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E.A. Bakirova[1,2*], **Zh.M. Kadirbayeva**[1,3], **A.N Nessipbayeva**[2]
[1]Institute of Mathematics and Mathematical Modeling, Kazakhstan, Almaty
[2]Kazakh National Women's Teacher Training University, Kazakhstan, Almaty
[3]International Information Technology University, Kazakhstan, Almaty
*e-mail: bakirova1974@mail.ru

MODIFICATION OF THE PARAMETRIZATION METHOD FOR SOLVING A BOUNDARY VALUE PROBLEM FOR LOADED DEPCAG

The functional differential equation plays important role in mathematical modeling of biological problems. In the present research work, we investigate a boundary value problem (BVP) for a functional differential equation. This equation includes loaded terms and a term with generalized piecewise constant argument. We apply a modified version of the Dzhumabaev parameterization method. The method's goal is to lead the original problem into an equivalent multi-point BVP for ordinary differential equations with parameters, which is composed of a problem with initial and additional conditions. The multi-point BVP is leaded to a system of linear algebraic equations in parameters, which are introduced as the values of the desired solution at the dividing points. The found parameters are plugged into auxiliary Cauchy problems on the partition subintervals, whose solutions are the restrictions of the solution to the original problem. The obtained results are verified by a numerical example. Numerical analysis showed high efficiency of the constructed modified version of the Dzhumabaev parameterization method.
Key words: load, piecewise-constant argument, two-point boundary value problem, parametrization method, numerical solution.

Э.А. Бакирова[1,2*], Ж.М. Қадырбаева[1,3], А.Н. Несiпбаева[2]
[1]Математика және математикалық моделдеу институты, Қазақстан, Алматы қ.
[2]Қазақ ұлттық қыздар педагогикалық университетi , Қазақстан, Алматы қ.
[3]Халықаралық ақпараттық технологиялар университетi, Қазақстан, Алматы қ.
*e-mail: bakirova1974@mail.ru

Жалпыланған түрдегi бөлiктi-тұрақты аргументi бар жүктелген дифференциалдық теңдеу үшiн шеттiк есептi шешудiң параметрлеу әдiсiнiң модификациясы

Функционалдық-дифференциалдық теңдеу биологиялық есептердi математикалық модельдеуде маңызды рөл атқарады. Осы жұмыста функционалдық-дифференциалдық теңдеу үшiн шеттiк есеп (ШЕ) қарастырылады. Бұл теңдеу жүктелген мүшелер мен жалпыланған түрдегi бөлiктi-тұрақты аргументi бар қосылғыштан тұрады. Жұмабаевтың параметрлеу әдiсiнiң модификацияланған нұсқасы қолданылады. Әдiстiң мақсаты - берiлген есептi бастапқы және қосымша шарттардан тұратын эквиваленттi параметрлерi бар жәй дифференциалдық теңдеулер жүйесi үшiн көп нүктелi ШЕ келтiрiлуi болып табылады. Көп нүктелi ШЕ бөлу нүктелерiнде iзделiндi шешiмнiң мәнi ретiнде енгiзiлетiн параметрлерi бар сызықтық алгебралық теңдеулер жүйесiне келтiрiледi. Табылған параметрлер бөлiктеудiң iшкi интервалдарындағы қосымша Коши есептерiне қойылады, олардың шешiмдерi бастапқы шеттiк есептiң шешiмдерiнiң сығылуы болып табылады. Алынған нәтижелер сандық мысалмен тексерiледi. Сандық талдау Жұмабаевтың параметрлеу әдiсiнiң құрастырылған модификациясының жоғары тиiмдiлiгiн көрсеттi.
Түйiн сөздер: жүктеу, бөлiктiтұрақты аргумент, екi нүктелi шеттiк есеп, параметрлеу әдiсi, сандық шешiм.

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Э.А. Бакирова[1,2*], Ж.М. Кадирбаева[1,3], А.Н. Несипбаева[2]
[1]Институт математики и математического моделирования, Казахстан, г. Алматы
[2]Казахский национальный женский педагогический университет, Казахстан, г. Алматы
[3]Международный университет информационных технологий, Казахстан, г. Алматы
*e-mail: bakirova1974@mail.ru

Модификация метода параметризации решения краевой задачи для нагруженных дифференциальных уравнений с кусочно-постоянным аргументом обобщенного типа

Функционально-дифференциальное уравнение играет важную роль в математическом моделировании биологических задач. В настоящей работе исследуется краевая задача (КЗ) для функционально-дифференциального уравнения. В это уравнение входят нагруженные члены и член с обобщенным кусочно-постоянным аргументом. Применим модифицированный вариант метода параметризации Джумабаева. Цель метода - привести исходную задачу к эквивалентной многоточечной КЗ для обыкновенных дифференциальных уравнений с параметрами, состоящей из задачи с начальными и дополнительными условиями. Многоточечная КЗ приводится к системе линейных алгебраических уравнений с параметрами, которые вводятся как значения искомого решения в точках деления. Найденные параметры подставляются во вспомогательные задачи Коши на подинтервалах разбиения, решения которых являются сужениями решения исходной задачи. Полученные результаты проверяются на численном примере. Численный анализ показал высокую эффективность построенной модифицированной версии метода параметризации Джумабаева.
Ключевые слова: нагрузка, кусочно-постоянный аргумент, двухточечная краевая задача, метод параметризации, численное решение.

1 Introduction and preliminaries

The theory's creators, K. Cook, J. Wiener and S. Busenberg, suggested using differential equations with the piecewise constant argument for investigations in [1], [2]. Within the final four decades, numerous interesting results have been found, and applications have been realized in this theory. Numerous additional theoretical issues, such as existence and uniqueness of solutions, oscillations and stability, integral manifolds and periodic solutions, as well as many more, have been thoroughly discussed. Information about differential equations with piecewise constant argument of generalized type (DEPCAG) can be found in books [3], [4] and papers [5], [6].

This article's basic objective is to broaden the modification of Dzhumabaev parametrization method [7], [8] to the boundary value problem for the system of loaded DEPCAG. For this purpose, we have developed computational method solving a boundary-value problem for the system of loaded DEPCAG.

Loaded differential equations (LDE) were investigated in [9], [10] and the references therewith. Numerous problems for LDE and methods for solving problems for LDE are considered in [11]– [16].

We consider the following system of loaded DEPCAG

$$\frac{dx}{dt} = A_0(t)x + K(t)x(\gamma(t)) + \sum_{i=1}^{m+1} M_i(t)x(\theta_{i-1}) + f(t), \quad x \in R^n, \quad t \in (0,T), \tag{1}$$

subject to the two-point boundary condition

$$B_0x(0) + C_0x(T) = d, \quad d \in R^n, \tag{2}$$

where $A_0(t)$, $K(t)$, $M_i(t)$, $(i=\overline{1,m+1})$, are of dimensions $(n\times n)$ and are continuous on $[0,T]$, and the n-vector-function $f(t)$ are piecewise continuous on $[0,T]$ with possible discontinuities of the first kind at the points $t=\theta_j$, $(j=\overline{1,m})$; B_0 and C_0 are $(n\times n)$ constant matrices, $\|x\|=\max\limits_{i=\overline{1,n}}|x_i|$.

The argument $\gamma(t)$ is a step function defined as $\gamma(t)=\xi_{i-1}$ if $t\in[\theta_{i-1},\theta_i)$, $i=\overline{1,m+1}$; $\theta_{i-1}<\xi_{i-1}<\theta_i$ for all $i=\overline{1,m+1}$; where $0=\theta_0<\theta_1<...<\theta_m<\theta_{m+1}=T$.

A function $x(t)$ is called a solution to problem (1) and (2) if:

(i) the $x(t)$ is continuous on $[0,T]$;

(ii) the $x(t)$ is differentiable on $[0,T]$ with the possible exception of the θ_j, $j=\overline{0,m}$, where the one-sided derivatives exist;

(iii) the $x(t)$ satisfies (1) on each interval (θ_{i-1},θ_i), $i=\overline{1,m+1}$; at the θ_j, Eq. (1) is satisfied by the right-hand derivatives of $x(t)$;

(iv) the $x(t)$ satisfies the boundary condition (2).

2 Materials and methods

We employ the approach proposed in [17] to solve the boundary-value problem for the system of loaded DEPCAG (1) and (2). This approach is based on the algorithms of the modified version of the Dzhumabaev parameterization method and numerical methods for solving Cauchy problems.

By using loading points, the interval $[0,T]$ is split into subintervals: $[0,T)=\bigcup\limits_{s=1}^{m+1}[\theta_{s-1},\theta_s)$.

$C([0,T],\theta,\mathbb{R}^{n(m+1)})$ be the space of functions systems $x[t]=(x_1(t),x_2(t),\dots,x_{m+1}(t))'$, where $x_s:[\theta_{s-1},\theta_s)\to\mathbb{R}^n$ are continuous and have finite left-hand side limits $\lim\limits_{t\to\theta_s-0}x_s(t)$, $s=\overline{1,m+1}$ with norm $\|x[\cdot]\|_2=\max\limits_{s=\overline{1,m+1}}\sup\limits_{t\in[\theta_{s-1},\theta_s)}|x_s(t)|$.

Denote by $x_r(t)$ a restriction of function $x(t)$ on r-th interval $[\theta_{r-1},\theta_r)$, i.e.

$x_r(t)=x(t)$ for $t\in[\theta_{r-1},\theta_r)$, $r=\overline{1,m+1}$.

Then the function system $x[t]=(x_1(t),x_2(t),\dots,x_{m+1}(t))\in C([0,T],\theta,R^{n(m+1)})$, and its elements $x_r(t)$, $r=\overline{1,m+1}$, satisfy the following boundary value problem for system of loaded DEPCAG

$$\frac{dx_r}{dt}=A_0(t)x_r+K(t)x_r(\xi_{r-1})+\sum_{i=1}^{m+1}M_i(t)x_i(\theta_{i-1})+f(t),\quad t\in[\theta_{r-1},\theta_r),\quad r=\overline{1,m+1},\quad (3)$$

$$B_0x_1(0)+C_0\lim_{t\to T-0}x_{m+1}(t)=d,\quad (4)$$

$$\lim_{t\to\theta_s-0}x_s(t)=x_{s+1}(\theta_s),\quad s=\overline{1,m}.\quad (5)$$

Introduce parameters $\lambda_r=x_r(\theta_{r-1})$ and $\mu_r=x_r(\xi_{r-1})$ for all $r=\overline{1,m+1}$. The following problem with parameters is obtained by substituting $v_r(t)=x_r(t)-\lambda_r$ on every r-th interval

$[\theta_{r-1},\theta_r)$:

$$\frac{dv_r}{dt}=A_0(t)(v_r+\lambda_r)+K(t)\mu_r+\sum_{i=1}^{m+1}M_i(t)\lambda_i+f(t),\quad t\in[\theta_{r-1},\theta_r), \tag{6}$$

$$v_r(\theta_{r-1})=0,\qquad r=\overline{1,m+1}, \tag{7}$$

$$B_0\lambda_1+C_0\lambda_{m+1}+C_0\lim_{t\to T-0}v_{m+1}(t)=d, \tag{8}$$

$$\lambda_s+\lim_{t\to\theta_s-0}v_s(t)=\lambda_{s+1},\quad s=\overline{1,m}, \tag{9}$$

$$\mu_r=v_r(\xi_{r-1})+\lambda_r,\qquad r=\overline{1,m+1}. \tag{10}$$

A solution to problem (6)–(10) is a triple $(\lambda^*,\mu^*,v^*[t])$, with elements $\lambda^*=(\lambda_1^*,\lambda_2^*,\ldots,\lambda_{m+1}^*)$, $\mu^*=(\mu_1^*,\mu_2^*,\ldots,\mu_{m+1}^*)$, $v^*[t]=\big(v_1^*(t),v_2^*(t),\ldots,v_{m+1}^*(t)\big)$, where $v_r^*(t)$ are continuously differentiable on $[\theta_{r-1},\theta_r)$, $r=\overline{1,m+1}$, and satisfying the system (6), conditions (7)- (10) at the $\lambda_r=\lambda_r^*$, $\mu_r=\mu_r^*$, $j=\overline{1,m+1}$.

The original problem (1), (2) and problem with parameters (6)–(10) are equivalent.

Let consider $\Phi_r(t)$ a fundamental matrix of the differential equation $\dfrac{dx_r}{dt}=A_0(t)x_r(t)$ on $[\theta_{r-1},\theta_r]$, $r=\overline{1,m+1}$.

Consequently, the solution to the Cauchy problem (6), (7) may be expressed as follows

$$v_r(t)=\Phi_r(t)\int_{\theta_{r-1}}^{t}\Phi_r^{-1}(\tau)\Big[A_0(\tau)\lambda_r+K(\tau)\mu_r+\sum_{i=1}^{m+1}M_i(\tau)\lambda_i\Big]d\tau+$$
$$+\Phi_r(t)\int_{\theta_{r-1}}^{t}\Phi_r^{-1}(\tau)f(\tau)d\tau,\quad t\in[\theta_{r-1},\theta_r),\quad r=\overline{1,m+1}. \tag{11}$$

Consider the Cauchy problems for ordinary differential equations on the subintervals

$$\frac{dy}{dt}=A_0(t)y+D(t),\qquad y(\theta_{r-1})=0,\qquad t\in[\theta_{r-1},\theta_r],\qquad r=\overline{1,m+1}, \tag{12}$$

where $P(t)$ is a $(n\times n)$-matrix or a n-vector, piecewise continuous on $[0,T]$ with possible discontinuities of the first kind at the $t=\theta_j$, $(j=\overline{1,m})$. On each r-th interval, denote by $P_r(D,t)$ a unique solution to the Cauchy problem (12). The uniqueness of the solution to the Cauchy problem yields

$$P_r(D,t)=\Phi_r(t)\int_{\theta_{r-1}}^{t}\Phi_r^{-1}(\tau)D(\tau)d\tau,\qquad t\in[\theta_{r-1},\theta_r],\qquad r=\overline{1,m+1}. \tag{13}$$

The following system of linear algebraic equations is get by substituting the right-hand side of (11) using (13) into conditions (8)-(10):

$$B_0\lambda_1 + C_0\lambda_{m+1} + C_0P_{m+1}(A_0,T)\lambda_{m+1} + C_0P_{m+1}(K,T)\mu_{m+1} + \\ + C_0\sum_{i=1}^{m+1} P_i(M_i,T)\lambda_i = d - C_0P_{m+1}(f,T), \quad (14)$$

$$\lambda_s + P_s(A_0,\theta_s)\lambda_s + P_s(K,\theta_s)\mu_s + \sum_{i=1}^{m+1} P_i(M_i,\theta_s)\lambda_i - \lambda_{s+1} = -P_s(f,\theta_s), \quad s=\overline{1,m}, \quad (15)$$

$$\mu_r - P_r(K,\xi_{r-1})\mu_r - P_r(A_0,\xi_{r-1})\lambda_r - \lambda_r - \\ - \sum_{i=1}^{m+1} P_i(M_i,\xi_{r-1})\lambda_i = P_r(f,\xi_{r-1}), \quad r=\overline{1,m+1}. \quad (16)$$

Symbolized by $Q(\theta)$ - $\Big(2n(m+1)\times 2n(m+1)\Big)$ matrix corresponding to the system's left side (14) - (16) and write the system as

$$Q(\theta)(\lambda,\mu) = F(\theta), \qquad \lambda\in\mathbb{R}^{n(m+1)}, \quad \mu\in\mathbb{R}^{n(m+1)}, \quad (17)$$

where $(\lambda,\mu) = \big(\lambda_1,\lambda_2,...,\lambda_{m+1},\mu_1,\mu_2,...,\mu_{m+1}\big)'$,
$F(\theta) = \big(d - C_0P_{m+1}(f,T), -P_1(f,\theta_1), -P_2(f,\theta_2),\ldots,-P_m(f,\theta_m), P_1(f,\xi_0), P_2(f,\xi_1),\ldots,$
$P_{m+1}(f,\xi_m)\big)\in\mathbb{R}^{2n(m+1)}$.

It is simple to establish that the solvability of the boundary value problem (1) and (2) is equivalent to the solvability of the system (17). The solution of the system (17) is a pair of vectors $(\lambda,\mu) = \Big(\lambda_1,\lambda_2,\ldots,\lambda_{m+1},\mu_1,\mu_2,...,\mu_{m+1}\Big)'\in R^{2n(m+1)}$ consists of the values of the solutions of the problem (1) and (2), i.e. $\lambda_r = x(\theta_{r-1})$, $\mu_r = x(\xi_{r-1})$, $r=\overline{1,m+1}$.

3 The Main results

We offer the following formulation of an algorithm for solving problem (1) and (2) based on the solving of Cauchy problems.

Step 1. Split up each r-th interval $[\theta_{r-1},\theta_r]$, $r=\overline{1,m+1}$, into N_r parts. Determine the approximate values of coefficients and system's right side (17) of via solutions to the following Cauchy matrix and vector problems obtained using the fourth-order Runge-Kutta method with step $h_r = (\theta_r-\theta_{r-1})/N_r$, $r=\overline{1,m+1}$:

$$\frac{dy}{dt} = A_0(t)y + A_0(t), \qquad y(\theta_{r-1}) = 0, \quad t\in[\theta_{r-1},\theta_r], \quad r=\overline{1,m+1},$$

$$\frac{dy}{dt} = A_0(t)y + K(t), \qquad y(\theta_{r-1}) = 0, \qquad t\in[\theta_{r-1},\theta_r], \quad r=\overline{1,m+1},$$

$$\frac{dy}{dt} = A_0(t)y + M_i(t), \qquad y(\theta_{r-1}) = 0, \quad t \in [\theta_{r-1}, \theta_r], \quad i = \overline{1, m+1}, \quad r = \overline{1, m+1},$$

$$\frac{dy}{dt} = A_0(t)y + f(t), \qquad y(\theta_{r-1}) = 0, \qquad t \in [\theta_{r-1}, \theta_r], \quad r = \overline{1, m+1}.$$

Step 2. Then we have the approximate system of algebraic equations with respect to parameters λ and μ:

$$Q_*(\theta)(\lambda^*, \mu^*) = F_*(\theta), \qquad \lambda^* \in \mathbb{R}^{n(m+1)}, \quad \mu^* \in \mathbb{R}^{n(m+1)}. \tag{18}$$

Solve the system (18) and we find $(\lambda^*, \mu^*) = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_{m+1}^*, \mu_1^*, \mu_2^*, \ldots, \mu_{m+1}^*)' \in \mathbb{R}^{2n(m+1)}$. Note that the elements of λ^* and μ^* are the values of the solution to problem (1) and (2): $\lambda_r^* = x^*(\theta_{r-1})$, $\mu_r^* = x^*(\xi_{r-1})$, $r = \overline{1, m+1}$.

Step 3. Solve the following Cauchy problems

$$\frac{dy}{dt} = A_0(t)y + K(t)\mu_r^* + \sum_{i=1}^{m+1} M_i(t)\lambda_i^* + f(t),$$

$$y(\theta_{r-1}) = \lambda_r^*, \qquad t \in [\theta_{r-1}, \theta_r), \quad r = \overline{1, m+1},$$

and determine the values of the solution $x^*(t)$ at the remaining points of the subintervals.

Hence, the offered algorithm provides us with the numerical solution to the problem for the system of loaded DEPCAG (1) and (2).

Consider the following example to demonstrate the proposed approach of the numerical solving of problem (1) and (2) based on the modification of Dzhumabaev parametrization method.

4 Example

We consider the problem for the system of loaded DEPCAG:

$$\frac{dx}{dt} = \begin{pmatrix} t & t^2 \\ 4t^3 & 4 \end{pmatrix} x + \begin{pmatrix} t^3 & t+3 \\ 2 & t^2 \end{pmatrix} x(\gamma(t)) +$$
$$+ \begin{pmatrix} 2 & t-4 \\ t^3 & 3t \end{pmatrix} x(\theta_0) + \begin{pmatrix} 6t^2 & 3 \\ -6t & 1 \end{pmatrix} x(\theta_1) + f(t), \quad x \in R^2, \quad t \in (0, T), \tag{19}$$

$$\begin{pmatrix} 2 & 5 \\ -7 & 1 \end{pmatrix} x(0) + \begin{pmatrix} -7 & 1 \\ 1 & 9 \end{pmatrix} x(T) = \begin{pmatrix} -36 \\ -29 \end{pmatrix}, \tag{20}$$

where $\theta_0 = 0, \quad \theta_1 = \frac{1}{2}, \quad \theta_2 = T = 1,$

$$\gamma(t) = \zeta_0 = \frac{1}{4}, \quad f(t) = \begin{pmatrix} \frac{57}{8}t^3 - 9t^4 + 18t^2 - \frac{129}{16}t + \frac{97}{16} \\ 16t^5 - 32t^6 - 5t^3 - \frac{49}{16}t^2 + 24t - 4 \end{pmatrix}, \quad t \in \left[0, \frac{1}{2}\right),$$

$$\gamma(t) = \zeta_1 = \frac{3}{4}, \quad f(t) = \begin{pmatrix} \frac{47}{8}t^3 - 9t^4 + 18t^2 - \frac{105}{16}t + \frac{169}{16} \\ 16t^5 - 32t^6 - 5t^3 - \frac{25}{16}t^2 + 24t - \frac{13}{2} \end{pmatrix}, \quad t \in \left[\frac{1}{2}, 1\right).$$

Here we have two subintervals: $\left[0,\frac{1}{2}\right)$, $\left[\frac{1}{2},1\right)$. Applying the scheme of the modification of Dzhumabaev parametrization method, introduce parameters $\lambda_1 = x_1(0)$, $\lambda_2 = x_2\left(\frac{1}{2}\right)$, $\mu_1 = x_1\left(\frac{1}{4}\right)$, $\mu_2 = x_2\left(\frac{3}{4}\right)$. Making the substitution

$$v_1(t) = x_1(t) - \lambda_1, \quad t \in \left[0,\frac{1}{2}\right), \quad v_2(t) = x_2(t) - \lambda_2, \quad t \in \left[\frac{1}{2},1\right),$$

we get the boundary value problem with parameters:

$$\frac{dv_1}{dt} = \begin{pmatrix} t & t^2 \\ 4t^3 & 4 \end{pmatrix}(v_1+\lambda_1) + \begin{pmatrix} t^3 & t+3 \\ 2 & t^2 \end{pmatrix}\mu_1 + \begin{pmatrix} 2 & t-4 \\ t^3 & 3t \end{pmatrix}\lambda_1 + \begin{pmatrix} 6t^2 & 3 \\ -6t & 1 \end{pmatrix}\lambda_2 + \\ + \begin{pmatrix} \frac{57}{8}t^3 - 9t^4 + 18t^2 - \frac{129}{16}t + \frac{97}{16} \\ 16t^5 - 32t^6 - 5t^3 - \frac{49}{16}t^2 + 24t - 4 \end{pmatrix}, \quad t \in \left[0,\frac{1}{2}\right), \tag{21}$$

$$v_1(0) = 0, \tag{22}$$

$$\frac{dv_2}{dt} = \begin{pmatrix} t & t^2 \\ 4t^3 & 4 \end{pmatrix}(v_2+\lambda_2) + \begin{pmatrix} t^3 & t+3 \\ 2 & t^2 \end{pmatrix}\mu_2 + \begin{pmatrix} 2 & t-4 \\ t^3 & 3t \end{pmatrix}\lambda_1 + \begin{pmatrix} 6t^2 & 3 \\ -6t & 1 \end{pmatrix}\lambda_2 + \\ + \begin{pmatrix} \frac{47}{8}t^3 - 9t^4 + 18t^2 - \frac{105}{16}t + \frac{169}{16} \\ 16t^5 - 32t^6 - 5t^3 - \frac{25}{16}t^2 + 24t - \frac{13}{2} \end{pmatrix}, \quad t \in \left[\frac{1}{2},1\right), \tag{23}$$

$$v_2\left(\frac{1}{2}\right) = 0, \tag{24}$$

$$\begin{pmatrix} 2 & 5 \\ -7 & 1 \end{pmatrix}\lambda_1 + \begin{pmatrix} -7 & 1 \\ 1 & 9 \end{pmatrix}\lambda_2 + \begin{pmatrix} 7 & 1 \\ 1 & 9 \end{pmatrix}\lim_{t\to 1-0} v_2(t) = \begin{pmatrix} -36 \\ -29 \end{pmatrix}, \tag{25}$$

$$\lambda_1 + \lim_{t\to\theta_1-0} v_1(t) = \lambda_2, \tag{26}$$

$$\mu_1 = v_1\left(\frac{1}{4}\right) + \lambda_1, \quad \mu_2 = v_2\left(\frac{3}{4}\right) + \lambda_2. \tag{27}$$

By dividing the subintervals $\left[0,\frac{1}{2}\right)$, $\left[\frac{1}{2},1\right)$, with step $h = 0.05$ we give the results of the numerical implementation of algorithm

Using equivalent problem (21)-(27) and solving the relevant system of linear algebraic equations (18) we get

$$\lambda_1^* = \begin{pmatrix} -0.999996216 \\ 0.000001946 \end{pmatrix}, \quad \lambda_2^* = \begin{pmatrix} 0.999995267 \\ -1.749996275 \end{pmatrix},$$

Таблица 1: Comparison of exact and numerical solutions to problem (19), (20)

t	$\lvert x_1^*(t)-\widetilde{x}_1(t)\rvert$	$\lvert x_2^*(t)-\widetilde{x}_2(t)\rvert$	t	$\lvert x_1^*(t)-\widetilde{x}_1(t)\rvert$	$\lvert x_2^*(t)-\widetilde{x}_2(t)\rvert$
0	0.000003784	0.000001946	0.5	0.000004733	0.000003725
0.05	0.000003775	0.000001785	0.55	0.000003786	0.000004755
0.1	0.000003772	0.000001679	0.6	0.000002852	0.00000587
0.15	0.00000378	0.000001638	0.65	0.000001937	0.000007011
0.2	0.000003807	0.000001674	0.7	0.000001054	0.000008079
0.25	0.000003858	0.000001796	0.75	0.00000023	0.000008925
0.3	0.000003941	0.000002011	0.8	0.000000489	0.000009324
0.35	0.000004063	0.000002324	0.85	0.000001025	0.000008942
0.4	0.00000423	0.000002728	0.9	0.000001251	0.000007283
0.45	0.000004451	0.000003207	0.95	0.000000967	0.000003617
0.5	0.000004733	0.000003725	1	0.00000014	0.000003144

$$\mu_1^* = \begin{pmatrix} 0.874996142 \\ -0.937498204 \end{pmatrix}, \quad \mu_2^* = \begin{pmatrix} 2.12499977 \\ -2.437491075 \end{pmatrix}.$$

Then, using the found values λ_1^*, λ_2^*, μ_1^*, μ_2^*, we solve the Cauchy problems by the fourth-order Runge-Kutta method

$$\frac{d\widetilde{x}_1}{dt} = \begin{pmatrix} t & t^2 \\ 4t^3 & 4 \end{pmatrix} \widetilde{x}_1 + \begin{pmatrix} t^3 & t+3 \\ 2 & t^2 \end{pmatrix} \cdot \begin{pmatrix} 0.874996142 \\ -0.937498204 \end{pmatrix} +$$
$$+ \begin{pmatrix} 2 & t-4 \\ t^3 & 3t \end{pmatrix} \cdot \begin{pmatrix} -0.999996216 \\ 0.000001946 \end{pmatrix} + \begin{pmatrix} 6t^2 & 3 \\ -6t & 1 \end{pmatrix} \cdot \begin{pmatrix} 0.999995267 \\ -1.749996275 \end{pmatrix} +$$
$$+ \begin{pmatrix} \frac{57}{8}t^3 - 9t^4 + 18t^2 - \frac{129}{16}t + \frac{97}{16} \\ 16t^5 - 32t^6 - 5t^3 - \frac{49}{16}t^2 + 24t - 4 \end{pmatrix}, \quad \widetilde{x}_1(0) = \begin{pmatrix} -0.999996216 \\ 0.000001946 \end{pmatrix}, \quad t \in \Big[0, \frac{1}{2}\Big),$$

$$\frac{d\widetilde{x}_2}{dt} = \begin{pmatrix} t & t^2 \\ 4t^3 & 4 \end{pmatrix} \widetilde{x}_2 + \begin{pmatrix} t^3 & t+3 \\ 2 & t^2 \end{pmatrix} \cdot \begin{pmatrix} 2.12499977 \\ -2.437491075 \end{pmatrix} +$$
$$+ \begin{pmatrix} 2 & t-4 \\ t^3 & 3t \end{pmatrix} \cdot \begin{pmatrix} -0.999996216 \\ 0.000001946 \end{pmatrix} + \begin{pmatrix} 6t^2 & 3 \\ -6t & 1 \end{pmatrix} \cdot \begin{pmatrix} 0.999995267 \\ -1.749996275 \end{pmatrix} +$$
$$+ \begin{pmatrix} \frac{47}{8}t^3 - 9t^4 + 18t^2 - \frac{105}{16}t + \frac{169}{16} \\ 16t^5 - 32t^6 - 5t^3 - \frac{25}{16}t^2 + 24t - \frac{13}{2} \end{pmatrix}, \quad \widetilde{x}_2\Big(\frac{1}{2}\Big) = \begin{pmatrix} 0.999995267 \\ -1.749996275 \end{pmatrix}, \quad t \in \Big[\frac{1}{2}, 1\Big).$$

and we find numerical solution of the problem (19) and (20).

Exact solution of the (19) and (20) is $x^*(t) = \begin{pmatrix} 8t^3 - 4t^2 + 1 \\ t^2 - 4t \end{pmatrix}$.

In Table 1, difference between the exact solution $x^*(t_k)$ and numerical solution $\widetilde{x}(t_k)$, $k = \overline{0, 20}$, are shown.

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References

[1] Busenberg S., Cooke K.L., "Models of vertically transmitted disease with sequential continuous dynamics" , *Lakshmikantham V (ed) Nonlinear phenomena in mathematical sciences,* (1982): 179-189.

[2] Cooke K.L., Wiener J., "Retarded differential equations with piecewise constant delays" , *J Math Anal Appl,* 99, (1984): 179-189.

[3] Akhmet M., Yilmaz E., "Neural Networks with Discontinuous/Impact Activations" , (Springer, New York, 2014).

[4] Akhmet M.U., "Nonlinear hybrid continuous/discrete time models" , (Atlantis, Amsterdam-Paris, 2011).

[5] Akhmet M.U., "Almost periodic solution of differential equations with piecewise-constant argument of generalized type" , *Nonlinear Analysis-Hybrid Systems,* 2, (2008): 456-467.

[6] Akhmet M.U., "On the reduction principle for differential equations with piecewise-constant argument of generalized type" , *J. Math. Anal. Appl.,* 1, (2007): 646-663.

[7] Dzhumabaev D.S., "Criteria for the unique solvability of a linear boundary-value problem for an ordinary differential equation" , *USSR Comput. Math. Math. Phys.,* 1, (1989): 34–46.

[8] Temesheva S.M., Dzhumabaev D.S., Kabdrakhova S.S. "On One Algorithm To Find a Solution to a Linear Two-Point Boundary Value Problem" , *Lobachevskii J. of Math.,* 42, (2021): 606-612.

[9] Nakhushev A.M., "Loaded equations and their applications" , (Nauka, Moscow, 2012) (in Russian).

[10] Nakhushev A.M., "An approximation method for solving boundary value problems for differential equations with applications to the dynamics of soil moisture and groundwater" , *Differential Equations,* 18, (1982): 72–81.

[11] Dzhenaliev M.T. "Loaded equations with periodic boundary conditions" , *Differential Equations,* 37, (2001): 51-57.

[12] Abdullaev V.M., Aida-zade K.R. "Numerical method of solution to loaded nonlocal boundary value problems for ordinary differential equations" , *Comput. Math. Math. Phys.,* 54, (2014): 1096-1109.

[13] Assanova A.T., Kadirbayeva Zh.M. "Periodic problem for an impulsive system of the loaded hyperbolic equations" , *Electronic Journal of Differential Equations,* 72, (2018): 1-8.

[14] Assanova A.T., Imanchiyev A.E., Kadirbayeva Zh.M. "Numerical solution of systems of loaded ordinary differential equations with multipoint conditions" , *Comput. Math. Math. Phys.,* 58, (2018): 508–516.

[15] Assanova A.T., Kadirbayeva Zh.M. "On the numerical algorithms of parametrization method for solving a two-point boundary-value problem for impulsive systems of loaded differential equations" , *Comp. and Applied Math.,* 37, (2018): 4966–4976.

[16] Kadirbayeva Zh.M., Kabdrakhova S.S., Mynbayeva S.T. "A Computational Method for Solving the Boundary Value Problem for Impulsive Systems of Essentially Loaded Differential Equations" , *Lobachevskii J. of Math.,* 42, (2021): 3675-3683.

[17] Dzhumabaev D.S. "On one approach to solve the linear boundary value problems for Fredholm integro-differential equations" , *J. of Comp. and Applied Math.,* 294, (2016): 342-357.