v\|^2_{L_3(\Omega_t)}, \ \ c > 0, \ \ \forall\, v \in L_3(\Omega_t), \tag{4}$$

$$\langle A_0(v), v\rangle \geq \alpha\|v\|^3_{L_3(\Omega_t)}, \ \ \alpha > 0, \ \ \forall\, v \in L_3(\Omega_t). \tag{5}$$

We have established the following theorems.

Theorem 1 (Main result) *Let*

$$f \in L_{3/2}((0, T); W^{-1}_{3/2}(\Omega_t)). \tag{6}$$

Then boundary value problem (1)–(2) *has a unique solution*

$$u \in L_3((0, T); L_3(\Omega_t)) \cap L_\infty((0, T); H^{-1}(\Omega_t)), \tag{7}$$

moreover, at $x \to 0+, \ x \to t - 0, \ t \to 0+$ *we have*

$$\begin{cases} u(x, t) = \mathcal{O}\left(x^{-\alpha_0}(t - x)^{-\alpha+\alpha_0}t^{-\beta}\right), \\ 0 < \alpha < \frac{1}{3}, \ \ \beta > 0, \ \ \alpha + \beta < \frac{2}{3}, \ \ 0 \leq \alpha_0 \leq \alpha. \end{cases} \tag{8}$$

Theorem 2 (On smoothness) *Let*

$$f \in L_{3/2}((0, T); L_{3/2}(\Omega_t)). \tag{9}$$

Then the solution of boundary value problem (1)–(2) *admits additional smoothness, i.e.,*

$$u \in L_\infty((0, T); L_2(\Omega_t)), \tag{10}$$

$$|u|^{1/2}u \in L_2((0, T); H^1_0(\Omega_t)), \tag{11}$$

$$\partial_t u \in L_{3/2}((0, T); W^{-1}_{3/2}(\Omega_t)). \tag{12}$$

2 Auxiliary initial boundary value problems in trapezoids

To prove Theorem 1, we first consider auxiliary initial boundary value problems. Let $\Omega_t = \{0 < x < t\}$ and $\partial\Omega_t$ be the domain of the Ω_t, $\varepsilon_m < t < T < \infty$, $\varepsilon_1 > \varepsilon_2 > ... > \varepsilon_m > ...$, $\varepsilon_m \to 0$ at $m \to \infty$. In the domain $Q_{xt}^m = \{x, t | \, x \in \Omega_t, \ t \in (\varepsilon_m, T)\}$, which is a trapezoid, we consider the following boundary value problems for a Boussinesq-type equation

$$\partial_t u_m - \partial_x \left(|u_m| \partial_x u_m \right) = f_m, \ \ \{x, t\} \in Q_{xt}^m, \tag{13}$$

with boundary

$$u_m = 0, \ \ \{x, t\} \in \Sigma_{xt}^m = \partial\Omega_t \times (\varepsilon_m, T), \tag{14}$$

and initial conditions

$$u_m = 0, \ \ x \in \Omega_{\varepsilon_m} = (0, \varepsilon_m), \tag{15}$$

where $f_m(x,t)$ are the narrowing of function $f(x,t)$ (6), which is given in the triangle Q_{xt}, into trapezoids Q_{xt}^m.

Earlier, in [1]– [2], we established the following theorems.

Theorem 3 *Let*

$$f_m \in L_{3/2}((\varepsilon_m, T); W_{3/2}^{-1}(\Omega_t)). \tag{16}$$

Then initial boundary value problem (13)–(15) *has a unique solution*

$$u_m \in L_3((\varepsilon_m, T); L_3(\Omega_t)) \cap L_\infty((\varepsilon_m, T); H^{-1}(\Omega_t)). \tag{17}$$

Theorem 4 *Let*

$$f_m \in L_{3/2}((\varepsilon_m, T); L_{3/2}(\Omega_t)). \tag{18}$$

Then the solution of initial boundary value problem (13)–(15) *admits additional smoothness, i.e.,*

$$u_m \in L_\infty((\varepsilon_m, T); L_2(\Omega_t)), \tag{19}$$

$$|u_m|^{1/2} u_m \in L_2((\varepsilon_m, T); H_0^1(\Omega_t)), \tag{20}$$

$$\partial_t u_m \in L_2((\varepsilon_m, T); W_{3/2}^{-1}(\Omega_t)). \tag{21}$$

Note that results similar to Theorem 4 for cylindrical domains are also available in [21]–[22].

3 Proof of Theorem 1. Existence

First of all, for each m and the corresponding given function $f_m(x,t)$, according to the statement of Theorem 3, we have established the existence of a unique solution $u_m(x,t)$ of initial boundary value problem (13)–(15).

We continue functions $u_m(x,t)$, $f_m(x,t)$ from the trapezoid Q_{xt}^m by zero to the entire triangle Q_{xt} and denote them by $\tilde{u}_m(x,t)$, $\tilde{f}_m(x,t)$. These functions will satisfy equations

$$\partial_t \tilde{u}_m - \partial_x \left(|\tilde{u}_m| \partial_x \tilde{u}_m \right) = \tilde{f}_m, \;\; \{x,t\} \in Q_{xt}, \tag{22}$$

with boundary conditions

$$\tilde{u}_m = 0, \;\; \{x,t\} \in \Sigma_{xt}. \tag{23}$$

From (22) we obtain

$$\langle \partial_t \tilde{u}_m(t), v \rangle + a_0(t, \tilde{u}_m(t), v) = \langle \tilde{f}_m(t), v \rangle, \;\; \forall\, v \in H^{-1}(\Omega_t), \;\; t \in (0,T), \tag{24}$$

where $a_0(t, \tilde{u}_m, v) = \langle A_0(t, \tilde{u}_m), v \rangle$, $A_0(t, \tilde{u}_m) = -\partial_x \left(|\tilde{u}_m| \partial_x \tilde{u}_m \right)$ and $\langle \cdot, \cdot \rangle$ is the scalar product defined by formula

$$\langle \varphi, \psi \rangle = \int_{\Omega_t} \varphi \left[\left(-d_x^{\,2} \right)^{-1} \psi \right] dx, \;\; \forall\, \varphi, \psi \in H^{-1}(\Omega_t), \;\; t \in (\varepsilon_m, T), \tag{25}$$

where $d_x^{\,2} = \frac{d^2}{dx^2}$, $\tilde{\psi} = \left(-d_x^{\,2} \right)^{-1} \psi : \; -d_x^{\,2} \tilde{\psi} = \psi$, $\tilde{\psi}(0) = \tilde{\psi}(t) = 0, \forall\, \psi \in H^{-1}(\Omega_t)$.

Note that concepts close to scalar product (25) have already been used in works [21], [22].

The operator $A_0(t, \tilde{u}_m)$ has the monotonicity property in accordance with scalar product (25). For solutions $\{\tilde{u}_m(t)\}_{m=1}^{\infty}$, we establish a priori estimates that are uniform in the index m. From (22)–(25) we will have:

$$\frac{1}{2} \|\tilde{u}_m(t)\|^2_{H^{-1}(\Omega_t)} + \alpha \int_0^t \|\tilde{u}_m(\tau)\|^3_{L_3(\Omega_t)} d\tau \le \int_0^t \|\tilde{f}_m(\tau)\|_{L_{3/2}(\Omega_t)} \|\tilde{u}_m(\tau)\|_{L_3(\Omega_t)} d\tau \le$$

$$\le \frac{2}{3} \sqrt{\frac{2}{3\,\alpha}} \int_0^t \|\tilde{f}_m(\tau)\|^{3/2}_{L_{3/2}(\Omega_t)} d\tau + \frac{\alpha}{2} \int_0^t \|\tilde{u}_m(\tau)\|^3_{L_3(\Omega_t)} d\tau \le$$

$$\le \frac{2}{3} \sqrt{\frac{2}{3\,\alpha}} \int_0^T \|f(t)\|^{3/2}_{L_{3/2}(\Omega_t)} dt + \frac{\alpha}{2} \int_0^t \|\tilde{u}_m(\tau)\|^3_{L_3(\Omega_t)} d\tau. \tag{26}$$

From here we get

$$\|\tilde{u}_m(t)\|^2_{H^{-1}(\Omega_t)} + \alpha \int_0^t \|\tilde{u}_m(\tau)\|^3_{L_3(\Omega_t)} d\tau \le \frac{4}{3} \sqrt{\frac{2}{3\,\alpha}} \|f(t)\|^{3/2}_{L_{3/2}(Q_{xt})}, \;\; t \in (0,T]. \tag{27}$$

In (26) we used the following relations

$$\frac{1}{2} \frac{d}{dt} \|\tilde{u}_m(t)\|^2_{H^{-1}(\Omega_t)} = \langle \tilde{u}'_m(t), \tilde{u}_m(t) \rangle, \;\; \text{since} \;\; \tilde{u}_m(t) \equiv 0 \text{ on } \Sigma_{xt},$$

$$\tilde{f}_m(t)\|_{L_{3/2}(\Omega_t)} \leq \|f(t)\|_{L_{3/2}(\Omega_t)},$$

as well as Young's inequality $(p^{-1}+q^{-1}=1)$:

$$|DE| = \left| \left(d^{1/p}D\right)\left(d^{1/q}\frac{E}{d}\right)\right| \leq \frac{d}{p}\,|D|^p + \frac{d}{qd^q}\,|E|^q\,,$$

where

$$D = \|w_m(t)\|_{L_{3/2}(\Omega)}\,,\ \ E = \|w_m(t)\|_{L_3(\Omega)}\,,\ \ d = \sqrt{\frac{2}{3\,\alpha}},\ \ p = 3/2,\ \ q = 3.$$

Finally, the relations

$$\tilde{u}_\mu \to u\ \ \star-\text{weak in}\ \ L_\infty((0,T);H^{-1}(\Omega_t)), \tag{28}$$

$$\tilde{u}_\mu \to u \text{ weak in }\ L_3(Q_{xt}), \tag{29}$$

$$\tilde{u}_\mu(T) \to \eta \text{ weak in }\ H^{-1}(\Omega_T), \tag{30}$$

$$A_0(t,\tilde{u}_\mu) \to h(t) \text{ weak in }\ L_{3/2}((0,T);L_{3/2}(\Omega_t)). \tag{31}$$

follow from (27) and inequality

$$\|A_0(t,\tilde{u}_\mu)\|_{L_{3/2}(\Omega_t)} \leq c\|\tilde{u}_\mu\|^2_{L_3(\Omega_t)}.$$

Now we continue functions $\tilde{u}_m(t),\ A_0(t,\tilde{u}_m(t)),...$, from domain Q_{xt} by zero to the infinite domain $\bar{Q}_{xt}$, where

$$\bar{Q}_{xt} = \begin{cases} x=0, & t\leq 0,\\ x\in\Omega_t, & t\in(0,T],\\ x\in\Omega_T, & t>T; \end{cases}$$

and denote these continuations by $\bar{\tilde{u}}_m(t),\ \bar{A}_0(t,\bar{\tilde{u}}_m(t)),...$, i.e.,

$$\bar{\tilde{u}}_m(t) = \begin{cases} 0, & t\leq 0,\\ \tilde{u}_m(t)\in H^{-1}(\Omega_t), & t\in(0,T],\\ 0, & t>T; \end{cases} \qquad \bar{v}(t) = \begin{cases} 0, & t\leq 0,\\ v(t)\in H^{-1}(\Omega_t), & t\in(0,T],\\ 0, & t>T. \end{cases} \tag{32}$$

As a result, for continuations (32) we will have:

$$\langle \bar{\tilde{u}}'_m(t),\bar{v}(t)\rangle + \langle \bar{A}_0(t,\bar{\tilde{u}}_m(t)),\bar{v}(t)\rangle = \langle \bar{\tilde{f}}_m(t),\bar{v}(t)\rangle - \langle \tilde{u}_m(T),\bar{v}(t)\rangle\delta(t-T),\ \ t\in R^1. \tag{33}$$

Further, choosing from $\{\bar{\tilde{u}}_m(t)\}_{m=1}^\infty$ a weakly convergent subsequence $\{\bar{\tilde{u}}_\mu(t)\}_{\mu=1}^\infty$ and passing to the limit at $\mu\to\infty$, we obtain

$$\langle \bar{u}'(t),\bar{v}(t)\rangle + \langle \bar{h}(t),\bar{v}(t)\rangle = \langle \bar{f}(t),\bar{v}(t)\rangle - \langle \eta,\bar{v}(t)\rangle\delta(t-T),\ \ t\in R^1,$$

where $\bar{u}(t)$, $\bar{h}(t)$ and $\bar{f}(t)$ are continuations of functions $u(t)$ (28), $h(t)$ (31) and $f(t)$ to R^1, that is, from here we get

$$\bar{u}'(t) + \bar{h}(t) = \bar{f}(t) - \eta\delta(t - T), \;\; t \in R^1. \tag{34}$$

Now, narrowing equality (34) to the time interval $(0, T)$, we obtain

$$u'(t) + h(t) = f(t), \;\; t \in (0, T), \tag{35}$$

$$u'(t) \in L_{3/2}((0, T); L_{3/2}(\Omega_t)). \tag{36}$$

Further, on the one hand, from the monotonicity condition of the operator $A_0(t, v)$ we will have

$$Y_\mu \equiv \int_0^T \langle A_0(t, \tilde{u}_\mu(t)) - A_0(t, v(t)), \tilde{u}_\mu(t) - v(t)\rangle \, dt \geq 0 \;\; \forall\, v \in L_3((0, T); L_3(\Omega_t)), \tag{37}$$

on the other hand, from (24) we get

$$\int_0^T \langle A_0(t, \tilde{u}_\mu(t)), \tilde{u}_\mu(t)\rangle \, dt = \int_0^T \langle \tilde{f}_\mu(t)), \tilde{u}_\mu(t)\rangle \, dt - \frac{1}{2}\|\tilde{u}_\mu(T)\|^2_{H^{-1}(\Omega_T)}. \tag{38}$$

Thus, it follows from relations (37)–(38) that

$$Y_\mu \equiv \int_0^T \langle \tilde{f}_\mu(t), \tilde{u}_\mu(t)\rangle \, dt - \frac{1}{2}\|\tilde{u}_\mu(T)\|^2_{H^{-1}(\Omega_T)} - \int_0^T \langle A_0(t, \tilde{u}_\mu(t)), v(t)\rangle \, dt -$$

$$- \int_0^T \langle A_0(t, v(t)), \tilde{u}_\mu(t) - v(t)\rangle \, dt \;\; \forall\, v \in L_3((0, T); L_3(\Omega_t)). \tag{39}$$

Now, using the property of weak lower semicontinuity of the norm in a Banach space

$$\liminf \|\tilde{u}_\mu(T)\|^2_{H^{-1}(\Omega_T)} \geq \|\tilde{u}(T)\|^2_{H^{-1}(\Omega_T)},$$

we have

$$0 \leq \limsup Y_\mu \leq \int_0^T \langle f(t), u(t)\rangle \, dt - \frac{1}{2}\|u(T)\|^2_{H^{-1}(\Omega_T)} - \int_0^T \langle h(t), v(t)\rangle \, dt -$$

$$- \int_0^T \langle A_0(t, v(t)), u(t) - v(t)\rangle \, dt \;\; \forall\, v \in L_3((0, T); L_3(\Omega_t)). \tag{40}$$

In turn, from (35) we get

$$\int_0^T \langle f(t), u(t)\rangle \, dt = \int_0^T \langle h(t), u(t)\rangle \, dt + \frac{1}{2}\|u(T)\|^2_{H^{-1}(\Omega_T)}. \tag{41}$$

Substituting the expression for $\int_0^T \langle f(t), u(t)\rangle\, dt$ from (41) into inequality (40), we establish the following inequality

$$\int_0^T \langle h(t) - A_0(t, v(t)), u(t) - v(t)\rangle\, dt \geq 0 \;\; \forall\, v(t) \in L_3((0,T); L_3(\Omega_t)). \tag{42}$$

Now, to complete the proof of Theorem 1, i.e. the existence of a solution to boundary value problem (1)–(2), our goal is: to show the validity of the following equality

$$h(t) = A_0(u(t)). \tag{43}$$

We use the property of hemicontinuity of the operator $A_0(t, v)$ (3). Replacing $v(t) = u(t) - \lambda w(t)$, $\lambda > 0$, $w \in L_3(Q_{xt})$ in (42), we obtain

$$\int_0^T \langle h(t) - A_0(t, u(t) - \lambda w(t)), w(t)\rangle\, dt \geq 0 \;\; \forall\, w(t) \in L_3(Q_{xt}).$$

Hence, at $\lambda \to 0+$, we obtain the required equality (43). The existence part of the solution in Theorem 1 is proved.

4 Proof of the Theorem 1. Uniqueness

Let us show that the operator $A_0(t, u)$ in problem (1)–(2) will have the property of monotonicity if the scalar product is introduced in an appropriate way. For this purpose, we take as the scalar product

$$\langle \varphi, \psi\rangle = \int_{\Omega_t} \varphi \left[\left(-d_x^{\,2}\right)^{-1} \psi\right] dy, \;\; \forall\, \varphi, \psi \in H^{-1}(\Omega_t), \;\; \forall\, t \in (0,T), \tag{44}$$

where $d_x^{\,2} = \frac{d^2}{dx^2}$, $\tilde{\psi} = \left(-d_x^{\,2}\right)^{-1} \psi \, : \;\; -d_x^{\,2}\tilde{\psi} = \psi,\, \tilde{\psi}(0) = \tilde{\psi}(t) = 0, \forall\, \psi \in H^{-1}(\Omega_t), \;\; \forall\, t \in (0,T)$.

The following lemma is valid.

Lemma 1 *Operator $A_0(t, u)$ is monotone in the sense of scalar product* (44) *in space $H^{-1}(\Omega_t)$, i.e. the following inequality is valid:*

$$\langle A_0(t, u_1) - A_0(t, u_2), u_1 - u_2\rangle \geq 0, \;\; \forall\, u_1, u_2 \in \mathcal{D}(\Omega_t), \;\; \forall\, t \in (0,T). \tag{45}$$

To the proof of Lemma 1 For each $t \in (0,T)$, operator $A_0(t, u) = -\partial_x \left(|u|\partial_x u\right)$ is monotonic and condition (45) is satisfied (according to [20], chap. 2, s. 3.1). Indeed, on the one hand, we have

$$\langle A_0(t, \varphi) - A_0(t, \psi), \varphi - \psi\rangle = \frac{1}{2} \int_{\Omega_t} \left(-d_x^2\right) \left(|\varphi|\varphi - |\psi|\psi\right) \left(-d_x^2\right)^{-1} (\varphi - \psi)\, dx =$$

$$= \frac{1}{2} \int_{\Omega_t} (|\varphi|\varphi - |\psi|\psi)(\varphi - \psi)\, dx, \;\; \forall\, \varphi, \psi \in \mathcal{D}(\Omega_t), \;\; \forall\, t \in (t_0, T).$$

On the other hand, from the convexity condition of the functional $J(t,\varphi) = \frac{1}{3}\int_{\Omega_t} |\varphi(x)|^3\, d\,x,\ \ \varphi \in \mathcal{D}(\Omega_t),\ \ \forall\, t \in (0,T)$, it follows

$$\langle J'(t,\varphi) - J'(t,\psi), \varphi - \psi\rangle \geq 0,\ \ \forall\, \varphi,\, \psi \in \mathcal{D}(\Omega_t),\ \ \forall\, t \in (0,T).$$

Thus, we get

$$\int\limits_{\Omega_t} (|\varphi|\varphi - |\psi|\psi)(\varphi - \psi)\, d\,x\, \geq 0,\ \ \forall\, \varphi,\, \psi \in \mathcal{D}(\Omega_t),\ \ \forall\, t \in (0,T),$$

that is, inequality (45) is established. Lemma 1 is proved.

Now we are ready to show the uniqueness of the solution in problem (1)–(2). Let $u_1(t)$ and $u_2(t)$ be two solutions to problem (1)–(2). Then their difference $u(t) = u_1(t) - u_2(t)$ satisfies the homogeneous problem:

$$u'(t) + A_0(t, u_1(t)) - A_0(t, u_2(t)) = 0,$$

$$\langle u'(t), u(t)\rangle + \langle (A_0(t, u_1(t)) - A_0(t, u_2(t)), u_1(t) - u_2(t)\rangle = 0$$

and, due to the monotonicity property of the operator $A_0(t,u)$, we have:

$$\langle u'(t), u(t)\rangle = \frac{d}{2\,dt}\|u(t)\|^2_{H^{-1}(\Omega_t)} \leq 0,\ \ \text{i.e.}\ \ u(t) \equiv 0.$$

The uniqueness of the solution to problem (1)–(2) is proved.

5 Proof of Theorem 1. Singularity of the solution

We show that the solution $u(x,t)$ of boundary value problem (1)–(2) having a singularity of the order specified in (8) will belong to the space $L_3(Q^{t_0}_{xt})$, where $Q^{t_0}_{xt} = \{x, t|\, 0 < x < t,\ 0 < t < t_0 \ll T\}$. For this purpose, it suffices to show that the following integral is bounded when $t_0 \to 0+$:

$$\int_{Q^{t_0}_{xt}} x^{-3\alpha_0}(t-x)^{-3\alpha+3\alpha_0} t^{-3\beta}\, dx\, dt. \tag{46}$$

We have

$$\int_0^{t_0} t^{-3\beta} \int_0^t x^{-3\alpha_0}(t-x)^{-3\alpha+3\alpha_0}\, dx\, dt = \left\| \begin{array}{l} x = t\sin^2\theta \\ 0 < \theta < \pi/2 \\ dx = 2\sin\theta\cos\theta\, d\theta \end{array} \right\| =$$

$$= 2\int_0^{t_0} t^{1-3\alpha-3\beta} \int_0^{\pi/2} \sin^{1-6\alpha_0}\theta \cos^{1-6\alpha+6\alpha_0}\theta\, d\theta\, dt.$$

It is not difficult to verify that under the conditions of Theorem 1 in the last expression, the inner integral takes a finite value. Calculating the outer integral, we have

$$\int_0^{t_0} t^{1-3\alpha-3\beta}\, dt = \frac{1}{2 - 3(\alpha+\beta)} t_0^{2-3(\alpha+\beta)},$$

which, under the conditions of Theorem 1, is also bounded from above.

Note that if the order of the singularity of solution $u(x,t)$ is higher than in (8), then this function is no longer an element of space $L_3(Q^{t_0}_{xt})$.

This completes the proof of Theorem 1.

6 Proof of Theorem 2

It suffice for us to show the existence of a solution, and the uniqueness follows from Theorem 1.

First of all, for each m and the corresponding given function $f_m(x,t)$, according to the statement of Theorem 4 we have established the existence of a smoother (than in Theorem 3) unique solution $u_m(x,t)$ of initial boundary value problem (13)–(15) for the corresponding trapezoid Q_{xt}^m.

We continue functions $u_m(x,t),\ f_m(x,t)$ from the trapezoid Q_{xt}^m by zero to the entire triangle Q_{xt} and denote them by $\tilde{u}_m(x,t),\ \tilde{f}_m(x,t)$. These functions will satisfy equations

$$\partial_t\tilde{u}_m - \partial_x\left(|\tilde{u}_m|\partial_x\tilde{u}_m\right) = \tilde{f}_m,\ \ \{x,t\}\in Q_{xt}, \tag{47}$$

with boundary conditions

$$\tilde{u}_m = 0,\ \ \{x,t\}\in\Sigma_{xt}. \tag{48}$$

From (47) we obtain

$$\langle\partial_t\tilde{u}_m(t),v\rangle + a_0(t,\tilde{u}_m(t),v) = \langle\tilde{f}_m(t),v\rangle,\ \ \forall\, v\in H^{-1}(\Omega_t),\ \ t\in(0,T), \tag{49}$$

where $a_0(t,\tilde{u}_m,v) = \langle A_0(t,\tilde{u}_m),v\rangle$, $A_0(t,\tilde{u}_m) = -\partial_x\left(|\tilde{u}_m|\partial_x\tilde{u}_m\right)$ and $\langle\cdot,\cdot\rangle$ is a scalar product

$$\langle\varphi,\psi\rangle = \int_{\Omega_t}\varphi\left[\left(-d_x^{\,2}\right)^{-1}\psi\right]dx,\ \ \forall\,\varphi,\psi\in H^{-1}(\Omega_t),\ \ t\in(\varepsilon_m,T),$$

where $d_x^{\,2} = \frac{d^2}{dx^2}$, $\tilde{\psi} = \left(-d_x^{\,2}\right)^{-1}\psi:\ -d_x^{\,2}\tilde{\psi} = \psi,\ \tilde{\psi}(0) = \tilde{\psi}(t) = 0, \forall\,\psi\in H^{-1}(\Omega_t)$.

Let us rewrite equation (49) in the form

$$\left(\partial_t\tilde{u}_m(t),\left(-\partial_x^{\,2}\right)^{-1}v\right) + \frac{1}{2}\left(|\tilde{u}_m(t)|\,\tilde{u}_m(t),v\right) = \left(\tilde{f}_m(t),\left(-\partial_x^{\,2}\right)^{-1}v\right),\ \ \forall\, v\in H^1_{0,\Delta}(\Omega_t),\ \ t\in(0,T),$$

where $H^1_{0,\Delta}(\Omega_t) = \{\varphi|\,\varphi,\ \partial_x^2\varphi\in H^1_0(\Omega_t)\}$, or

$$\left(\partial_t\tilde{u}_m(t),\tilde{v}\right) + \frac{1}{2}\left(|\tilde{u}_m(t)|\,\tilde{u}_m(t),v\right) = \left(\tilde{f}_m(t),\tilde{v}\right),\ \ \forall\,\tilde{v} = \left(-\partial_x^{\,2}\right)^{-1}v\in H^1_0(\Omega_t),\ \ t\in(0,T). \tag{50}$$

Further, from (50) we obtain the following equality

$$\langle\partial_t\tilde{u}_m(t),\tilde{u}_m(t)\rangle + \frac{1}{2}\left(|\tilde{u}_m(t)|\,\tilde{u}_m(t),-\partial_x^2\tilde{u}_m(t)\right) = \langle\tilde{f}_m(t),\tilde{u}_m(t)\rangle,\ \ t\in(0,T), \tag{51}$$

and from (51), therefore, we will have

$$\frac{1}{2}\frac{d}{dt}\|\tilde{u}_m(t)\|^2_{L_2(\Omega_t)} + \frac{4}{9}\int_{\Omega_t}\left[\partial_x\left(|\tilde{u}_m(t)|^{1/2}\,\tilde{u}_m(t)\right)\right]^2dx = \langle\tilde{f}_m(t),\tilde{u}_m(t)\rangle,\ \ t\in(0,T),$$

or

$$\frac{1}{2}\|\tilde{u}_m(t)\|^2_{L_2(\Omega_t)} + \frac{4}{9}\int_0^t\int_{\Omega_\tau}\left[\partial_x\left(|\tilde{u}_m(\tau)|^{1/2}\,\tilde{u}_m(\tau)\right)\right]^2dx\,d\tau = \int_0^t\langle\tilde{f}_m(\tau),\tilde{u}_m(\tau)\rangle\,d\tau,\ \ t\in(0,T).$$

(52)

Here we use the following equality

$$-\frac{1}{2}\int_{\Omega_t}|\tilde{u}_m(t)|\tilde{u}_m(t)\partial_x^2\tilde{u}_m(t)\,dx=\frac{4}{9}\int_{\Omega_t}\left[\partial_x\left(|\tilde{u}_m(t)|^{1/2}\tilde{u}_m(t)\right)\right]^2dx,\ \ t\in(0,T). \tag{53}$$

Let us show its justice. First, we transform the left side of equality (53). Let us show that equality

$$-\frac{1}{2}\int_{\Omega_t}|\tilde{u}_m(t)|\tilde{u}_m(t)\partial_x^2\tilde{u}_m(t)\,dx=\int_{\Omega_t}|\tilde{u}_m(t)|\left[\partial_x\tilde{u}_m(t)\right]^2\,dx, \tag{54}$$

holds. Indeed, we have:

$$|\tilde{u}_m|\tilde{u}_m=\begin{cases}[\tilde{u}_m]^2, & \text{at } \tilde{u}_m>0,\\ 0, & \text{at } \tilde{u}_m(t)=0,\\ -[-\tilde{u}_m]^2, & \text{at } \tilde{u}_m<0,\end{cases}\quad \partial_x\left(|\tilde{u}_m|\tilde{u}_m\right)=\begin{cases}2\,\tilde{u}_m\partial_x\tilde{u}_m, & \text{at } \tilde{u}_m>0,\\ 0, & \text{at } \tilde{u}_m(t)=0,\\ 2\,[-\tilde{u}_m]\partial_x\tilde{u}_m, & \text{at } \tilde{u}_m<0.\end{cases}$$

Thus, from here we obtain: $\partial_x\left(|\tilde{u}_m(t)|\tilde{u}_m(t)\right)=2\,|\tilde{u}_m(t)|\partial_x\tilde{u}_m(t)$, i.e. equality (54).

The same holds for the right side of equality (53). We get

$$|\tilde{u}_m|^{1/2}\tilde{u}_m=\begin{cases}[\tilde{u}_m]^{3/2}, & \text{at } \tilde{u}_m>0,\\ 0, & \text{at } \tilde{u}_m=0,\\ -[-\tilde{u}_m]^{3/2}, & \text{at } \tilde{u}_m<0,\end{cases}\quad \partial_x\left(|\tilde{u}_m|^{1/2}\tilde{u}_m\right)=\begin{cases}\frac{3}{2}[\tilde{u}_m]^{1/2}\partial_x\tilde{u}_m, & \text{at } \tilde{u}_m>0,\\ 0, & \text{at } \tilde{u}_m=0,\\ \frac{3}{2}[-\tilde{u}_m]^{1/2}\partial_x\tilde{u}_m, & \text{at } \tilde{u}_m<0.\end{cases}$$

Thus, from here we get: $\partial_x\left(|\tilde{u}_m(t)|^{1/2}\tilde{u}_m(t)\right)=\frac{3}{2}|\tilde{u}_m(t)|^{1/2}\partial_x\tilde{u}_m(t)$, that is, the following equality is true:

$$\frac{4}{9}\int_{\Omega_t}\left[\partial_x\left(|\tilde{u}_m(t)|^{1/2}\tilde{u}_m(t)\right)\right]^2dx=\int_{\Omega_t}|\tilde{u}_m(t)|\left[\partial_x\tilde{u}_m(t)\right]^2dx.$$

Thus, we have shown the validity of equality (53).

Since from Theorem 3 we have that the functions $\tilde{u}_m(t)$ are bounded in $L_3(Q_{xt})$, therefore the right part of (52) is bounded when condition (6) of Theorem 1 is fulfilled. Hence from (52) we deduce that

$$\tilde{u}_m \ \text{ are bounded in } \ L_\infty((0,T);L_2(\Omega_t)), \tag{55}$$

$$\partial_x\left(|\tilde{u}_m|\,\tilde{u}_m\right) \ \text{ are bounded in } \ L_2(Q_{xt}),\ \text{ i.e. } \ |\tilde{u}_m|^{1/2}\,\tilde{u}_m\in L_2((0,T);H_0^1(\Omega_t)). \tag{56}$$

From relations (55)–(56), equation (47) and conditions (4), (18) we establish an estimate for the time derivative t

$$\partial_t\tilde{u}_m \ \text{ are bounded in } \ L_{3/2}((0,T);W_{3/2}^{-1}(\Omega_t)). \tag{57}$$

Hence, we can write

$$\tilde{u}_m \to u \text{ weakly in } L_\infty((0,T); L_2(\Omega_t)), \tag{58}$$

$$|\tilde{u}_m|^{1/2}\tilde{u}_m \to \chi \text{ weakly in } L_2((0,T); H_0^1(\Omega_t)). \tag{59}$$

Thus, on the basis of relations (57)–(59) we establish

$$\tilde{u}_m \to u \text{ strongly in } L_3((0,T); L_3(\Omega_t)) \text{ and almost everywhere,}$$

and, further, using (56) and applying Theorem 12.1 and Proposition 12.1 from ([20], chapter 1, 12.2), as well as Lemma 1.3 from ([20], chapter 1, 1.4), as a result we have

$$|\tilde{u}_m|^{1/2}\tilde{u}_m \to |u|^{1/2}u \text{ weakly in } L_2((0,T); H_0^1(\Omega_t)), \text{ i.e. } \chi = |u|^{1/2}u. \tag{60}$$

Lemma 1.3 ([20], chapter 1, 1.4). *Let $\mathcal{O}$ is a bounded domain in $\mathbb{R}_x^n \times \mathbb{R}_t^1$, g_μ and g are functions from $L_q(\mathcal{O})$, $1 < q < \infty$, such that*

$$\|g_\mu\|_{L_q(\mathcal{O})} \leq C, \;\; g_\mu \to g \;\; a.e. \text{ in } \mathcal{O}.$$

Then $g_\mu \to g$ weakly in $L_q(\mathcal{O})$.

From (57), (59) and (60) we obtain the required statement (10)–(12). Theorem 2 is completely proved.

Conclusion

In this paper, we study boundary problems for a one-dimensional Boussinesq-type equation in a domain that is a triangle. Using the methods of the theory of monotone operators and a priori estimates, we prove theorems on their unique weak solvability in Sobolev classes, as well as theorems on improving the smoothness of a weak solution.

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