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A.B. Birgebaev[1*], **M.B. Muratbekov**[2]
[1]Abai Kazakh National Pedagogical University, Kazakhstan, Almaty
[2]M.Kh. Dulati Taraz Regional University, Kazakhstan, Taraz
*e-mail: ahtai@mail.ru

SMOOTHNESS OF SOLUTIONS (SEPARABILITY) OF THE NONLINEAR STATIONARY SCHRÖDINGER EQUATION

The equation of motion of a microparticle in various force fields is the Schrödinger wave equation. Many questions of quantum mechanics, in particular the thermal radiation of electromagnetic waves, lead to the problem of separability of singular differential operators. One such operator is the above Schrödinger operator. In this paper, the named operator is studied by the methods of functional analysis. Found sufficient conditions for the existence of a solution and the separability of an operator in a Hilbert space. All theorems were originally proved for the model Sturm-Liouville equation and extended to a more general case.
In §1-2, for the nonlinear Sturm-Liouville equation, sufficient conditions are found that ensure the existence of an estimate for coercivity, and estimates of weight norms are obtained for the first derivative of the solution. In Sections 3-4 the results of Sections 1-2 are generalized for the Schrödinger equation in the case $m = 3$.
Key words: Nonlinear equations, continuous operator, equivalence, potential function.

А.Б. Биргебаев[1*], М.Б. Муратбеков[2]
[1]Абай атындағы Қазақ ұлттық педагогикалық университеті, Қазақстан, Алматы қ.
[3]М.Х. Дулати атындағы Тараз өңірлік университеті, Қазақстан, Тараз қ.
*e-mail: ahtai@mail.ru

Шредингер теңдеуінің сызықты емес стационарлық теңдеуінің шешімдерінің тегістілігі (бөлімділігі)

Микробөлшектердің әртүрлі күш өрістеріндегі қозғалыс теңдеуі Шредингер толқынының теңдеуі болып табылады. Кванттық механиканың көптеген сұрақтары, атап айтқанда электромагниттік толқындардың жылулық сәулеленуі сингулярлы дифференциалдық операторлардың бөліну мәселесіне әкеледі. Осындай операторлардың бірі жоғарыдағы Шредингер операторы болып табылады. Бұл жұмыста аталған оператор функционалдық талдау әдістерімен зерттеледі. Шешімнің болуы және Гильберт кеңістігіндегі оператордың бөлінуі үшін жеткілікті шарттар табылды. Барлық теоремалар бастапқыда Штурм-Лиувилл теңдеуінің үлгісі үшін дәлелденді және жалпы жағдайға дейін кеңейтілді.
§1-2-де сызықты емес Штурм-Лиувилл теңдеуі үшін коэрцивтілік бағасының болуын қамтамасыз ететін жеткілікті шарттар табылды және шешімнің бірінші туындысы үшін салмақ нормаларының бағалаулары алынды. 3-4 бөлімдерде 1-2 бөлімдердің нәтижелері $m = 3$ жағдайындағы Шредингер теңдеуі үшін жалпыланған.
Түйін сөздер: Сызықты емес теңдеулер, үздіксіз оператор, эквиваленттілік, потенциалдық функция.

А.Б. Біргебаев[1*], М.Б. Муратбеков[2]
[1]Казахский национальный педагогический университет имени Абая, Казахстан, г. Алматы
[2]Таразский региональный университет имени М.Х. Дулати, Казахстан, г. Тараз
*e-mail: ahtai@mail.ru

Гладкость решений (разделимость) нелинейного стационарного уравнения Шредингера

Уравнением движения микрочастицы в различных силовых полях является волновое уравнение Шредингера. Многие вопросы квантовой механики в частности тепловое излучение электромагнитных волн приводят к задаче разделимости сингулярных дифференциальных операторов. Одним из таких операторов является вышеуказанный оператор Шредингера. Данной работе исследуется названный оператор методами функционального анализа. Найденный достаточные условия существовании решении и разделимости оператора в Гильбербовом пространстве. Все теоремы первоначально доказаны для модельного уравнение Штурма -Лиувилля и распространено на более общий случай.
В §1-2 для нелинейного уравнения Штурма-Лиувилля найдены достаточные условия, обеспечивающие наличие оценки коэрцитивности, а для первой производной решения получены оценки весовых норм. В §3-4 обобщены результаты §1-2 для уравнения Шредингера в случае $m = 3$.
Ключевые слова: Нелинейные уравнения, непрерывный оператор, эквивалентность, потенциальная функция.

1 Introduction

In this paper, the smoothness of solutions to the nonlinear equation is considered

$$Lu = -\Delta u + q(x,u)u = f(x) \in L_2(R^m)$$

In [1,2] for the nonlinear Sturm-Liouville equation, sufficient conditions are found that ensure the existence of an estimate for the coecitivity, and for the first derivative of the solution, estimates for the weight norms were obtained. In [1,2] generalized the results of §1-2 for the Schrödinger equation in the case $m = 3$.

2 Materials and methods

For simplicity, we present one result for the Sturm-Liouville equation.

Theorem 1 *Let the following conditions are satisfied:*
a) $q(x,y) \geq \delta\rangle 0$;
b) $q(x,y)$ *is a continuous function on the set of variables in* R^2;
c) $\sup\limits_{[x-\eta)\leq 1} \quad \sup\limits_{|C_0-C_1|\leq A} \quad \sup\limits_{|C_0|\leq A} \dfrac{q(x,C_0)}{q(x,C_1)} < \infty$, *where is any finite value. Then for any* $f(x) \in L_2(R^m)$ *there is a solution* (x) *to the equation*

$$Ly = -y''(x) + q(x,y)y = f,$$

which has quadratically summable second derivative, i.e. $y''(x) \in L_2(R^m)$.

The proof of this theorem belongs to Muratbekov M.B. [3]. Unfortunately, in the work [8] the author was incorrectly specified. Please apologize for inaccuracy. As we will see later (in Section 2.4), such results hold for a wide class of nonlinear operators. For linear operators of similar work was considered in [1-3, 5-7, 9, 11, 12, 13]

Let us enter the following designations: R^m is Euclidean m-dimensional real space of points $x = (x_1, x_2, \ldots, x_m)$. $\bar{\Omega}$ is a closure of Ω where Ω is an open set in R^m, $\|\cdot\|_{p,\Omega}$. is a norm of the element $L_\rho(\Omega)$. Instead of $\|\cdot\|_{p,\Omega}$ at $\Omega = R^m$ we will write $\|\cdot\|_\rho$, if $p = 2$ in designations $\|\cdot\|_{\rho,\Omega}$ and $\|\cdot\|_p$ we will omit ρ.

$$D_u^\alpha = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \ldots \partial x_m^{\alpha_m}},$$

$\alpha = (\alpha_1, \ldots, \alpha_m)$ – multiindex, $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_m$. $C_1, C_2, \ldots$ are various constants constants, the exact value of which does not interest us.

2.1 Existence of the solution

In the given section the following equation is considered

$$Ly = -y''(x) + q(x, y)y = f(x) \in L_2(R), \tag{1}$$

where $R = (-\infty, \infty)$.

The function $y \in L_2(R)$ is called the weak solution of equation (1), if there is a sequence $\{y_n\} \subset W_2^1(R) \bigcap W_{2,l>c}^2(R)$ such that

$$\|y_n - y\|_{\alpha_{2,loc(R)}} \to 0, \quad \|Ly_n - f\|_{L_{2,loc(R)}} \to 0, \quad n \to \infty.$$

It is said that the sequence $\{\eta_n\}_{n=1}^\infty$ of basic functions from $C_0^\infty(R^m)$ converges to (1) in R^m, if:

a) for any compact $K \subset R^m$ there will be such a number N, that $\eta_n(x) = 1$ at all $x \in K$ and $n \geq N$

b) functions $\{\eta_n\}$ uniformly limited in R^m, $|\eta_n(x)| \leq 1$, $x \in R^m$, $n = 1, 2, \ldots$ [8].

Lemma 1 *Let $q(x, y) \geq \delta < 0$ and is continuous on both arguments in R^2, then for any $f \in L_2(R)$ there is a weak solution of the equation (1) in the space $W_2^1(R)$.*

Proof. Since, according to the assumption, the function $q(x, y)$ is limited from below, then, without losing the generality of reasoning, we can assume that the condition $q(x, y) \geq 1$ is hold.

First, we will be engaged in proving the existence of a solution to the first boundary value problem

$$L_{n_\varepsilon} y_{n_\varepsilon} = -y''_{n_\varepsilon} + y_{n_\varepsilon} + \frac{(q(x, y_{n_\varepsilon}) - 1)y_{n_\varepsilon}}{(1 + \varepsilon(q)(x, y_{n_\varepsilon}) - 1) + \varepsilon\|b(x, y_{n_\varepsilon})\|_{2,(-a_n,a_n)}} = f\eta_n, \tag{2}$$

$$y_{n_\varepsilon}(+a) = y_{n_\varepsilon}(a) = 0, \tag{3}$$

where $[-a_n, a_n]$ – $\sup p\eta_n$, and $b(x, y_{n_e}) = (q(x, y_{n_\varepsilon}) - 1)y_{n_\varepsilon}$ in the space $W_{2,0}^2[-a_n, a_n]$; $W_{2,0}^2[-a_n, a_n]$ – is space of functions $z \in W_2^2$ и $z(-a_n) = z(a_n) = 0$.

We will reduce problem (2) – (3) to an equivalent integral equation, to which we then apply the Schauder principle [9].

Let us denote by L_0 the operator defined on $W_{2,0}^2[-a_n, a_n]$ with the equality

$$L_0 y = -y''(x) + y(x).$$

Due to the known theorems for the Sturm-Liouville operator there is a completely continuous inverse operator L_0^{-1}, defined all over space $L_2[-a_n, a_n]$. We need Lemma.

Lemma 2 *The problem (2) – (3) is equivalent to the integral equation*

$$z_{n_\varepsilon} = \frac{(q(x, L_0^{-1} z_{n_\varepsilon}) - 1) L_0^{-1} z_{n_\varepsilon}}{1 + \varepsilon (q(x, L_0^{-1} z_{n_\varepsilon}) - 1) + \varepsilon \| b(x, L_0^{-1} z_{n_\varepsilon}) \|_2^2} + f\eta_n, \tag{4}$$

$$z_{n_\varepsilon}, f\eta_n \in L_2[-a_n, a_n].$$

The proof is obvious.

Let us denote by A the operator which acts on the following formula:

$$A(z) = \frac{(q(x, L_0^{-1} z) - 1) L_0^{-1} z}{1 + \varepsilon (q(x, L_0^{-1} z) - 1) + \varepsilon \| b(x, L_0^{-1} z) \|_{2,[-a_n,a_n]}^2} + f\eta_n.$$

Further we denote

$$\overline{S}(0; N) = \left\{ \vartheta \in L_2(-a_n, a_n) : \|\vartheta\|_2 \le N = \frac{1}{\sqrt{\varepsilon}} \right\},$$

where $\vartheta = z - f\eta_n$. Consider the operator on this ball

$$A(\vartheta) = A(z) - f\eta_n = A(\vartheta + f\eta_n) - f\eta_n =$$

$$= \frac{(q(x, L_0^{-1}(\vartheta + f\eta_n)) - 1) L_0^{-1}(\vartheta + f\eta_n)}{1 + \varepsilon (q(x, L_0^{-1}(\vartheta + f\eta_n)) - 1) + \varepsilon \| b(x, L_0^{-1}(\vartheta + f\eta_n)) \|_{2,(-a_n,a_n)}^2}.$$

It is obvious that, if ϑ_0 – is a fixed point of operator $_m$, then $\vartheta_0 + f\eta_n$ – is a fixed point of operator . Therefore, in the future instead of operator A, it is enough to consider A_0.

Let us prove that $_0$ reflects the ball $\overline{S}(0; N) \in L_2[-a_n, a_n]$ in itself. Let $\vartheta \in \overline{S}(0; N)$. We will consider two cases:

1.

$$\| (q(x, L_0^{-1}(\vartheta + f\eta_n)) - 1) L_0^{-1}(\vartheta + f\eta_n) \|_{2,(-a_n,a_n)}^2 \le N = \frac{1}{\sqrt{\varepsilon}}.$$

Then

$$\|A_0(\vartheta)\|_2 = \left\| \frac{(q(x, L_0^{-1} z) - 1) L_0^{-1} z}{1 + \varepsilon (q(x, L_0^{-1} z) - 1) + \varepsilon \| b(x, L_0^{-1} z) \|_2^2} \right\|_{2,(-a_n,a_n)} \le$$

$$\le \| (q(x, L_0^{-1}(\vartheta + f\eta_n)) - 1) L_0^{-1}(\vartheta + f\eta_n) \| \le N = \frac{1}{\sqrt{\varepsilon}}$$

2.

$$(q(x, L_0^{-1}(\vartheta + f\eta_n)) - 1) L_0^{-1})(\vartheta + f\eta_{n)} \| \ge N.$$

Then

$$A_0(\vartheta)_2 \le \frac{\| (q(x, L_0^{-1}(\vartheta + f\eta_n)) - 1) L_0^{-1}(\vartheta + f\eta_n) \|_{2,(-a_n,a_n)}}{\varepsilon \| (q(x, L_0^{-1}(\vartheta + f\eta_n)) - 1) L_0^{-1}(\vartheta + f\eta_n) \|_{2,(-a_n,a_n)}^2} =$$

$$= \frac{1}{\varepsilon \| (q(x, L_0^{-1}(\vartheta + f\eta_n)) - 1) L_0^{-1}(\vartheta + f\eta_n) \|_{2,(-a_n,a_n)}} \le \frac{1}{\varepsilon N} = \frac{1}{\sqrt{\varepsilon}}.$$

Therefore,

$$\|A(\vartheta)\|_{2,(-a_n,a_n)} \le N, \quad \forall \vartheta \in \overline{S}(0; N). \tag{5}$$

Now we will show that $_m$ – is completely continuous operator at $\overline{S}(0;N)$. Continuity is obvious. Further, by virtue of Riesz theorem, it is enough to prove that the set of functions $\{A_0\vartheta : \vartheta \in \overline{S}(0;N)\}$ is uniformly limited and the relation is performed

$$\lim_{h\to 0} \|(A_0(\vartheta))(x+h) + (A_0(\vartheta))(x)\|_{2,(-a_n,a_n)} = 0$$

uniformly on $\vartheta \in \overline{S}$.

Due to estimate (5) the set of functions $\{A_0(\vartheta) : \vartheta \in \overline{S}(0;N)\}$ is uniformly bounded.

Due to the continuity of $q(x,y)$ on combination of variables and properties of the operator L_0^{-1}, the relation $q(x,y)$

$$\|(A_0(\vartheta)(x+h) - A_0(\vartheta))(x)\|^2_{2,(-a_n,a_n)} \to 0$$

uniformly at $h \to 0$ on $\vartheta \in \overline{S}(0;N)$.

Thus, the operator A_m is completely continuous and reflects $\overline{S}(0;N)$ in itself. Therefore, according to the Schauder principle; integral equation (4) has at least one solution in the ball $\overline{S}(0;N)$. Hence, by virtue of Lemma 2, it follows that there exists a solution to problem (2) – (3) belonging to the space W_2^2.

Further $\|y_{n_\varepsilon}\|_{W_2^1[-a_n,a_n]}$ is estimated from above by constant independent of n, ε.

To prove this fact, let us take the linear operator

$$\ell_{n_\varepsilon} y = y''(x) + (1 + \frac{\tilde{q}(x) - 1}{1 + \varepsilon(\tilde{q}(x) - 1) + \varepsilon\|(q(x,y_{n_\varepsilon}) - 1)y_{n_\varepsilon}\|_2^2})y(x),$$

Defined on a set $W^2_{2,0}(-a_n,a_n)$, where $\tilde{q}(x) = q(x, y_{n_\varepsilon})$, and y_{n_ε} – is a solution of the problem (2) – (3) with the right side $f\eta_n$. Let us construct a scalar product $\langle \ell n_\varepsilon, y_{n_\varepsilon}, y_{n_\varepsilon}\rangle$. Integrating in parts and taking into account that non-integral members disappear due to (3), we obtain

$$\|y_{n_\varepsilon}\|_{W_2^1[-a_n,a_n]} \le 2^{1/2} \left(\int\limits_{-\infty}^{\infty} |f|^2 dx \right)^{1/2}.$$

Assume that $C = 2^{1/2} \left(\int\limits_{-\infty}^{\infty} |f|^2 dx \right)^{1/2}$, then

$$\|y_{n_\varepsilon}\|_{W_2^1[-a_n,a_n]} \le C. \tag{6}$$

Let us choose some sequence $\{y_{n_{\varepsilon_k}}\}$ of solutions belonging to a bounded set $\{y_{n_\varepsilon}\}$, so that

$$\|y_{n_{\varepsilon_k}}\|_{W_2^1[-a_n,a_n]} \le C, \tag{7}$$

where $\varepsilon_k \to 0$ at $k \to \infty$.

By virtue of (7) from the sequence $\{y_{n_{\varepsilon_k}}\}$ we can select subsequence, denote it again by $\{y_{n_{\varepsilon_k}}\}$, so that

$$y_{n_{\varepsilon_k}} \to y_n \quad \text{weakly in} \quad W_2^1(-a_n,a_n),$$

$$y_{n_{\varepsilon_k}} \to y_n \text{ weakly in } L_2(-a_n, a_n).$$

From (7) we have $\|y_n\|_{W_2^1(-a_n,a_n)} \leq C$, and it is not difficult to see that y_n satisfies the equation

$$L_n y_n = -y_n''(x) + q(x, y_n)y_n = f\eta_n \text{ and } y_n(-a_n) = y_n(a_n) = 0.$$

Next, each y_n we continue with zero outside of $[-a_n, a_n]$, continuation denote by $\tilde{y}_n$. With this continuation, we obtain elements $W_2^1(R)$, norms of which are limited:

$$\|\tilde{y}_{n_\varepsilon}\|_{W_2^1(R)} \leq C.$$

Therefore, from the sequence, we can select a subsequence $\tilde{y}_{n_k}$, such that

$$\tilde{y}_{n_k} \to y \text{ weakly in } W_2^1(R) \tag{8}$$

$$\tilde{y}_{n_k} \to y \text{ weakly in } L_{2,\ell oc}(R), \tag{9}$$

and besides

$$\|y\|_{W_2^1(R)} \leq C. \tag{10}$$

Let $[\alpha, \beta]$ is any fixed segment in R. Then for any $\varepsilon\rangle 0$ there exists such number N, that at $k = N(\alpha, \beta) \in \sup p\tilde{y}_{n_k}$ and by virtue (8)

$$\|L\tilde{y}_{n_k} - f\|_{2,(\alpha,\beta)}\langle\varepsilon.$$

From here and (9) we get that $y(x)$ is a weal solution of the equation (1). Lemma is proved.

2.2 Smoothness of the solution

In this section we will show that all solutions from $W_2^1(R)$ will be elements from $W_2^2(R)$, as soon as a potential function known in it has some properties.

Theorem 2 *Let the following conditions hold;*

a) $q(x, y) \geq \delta\rangle 0$;

b) $q(x, y)$ *is continuous function on a set of variables in* R^2;

c) $\sup\limits_{|x-\eta|\leq 1} \sup\limits_{|C_1-C_2|\leq A} \sup\limits_{|C_1|\leq A} \dfrac{q(x, C_1)}{q(x, C_2)} < \infty$,

where A *is any finite value. Then for any* $f \in L_2(R)$ *there exists the solution* $y(x) \in L_2(R)$ *of the equation (1), such that* $y''(x) \in L_2(R)$.

Theorem 3 *Let the conditions hold:*

a) $q(x, y) \geq \delta\rangle 0$;

b) $q(x, y)$ *are continuous on a set of variables in* R^2;

c) $\sup\limits_{x\in R} \sup\limits_{|C_1-C_2|\leq A} \sup\limits_{|C_2|\leq A} \dfrac{q(x, c_1)}{\theta^2(x, c_2)} < \infty$, *where*

$$\theta(x, C_1) = \inf_{d\rangle 0|x-t|\leq 10} (d^{-1} + \int\limits_{|t-h|\leq d} q(\eta, C_2)d\eta),$$

A is any finite value. Then for any $f \in L_2(R)$ there exists the solution $y(x) \in L_2(R)$ of the equation (1) such that $y''(x) \in L_2(R)$.

Theorem 4 *Let the conditions a)-c) of theorem 2 are held and $r(x)$ is a continuous, such that* $\sup\limits_{|x-y|\le 1} \dfrac{r(y)}{r(x)} < \infty$.

If for any $k > 0$ the value

$$B = \sup_{x\in R} \sup_{|C_1|\le K} \sup_{0<\eta\le m^{-1}(x,C_1)} \left[\eta^{-p} \int\limits_{|t-x|\le\eta} |r(t)|^0 dt\right]^{1/\theta}$$

is finite, then for any $f \in L_2(R)$ function

$$r(x)\frac{d}{dx}y(x) \in L_2(R), \quad (2 \le \theta < \infty,\ p = -\frac{\theta}{2}, m(x,C_1) = (q(x,C_1))^{1/q}),$$

here $y(x)$ is the solution of the equation (1) from $L_2(R)$.

Proof of Theorems 2-4. At any function $f \in L_2(R)$ by virtue of Lemma 1 for the equation there exists a solution $y(x)$ such that $y(x) \in W_2^1(R)$. Therefore, by Sobolev's embedding theorem [10] $y(x) \in C(R)$. Then according to the condition b)

$$q(x, y(x)) \in C_{\ell oc}(R). \tag{11}$$

Let $y_0(x)$ is a weak solution of the equation (1) with the right side $f_0 \in L_2(R)$. Since $y_0(x) \in W_2^1(R)$, then

$$y_0(t) - y_0(\eta) = \int\limits_\eta^t \frac{dy_0}{dx} dx.$$

By the Bunyakovsky inequality and by (10), we have

$$|y_0(t) - y_0(\eta)| \le (|t-\eta|)^{1/2} \|f\|_{2,R}. \tag{12}$$

Assume that $\tilde q(x) = q(x, y_0(x))$ and denote by $\tilde L$ closure in norm of L_2 operator, given on $C_0^\infty(R)$ by equality $L_0 y = -y''(x) + \tilde q(x)y$.

Lemma 3 *Operator $\tilde L$ is self adjoint and positive defined.*

Proof. The positive definiteness of $\tilde L$ follows from condition a) of Theorem 2. Self-adjointness follows from (2) and from the results of [2]. The lemma is proved.

Now, assuming that $y_0(t) = C_2$, $y_0(\eta) = C_1$, $A = 2\|f\|_2 \ge \sqrt{A\eta}\|f\|_2$, from (12) we obtain $|C_2 - C_1| \le A$. From here, due to conditions a)-c) of Theorem 2, for operator $\tilde L$ all conditions of the Theorem 3, 4 are satisfied. Therefore, the operator L is separable, i.e.

$$\|y''\|_2 + \|\tilde q(x)y\|_2 \le C(\|\tilde L y\| + \|y\|_2),$$

where does not depend on $y \in D(\tilde L)$, where $D(\cdot)$ is the definition area, and $\|\cdot\|$ is the norm in $L_2(D)$.

It remains for us to show that $y_0(x) \in D(\tilde{L})$. Suppose the contrary, that $y_0(x) \notin D(\tilde{L})$. By virtue of Lemma 2, there exists $y_1(x) \in W_2^1(R)$ such that $y_1(x) = \tilde{L}^{-1} f_0$. So, it is assumed that $y_0(x) \in W_2^1(R)$ is a solution of equation (1) with the right side of $f_o(x)$, then

$$\tilde{L} y_2 = 0, y_2 = y_1 - y_0 \in L_2(R).$$

To complete the proof of the theorem, we need a lemma.

Lemma 4 *Let the conditions a) and b) of theorem 2 be satisfied. Then the equation* $\tilde{L}y = 0$ *does not have a solution* $y(x) \in L_2(R)$.

Proof. It is well known that if $\tilde{q}(x) \geq \delta > 0$, then the solution of the equation $y''(x) = q(x)y$ exponentially grows both at $x \to -\infty$, and at $x \to +\infty$. Therefore, this solution cannot belong to $L_2(R)$. The Lemma is proved.

From this lemma we obtain that $y_0(x) = y_1(x)$. We get a contradiction. The theorem 2. is completely proved.

Theorems 3, 4 are proved in the same way.

2.3 Nonlinear Schrödinger-type operator in $L_2(R^3)$

Now let us consider the equation

$$-\Delta u + q(x,u)u = f(x) \tag{13}$$

in the space $L_2(R^3)$.

Lemma 5 *Let* $q(x,u) \geq \delta > 0$ *and is continuous on both arguments in* R^2*, then for each* $f \in L_3(R^3)$ *there is a weak solution to equation* (13) *in space* $W_2^1(R^3)$.

This lemma is proved in the same way as the lemma 1.

Lemma 6 *Let* $q(x,u) \geq \delta > 0$ *and is continuous on both arguments in* R^2*, then for each* $f \in L_2(R^3)$ *there is a weak solution to equation* (13) *and the following inequality holds*

$$\|u\|_{L_\infty(R^3)} + \|u\|_{W_2^1(R^3)} \leq C\|f\|_{L_2(R^3)}, \tag{14}$$

Where the constant C *does not depend on* u *and* f.

Proof. Let

$$q_N(x,u) = \begin{cases} q(x,u), & \text{if} \quad q(x,u) \leq N, \\ N, & \text{if} \quad q(x,u) \geq N \end{cases}$$

The existence of a solution to the equation

$$-\Delta u + q_N(x,u)u = f_N \tag{15}$$

follows from lemma 5.

Let $u_x \in W_2^1(R^3)$is a solution to equation (15). Let us consider the equation

$$L_u = f_N, \tag{16}$$

where $L = -\Delta + \tilde{q}_N(x)$,

Since $q_N(x, u_N)$ are limited and $\tilde{q}_N(x)$, then on the theorem (3), [see 11] operator L is self-adjoint and the equation (16) has a unique solution that coincides with u_N.

It is known, if $q_1(x) \leq q_2(x)$, then $Q_1(x, y) \geq 0$ and $Q_2(x, y) \geq 0$, and $Q_1(x, y) \geq Q_2(x, y)$, where $Q_1(x, y)$ and $\tilde{N}_2(x, y)$ are Green functions of operators $-\Delta + q_1(x)$, $-\Delta + q_2(x)$.

Let $Q_N(x, y)$is the Green function of the operator L, then it follows from the above fact that

$$Q_N(x, y) \leq Q_0(x, y), \tag{17}$$

where $Q_0(x, y)$ is Green function of the operator $-\Delta + 1$. It follows from this and (17) that

$$|u_x(x)| = \left| \int\limits_{R^3} Q_N(x, y) f(y) dy \right| \leq \int\limits_{R^3} Q_N(x, y) f(y) dy \leq \int\limits_{R^3} Q_0(x, y) |f(y)| dy.$$

It is known that the operator

$$(Qf)(x) = u_0(x) = \int\limits_{R^3} Q_0(x, y) |f(y)| dy \tag{18}$$

acts from $L_2(R^3)$ in $W_2^2(R^3)$. Therefore, by virtue of the Sobolev embedding theorems [10], we have

$$\|u_N(x)\|_{L_\infty(R^3)} \leq C_0 \|f\|_{L_2(R^3)}, \tag{19}$$

where C_0 does not depend on N and f.

On the other hand, here is an estimation

$$\|u_N(x)\|_{W_2^1(R^3)} \leq C_1 \|f\|_{L_2(R^3)}, \tag{20}$$

where C_1 does not depend on N and f.

Indeed, we will compose a scalar product $\langle Lu_N, u_N \rangle$. Integrating in parts, we obtain (20).

From (19) and (20) we will have

$$\|u_N(x)\|_{L_\infty(R^3)} + \|u_N\|_{W_2^1(R^3)} \leq C_2 \|f\|, \tag{21}$$

where $C_2 = \max(C_1, C_2)$.

Moving to limit at $N \to \infty$ we get

$$\|u(x)\|_{L_\infty(R^3)} + \|u(x)\|_{W_2^1(R^3)} \leq C_2 \|f\|_{L_2(R^3)}.$$

It is not difficult to check that $u(x)$ is the weak solution to equation (13) (see lemma 2). The lemma is proved.

2.4 Smoothness of the solution

Theorem 5 *Let the following conditions be satisfied: a)* $q(x,y) \geq \delta > 0$; *b)* $q(x,y)$ *is a continuous function on a set of variables in* R^2 *and*

$$\sup_{|x-y|\leq 1} \sup_{|C_1-C_2|\leq A |C_1|\leq A} \frac{q(x,C_1)}{q(y,C_2)} < \infty,$$

where A *is any finite value. Then: a) for any right side of* $f \in L_2(R^3)$ *there exists a solution* $u(x)$ *of the equation* (13) *such that* $\Delta u \in L_2(R^3)$; *b) let* $r(x)$ *is continuous function in* R^3, *if for any* $k\rangle 0$ *the value*

$$B = \sup_{x\in R} \sup_{|C_1|\leq K} \sup_{0<\eta\leq m^{-1}(x,C_1)} \left[\eta^{-p} \int\limits_{|t-x|<\eta} |r(t)|^\theta dt \right]^{1/\theta}$$

Is finite, then

$$r(x)D^2 u(x) = L_\theta(R^3),$$

$$(2 \leq \theta < \infty, \quad p = -\frac{\theta}{2}, \quad m(x,C_1) = (q(x,C_1))^{1/2}.$$

Let us enter the function

$$q_\varepsilon^*(t,C_0) = \inf \left\{ d^{-1}; d \geq \inf_{e\in F_d^{(\varepsilon)}(t)} \int\limits_{\theta_d(t)|_e} q(x,C_0)dx \right\},$$

where $F_d^{(\varepsilon)}(t)$ *is a set of all compact subsets of cube* $\theta_d(t)$, *satisfying the following inequality*

$$mes e \leq \varepsilon d^n, \quad \varepsilon \in (0,1).$$

Theorem 6 *Let the conditions a), b) of the theorem 5 be satisfied and*

$$\sup_{|x-y|\leq 1} \sup_{|C_0-C_1|\leq A} \frac{q_\varepsilon^*(x,C_0)}{q_\varepsilon^*(x,C_1)} < \infty,$$

Let us denote $m(x,C_0) = q_\varepsilon^*(x,C_0)$, *and by* $A_p(x,C_0)$ *– the function which is defined with the equality*

$$A_p(x,C_0) = m^{-1-\beta}(x,C_0) \sup_{|C_1|\leq K} \sup_{0<\eta<m^{-1}(x,C_1)} \eta^{-\beta} \int\limits_{|x-t|<\eta} q(t,C_1)dt,$$

where κ *is any value,* $\beta = 2(\frac{3}{p} - 1)$, p *– is any number from the interval (1,2). Then, if at some* $p \in (1,2)$ *the value*

$$A_p = \sup_{|C_0|\leq K} \sup_{x\in R^3} A_p(x,C_0)$$

Is finite, then for any $f(x) \in L_2(R^3)$ *there exists a solution* $u(x) \in L_2(R^3)$ *of the equation* (13), *such that* $\Delta u \in L_2(R^3)$.

Theorems 5, 6 are proved in the same way as theorems 2-4, based on results of work [7].

3 Discussion

For differential equations one of the important questions is finding solutions in function spaces. In this paper, using operator methods, a sufficient condition for the existence of solutions to the nonlinear Sturm-Liouville and Schrodinger equations is found. Research methods and results can be used in the study of other nonlinear differential equations.

4 Conclusion

The issues of separability of operators and coercive estimates, and also the existence of a solution to differential equations, are solved in combination. The results of this work are new and generalize previously published works.

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