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MODELLING OF HORIZONTAL DRILL STRING MOTION BY THE LUMPED-PARAMETER METHOD

The motion of drill strings is modeled in the drilling of geotechnological wells in the mining industry by the Lumped-Parameter Method (LPM). This method is widely used in structural mechanics and is most justified in modeling dynamic systems with a variable structure. On the example of horizontal drilling of geotechnological wells, longitudinal vibrations of a drill string with a static compressive load at the left end are considered [1]. The contact interaction of the drill string with the borehole walls and the inertia force of the bit on the destructible rock at the right end of the string are taken into account. The analysis of the column splits number, which specifies the dimension of the system of discrete equations, is carried out by verifying the obtained results with the previously known data [1]. For verification, the developed C# software was used, allowed to determine the error of the column splits in comparison with the test data. The optimal number of the drill string splits in terms of "implementation time – calculation error" by the LPM was identified. The numerical implementation of the model is conducted by the fourth-order Runge-Kutta method. In connection with the increase in the implementation time of the program code due to the increase in the dimension of the system, the numerical algorithm is optimized using the parallel programming tools. The expediency of this optimization is analyzed.
Key words: drill string, nonlinear, vibrations, lumped-parameter method, parallel programming.

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Жиынтық параметрлер әдісі бойынша бұрғылау бағананың көлденден қозғалуын модельдеу

Бұрғылау бағананың қозғалысы тау-кен өнеркәсібінде геотехнологиялық ұңғымаларды игеруде жиынтық параметрлер әдісімен (ЖПӘ) модельденеді. Бұл әдіс құрылымдық механикада кеңінен қолданылады және айнымалы құрылымы бар динамикалық жүйелерді модельдеуде барынша негізделген. Геотехнологиялық ұңғымаларды көлденең бұрғылау мысалында оның сол жақ шетінде статикалық қысу жүктемесі бар бұрғылау бағананың бойлық тербелістері қарастырылған [1]. Бұрғылау бағананың ұңғыма қабырғаларымен жанасу әрекеті және тізбенің оң жақ шетінде жойылатын жынысқа қашау инерция күші ескеріледі. Алынған нәтижелерді бұрын белгілі [1] деректермен тексеру арқылы, дискретті теңдеулер жүйесінің өлшемін белгілейтін бағананың бөлімдер санының талдауы жүргізіледі. Тексеру үшін C# тілінде әзірленген бағдарламалық қамтамасыз ету пайдаланылады, бұл сынақ деректерімен салыстырғанда шығарылған бағана бөлімдерінің қателігін анықтауға мүмкіндік береді. Бағананың бөлімдерінің оңтайлы саны «есептеу уақыты-есептеу қатесі» тұрғысынан анықталады. Модельдің сандық орындалуы 4-ші ретті Рунге-Кутта әдісімен жүзеге асырылды. Жүйе өлшемінің өсуі мен программалық кодты орындау уақытының ұлғаюына байланысты параллельді бағдарламалау құралдарының көмегімен сандық алгоритм оңтайландырылды. Осы оңтайландырудың орындылығына талдау жүргізілді.
Түйін сөздер: бұрғылау бағана, бейсызықтылық, тербелістер, жиынтық параметр әдісі, параллельді бағдарламалау.

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Моделирование горизонтального движения буровой колонны методом сосредоточенных параметров

Моделируется движение буровых колонн при разработке геотехнологических скважин в добывающей промышленности методом сосредоточенных параметров (МСП). Данный метод широко применяется в строительной механике и наиболее оправдан при моделировании динамических систем с переменной структурой. На примере горизонтального бурения геотехнологических скважин рассмотрены продольные колебания буровой колонны со статической сжимающей нагрузкой на ее левом конце [1]. Учтены контактное взаимодействие буровой колонны со стенками скважины и сила инерции долота на разрушаемую породу на правом конце колонны. Посредством верификации полученных результатов с ранее известными данными [1] проведен анализ числа разбиений колонны, задающий размерность системы дискретных уравнений. Для верификации использовалось разработанное программного обеспечение на C#, позволяющее определить погрешность производимых разбиений колонны в сравнении тестовыми данными. Определено оптимальное число разбиений колонны с точки зрения «вычислительное время-погрешность расчета». Численная реализация модели осуществлена методом Рунге-Кутта 4-го порядка. В связи с увеличением времени реализации программного кода за счет роста размерности системы произведена оптимизация численного алгоритма с применением средств параллельного программирования. Проведен анализ целесообразности данной оптимизации.

Ключевые слова: буровая колонна, нелинейность, колебания, метод сосредоточенных параметров, параллельное программирование.

1 Introduction

In the complex process of drilling geotechnological wells in the mining industry, horizontal drilling has become widespread [2–4]. Research in the field of modelling the motion of horizontal drill strings from the point of view of the influence of stochastic processes on the dynamics of drilling equipment was carried out by Ritto T.G. with a group of scientists [1, 5, 6] and the authors of [7]. The authors of [8, 9] created an experimental setup based on the principle of mechanical similarity, and analyzed the accuracy of the theoretical models in accordance with the obtained experimental data. The authors of [10] studied the importance of drilling fluid formulations when drilling horizontal wells and proposed the use of biopolymer-based drilling fluids. In [11], the longitudinal vibrations of the column were modeled by the method of summation of modes, the analysis of the influence of the modes number on the dynamics of the system and their convergence was carried out. The authors of [12] developed a model that takes into account the geometric nonlinearity and the contact of the drill string with the well, based on the geometrically exact beam theory and the method of quadrature elements.

The search and application of alternative solutions in modelling are of scientific and practical interest, since they allow verifying the correctness of the already available results and expanding the class of problems under study. In particular, today, little-studied problems of modelling the dynamics of industrial equipment and machines in complicated conditions are relevant, namely due to the inhomogeneity of physical and mechanical properties, the variable structure of the research object, local and point loads.

Sadler J.P. in his work [13] considered the lumped-parameter method (LPM) for kinetic-elastodynamic analysis of mechanisms, later successfully used for the analysis of nonlinear vibrations of elastic multi-link mechanisms [14–16]. LPM is a special case of the finite element method, when the equation of a one-dimensional continuous medium is replaced by its discrete analogue. The essence of the method lies in the convertion from the model of a continuous medium to its discrete representation at the nodes by a system of ordinary differential equations. This method is widely known in structural mechanics, as well as in the study of the dynamics of flat beam structures [17–19]. Its application is most justified when modelling nonlinear systems with elements of heterogeneous material, variability of cross sections, loading, etc.

The purpose of this work is to identify the optimal number of drill string splits from the point of view of "implementation time-calculation error" by the LPM using parallel programming tools.

2 Mathematical model and its discretization

The horizontal motion of a drill string [1] under the action of a static compressive load at its left end, friction forces of the drill string against the rock, a variable harmonic force, gravitational forces, as well as an interaction force between the bit and the rock at the right end is considered

Figure 1. The lumped-parameter method (LPM) for solving the problem of the dynamics of drilling equipment was applied.

The equation of motion of the drill string with a length L is given in a general form [1]:

$$\begin{aligned} \rho A\frac{\partial^2 u(x,t)}{\partial t^2} - EA\frac{\partial^2 u(x,t)}{\partial x^2} = f_{sta}(x,t) + f_{har}(x,t) + \\ + f_{bit}(\dot{u}(x,t)) + f_{fric}(\dot{u}(x,t)) + f_{mass}(\ddot{u}(x,t)) \end{aligned} \tag{1}$$

where $u(x,t)$ is the longitudinal displacement of the drill string, ρ is density of the column material, A is the cross-sectional area, E is Young's modulus. The right-hand side of Eq. (1) contains the forces acting on the drill string.

The constant force f_{sta} acts on the left end of the drill string ($x = 0$) and it is given by

$$f_{sta}(x,t) = F_{sta}\delta(x), \tag{2}$$

where F_{sta} is an amplitude, $\delta(x)$ is the Dirac delta function.

The harmonic force f_{har} is given as:

$$f_{har}(x,t) = F_0 sin(\omega_f t)\delta(x-L), \tag{3}$$

where F_0 is an amplitude, ω_f is the harmonic force frequency.

The bit inertia force and the drill string friction force on the rock are defined, respectively, as:

$$\begin{aligned} f_{mass}(\ddot{u}(x,t)) = -m_{bit}\ddot{u}(x,t)\delta(x-L), \\ f_{fric}(\dot{u}(x,t)) = -\mu(x)(\rho A)g\ sgn(\dot{u}(x,t)), \end{aligned} \tag{4}$$

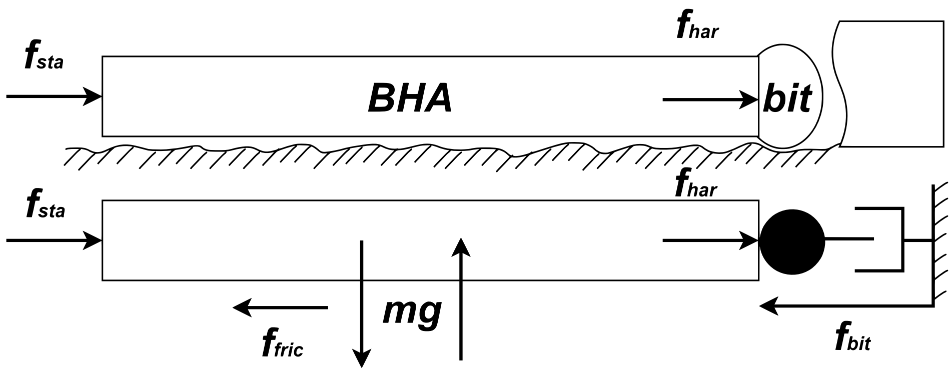

Figure 1: The sketch of forces acting on a drill string

where m_{bit} is the mass of the bit, concentrated at the point $x = L$, $\mu(x)$ is the coefficient of friction against the rock, g is gravitational acceleration.

The static compressive force at the right end of the column is determined in an exponential form as

$$f_{bit}(\dot{u}(x,t)) = \begin{cases} (c_1 exp(-c_2\dot{u}(x,t)) - c_1)\delta(x-L) & for\ \dot{u}(L,t) > 0, \\ 0 & for\ \dot{u}(L,t) \le 0 \end{cases} \tag{5}$$

where c_1, c_2 are the coefficients of the bit-rock interaction.

The mathematical model Eq. (1-5) was solved by T.G. Ritto et al. in the work [1] by the finite element method. Here, the authors of the work, as in [20], use LPM, which is an effective method for the numerical analysis of such dynamical systems. Due to the inhomogeneity of the drill string loading, the mathematical model is written in accordance with the drilling equipment loading scheme (Figure 1) as follows:

$$\rho A \frac{\partial^2 u(x,t)}{\partial t^2} - EA \frac{\partial^2 u(x,t)}{\partial x^2} = f_{fric}(\dot{u}(x,t)) \tag{6}$$

with the boundary conditions

$$\begin{aligned} x = 0: & \qquad EA\frac{\partial u}{\partial x} = -F_{sta}, \\ x = L: & \quad EA\frac{\partial u}{\partial x} = f_{har}(x,t)) + f_{mass}(\ddot{u}(x,t)) + f_{bit}(\dot{u}(x,t)). \end{aligned} \tag{7}$$

The metric is introduced in spatial and time coordinates:

$$u = L\text{U},\ x = L\text{X},\ t = \frac{\tau}{c},\ c = \sqrt{\frac{E}{\rho L^2}} \tag{8}$$

Approximate the derivatives according to the LPM used here:

$$\left(\frac{\partial^2 \text{U}}{\partial \text{X}^2}\right)_j = 2\frac{\Delta \text{X}_{j+1}\text{U}_{j-1} - (\Delta \text{X}_j + \Delta \text{X}_{j+1})\text{U}_j + \Delta \text{X}_j \text{U}_{j+1}}{\Delta \text{X}_{j+1}\Delta \text{X}_j(\Delta \text{X}_j + \Delta \text{X}_{j+1})} \tag{9}$$

$$\left(\frac{\partial \text{U}}{\partial \text{X}}\right)_j = \frac{\text{U}_j - \text{U}_{j-1}}{\Delta \text{X}_j}, \ \left(\frac{\partial \text{U}}{\partial \text{X}}\right)_j = \frac{\text{U}_{j+1} - \text{U}_j}{\Delta \text{X}_{j+1}} \tag{10}$$

$$\left(\frac{\partial \text{U}}{\partial \text{X}}\right)_j = \frac{\text{U}_{j+1} - \text{U}_{j-1}}{\Delta \text{X}_{j+1} + \Delta \text{X}_j} \tag{11}$$

where $\Delta \text{X}_j = \text{X}_j - \text{X}_{j-1}$, $\text{X}_j = \begin{cases} (2j-1)l & for\ 1 \le j \le N \\ 1 & for\ j = N+1 \end{cases}$, $2l = \dfrac{1}{N-1}$, N is the number of the drill string splits.

The model Eq. (6) and its boundary conditions Eq. (7) are represented in the discrete form:

$$\begin{aligned} \frac{\partial^2 \text{U}_1}{\partial \tau^2} - \frac{1}{3l^2}(2\text{U}_0 - 3\text{U}_1 + \text{U}_2) &= -\frac{\mu g}{Lc^2} sgn(\dot{\text{U}}_1)\ for\ j = 1, \\ \frac{\partial^2 \text{U}_j}{\partial \tau^2} - \frac{1}{4l^2}(\text{U}_{j-1} - 2\text{U}_j + \text{U}_{j+1}) &= -\frac{\mu g}{Lc^2} sgn(\dot{\text{U}}_j)\ for\ j = \overline{2, N-2} \\ \frac{\partial^2 \text{U}_{N-1}}{\partial \tau^2} - \frac{1}{3l^2}(\text{U}_{N-2} - 3\text{U}_{N-1} + 2\text{U}_N) &= -\frac{\mu g}{Lc^2} sgn(\dot{\text{U}}_N)\ for\ j = N-1 \end{aligned} \tag{12}$$

$$\begin{aligned} \text{X} = 0 : \text{U}_1 - \text{U}_0 &= -\frac{lF_{sta}}{EA} \\ \text{X} = 1 : \frac{\partial^2 \text{U}_N}{\partial \tau^2} + \frac{(\rho A)L}{m_{bit}} \frac{(\text{U}_N - \text{U}_{N-1})}{l} &= \frac{F_0}{m_{bit} Lc^2} sin\left(\frac{\omega_f}{c}\tau\right) + \frac{1}{m_{bit} Lc^2} f_{bit}(Lc\dot{\text{U}}_N) \end{aligned} \tag{13}$$

As a result, the system of N nonlinear second-order ordinary differential equations with respect to time with one algebraic expression is obtained.

3 Numerical analysis of the model

The numerical analysis of the model was carried out by the fourth-order Runge-Kutta method. The algorithm and the program code for numerical modelling have been developed in the C++ programming language.

The values of the physical and geometric parameters of the drill string, the indicators of the acting loads were taken in accordance with the author's values of [1]: $E = 2.1 \cdot 10^{11}$Pa, $\rho = 7850$kg·m^{-3}, $g = 9.81$m·s^{-2}, $D_i = 0.10$m (inner diameter), $D_o = 0.15$m (outer diameter), $\dfrac{L}{D_o} = 400$, $m_{bit} = 20$kg, $c_1 = 1.4 \cdot 10^3$N, $c_2 = 400$, $\mu = 0.1$, $\omega_f = 100 \cdot \dfrac{2\pi}{60}$rad·s^{-1}, $t \in [0, 10]$s, $\Delta t = 0.0001$s, $f_{sta} = 5500$N, $F_0 = 550$N.

To evaluate the efficiency of the drilling rig, the ratio of the input power of the drill to the power of the drill at the output was used:

$$\begin{aligned} p_{in}(t) &= f_{sta}\dot{u}(0,t) + f_{har}\dot{u}(L,t) \\ p_{out}(t) &= f_{bit}\dot{u}(L,t) \end{aligned} \tag{14}$$

where $p_{in}(t)$ is the input power, $p_{out}(t)$ is the output power.

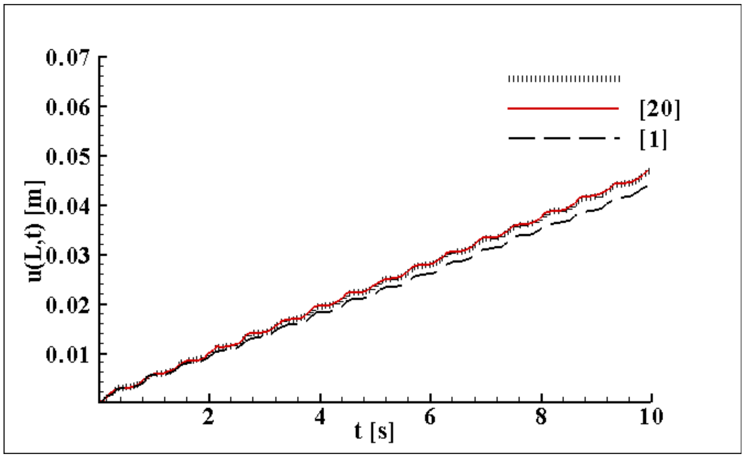

Figure 2: Verification of the obtained results of the longitudinal displacement.

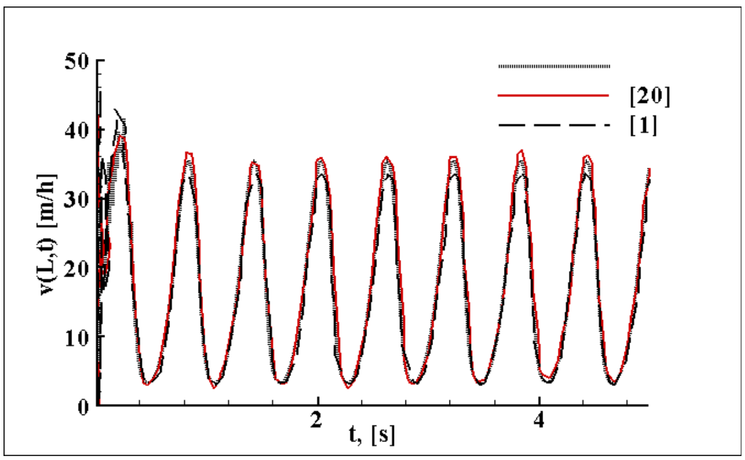

Figure 3: Verification of the obtained results of the bit speed.

The number of the drill string splitting nodes was taken, as in [20], equal to $N = 101$.

The research results, which are longitudinal displacement of the drill string at an interval of 10 s, are shown in Figure 2. The data of the bit speed are demonstrated in Figure 3.

The verification of the obtained results with the results of works [1] and [20] was carried out. In [1], the numerical modelling of the drill string motion was realized by the finite element method. The authors of [20] used LPM, and the numerical solution of the mathematical model was found in the symbolic mathematics package Wolfram Mathematica (WM). Here, numerical modelling was conducted in C++.

It was found that the longitudinal displacement of the drill string at the point $x = L$ increases with time, and the speed of the drill string at the right end is oscillating. It is caused by the presence of loads on the drilling equipment in the model.

The dashed black line shows the results of T.G. Ritto [1], solid red line is the results of L. Khajiyeva, A. Sergaliyev [20], dotted black one is the results of this work.

It is visually clear that the graphs in both figures are qualitatively convergent.

Longitudinal displacements grown by red and dotted black lines coincide completely, while in comparison with dashed black one, the error increases with time. The amplitudes of the speed of motion depicted by red and dotted black lines are slightly higher than the amplitude of dashed black line, which may be caused by the digitization error, the use of various numerical methods, or an insufficient number of nodes in the discrete model.

In Figure 4 the change in the ratio between the output and input power is shown. The higher this ratio, the more efficient the drilling rig is. It can be seen from the graph that this indicator of the drill string does not exceed 25%, which is explained by the fact that the model takes into account the loads affecting the equipment, which are friction forces, the reaction force of the rock on the drill, static compressive force, gravitational forces, etc. The dashed black line shows the results of T.G. Ritto, solid red line is the results of L. Khajiyeva, A. Sergaliyev, dotted black one is the results of this work. Good consistency of the results is observed.

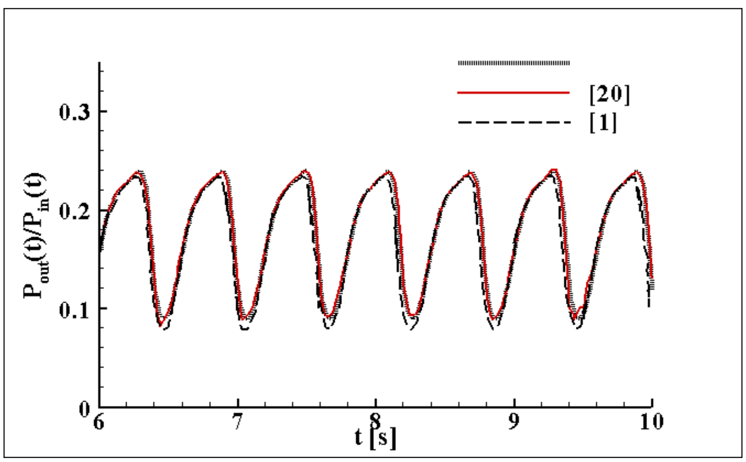

Figure 4: Verification of the obtained results of the ratio between the input and output power

4 Dimension analysis of the discrete ODE system

Obviously, the calculation accuracy depends on the choice of the number of points for dividing the drill string along the length: the spatial steps l decrease with an increase in the nodes in space, the discrete system tends to the continuity equation. However, with an increase in the number of partitions, the program implementation time also increases. This requires additional analysis of the dependence of the computational accuracy on the number of nodes N and the time spent on executing the program code.

The results of T.G. Ritto, who first considered this problem, were taken as a sample to estimate the calculation error. For algebraic verification, a WPF Application was written in the C# language. It compares the digitized data of the work [1] with the results of this work and finds the difference in the data of the loaded files at the closest possible time points.

The results of the longitudinal displacement of the drill string were taken as comparative data. Tables 1 presents the results showing the effect of the number of split points on the calculation error. It is relevant to notice, the accuracy of the results is influenced by the quality of the digitized data from the test graph; the error of the time points at which the difference in the results is located (this indicator does not exceed the time step $dt = 1e-5$s); the error of the used numerical methods.

Tables 1 shows that the best convergence values were obtained by splitting the column into 1000 segments: the maximum error does not exceed 0.39 mm, while the computation time is no more than 8.5 minutes.

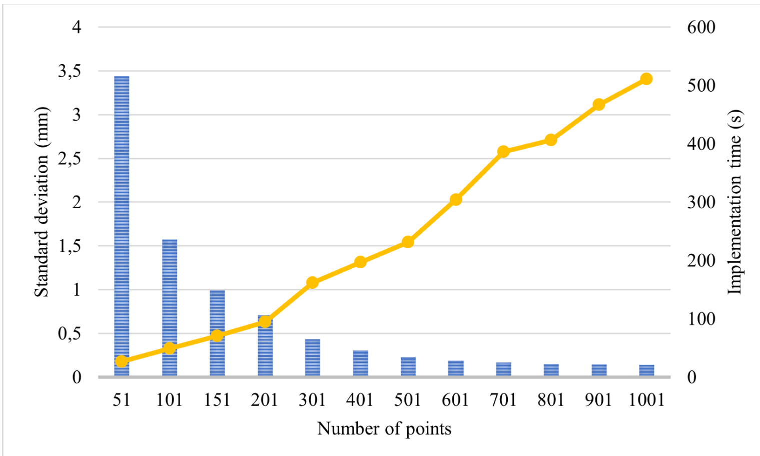

Figure 5: Influence of the number of column splits on the error and implementation time

Table 1: Analysis of the influence of the number of partitioning nodes on time and computation error.

The number of nodes	11	101	201	401	601	801	1001
Maximum error (mm)	47.3475	2.837	1.3428	0.68111	0.46391	0.3847	0.3922
Standard deviation (mm)	26.5089	1.573	0.7076	0.304	0.19071	0.1513	0.1392
Time implementation (s)	6.667	48.9	94.837	196.707	304.165	405.95	510.85

Figure 5 clearly demonstrates the need to use more points, where the bar graph corresponds to the standard deviation for a particular number of splits, and the graph depicts the implementation time. Note that the error for 101 points is more than 1.5 mm, therefore, more than 300 nodes are required to obtain quantitatively accurate values.

If the priority of the research is the accuracy of the calculation with a sufficient amount of time resources, splitting into 1000 or more parts is the most appropriate.

5 Optimization of the numerical algorithm using parallel programming tools

A small time step, the need to use a large number of partitions and, as a consequence, a large number of iterations served as factors for the next stage of the study which is optimization of the program code using parallel programming tools. Parallelization of the C++ code was implemented using the Open Multi-Processing (OpenMP) API. The OpenMP technology, designed for shared memory systems, implements parallelism of calculations due

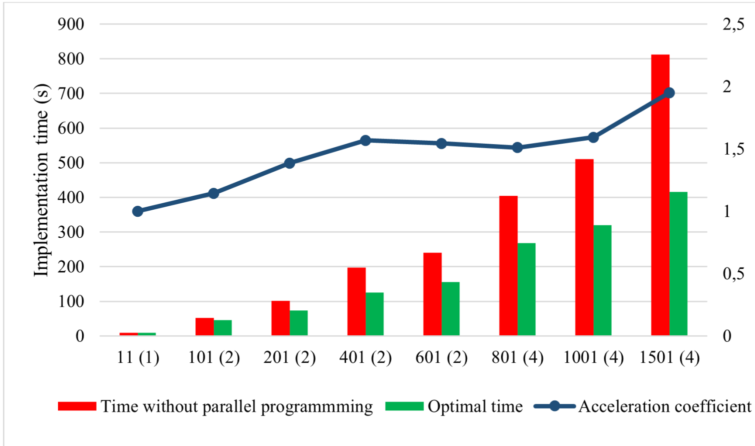

Figure 6: Acceleration coefficient for a different number of points

to multithreading. The master thread creates a number of threads, the task is distributed among them. Due to this technology, the logic of the code does not change compared with MPI, oriented to distributed memory systems, where it is necessary to determine connections between processes. OpenMP allows to find "vulnerable"places in the program and significantly speed up the execution of these blocks, alternating them with a sequential part. In particular, this approach is applied to linear algorithms, which include the fourth-order Runge-Kutta method.

To analyze the advantages of using parallel programming, spent time resources, and the optimal number of threads, the program code was tested for various values of the parameters of the partition nodes and number of threads on the interval of $t = 10$s.

The test results are clearly shown in Figure 6, where the values of the acceleration factor of the program using the OpenMP library are presented for a different number of points. The bar chart shows the implementation time of the code, where the red columns correspond to the execution time of the code by one thread, that is, without using parallel computations, the green columns correspond to the time of the optimal number of threads (in parentheses next to the number of nodes). The line graph shows the acceleration factor as the ratio of the time taken by one thread at the optimal time. Thus, for a smaller number of points, one stream is optimal, but with an increase in the number of points, the use of parallel computation is justified.

It is worth noting that in the further, considering a more complex model and complicating the computational algorithm, using a larger number of nodes, the efficiency indicators will increase accordingly.

6 Conclusion

During the research of the dynamics of longitudinal vibrations of a horizontal drill string the optimal number of the drill string splits by LPM using the developed software in the C# language and parallel programming tools was found. The optimal number of the drill string splits in terms of “implementation time-calculation error” varies within the range of 400-600 nodes. It improves the accuracy of the solution in comparison with the case of splitting the string into 100 elements [20]. A good agreement between the obtained results and the results of T.G. Ritto's work [1] based on the FEM has been established.

In addition, the numerical algorithm implemented in the C++ language allows further refinement of solutions by increasing the number of the drill string splits. At the same time, the increase in the dimension of the discrete lumped model is successfully implemented through the use of parallel programming. Comparative analysis showed the justification of its application for optimization of the numerical algorithm.

In the future, this work of the authors is seen in the use of LPM in modelling nonlinear vibrations of vertical drill strings with spatial type of deformation, inhomogeneous structure, inhomogeneity of loading due to local and point loads, etc.

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