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# GENERALIZED FORMULA FOR ESTIMATING THE OSCILLATION FREQUENCY RESPONSE OF A CANTILEVER BAR WITH POINT MASSES

This paper presents a study of natural oscillations of a cantilever bar with five point masses with variable geometric and stiffness parameters (distances between locations of the masses, coefficients of variability of the bending stiffness of the bar sections). Using the exact method of forces based on the Mohr formula, there have been obtained expressions in general form for calculating the main unit coefficients of the secular equation, which makes it possible to perform calculations and to determine the oscillation frequency response of natural oscillations with a wide range of changes in the initial parameters of the physical and geometric state of cantilever bars. A numerical example has been given to illustrate the proposed theoretical approaches. The results have been compared with the results based on calculating a similar cantilever bar with one (reduced by masses) degree of freedom. A graphical dependence of the oscillation frequency response value on changing the value of the bending stiffness along the length of the cantilever bar gas been obtained. The theoretical provisions and applied results presented in the work will be widely used both in the practical design of bar systems and in scientific research in the field of mechanics of a deformable solid body.

**Key words**: cantilever bar, point masses, variable bending stiffness, main unit coefficients, oscillations frequency response for natural oscillations, graphical dependence of the oscillation frequency response, reduced mass, calculation reliability, calculation nomogram.

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# НҮКТЕЛІК МАССАЛАРЫ БАР КОНСОЛЬДІ ӨЗЕКТІҢ НЕГІЗГІ ЖАҒДАЙЫН БАҒАЛАУҒА АРНАЛҒАН ЖАЛПЫЛАНҒАН ФОРМУЛА

Бұл мақалада айнымалы геометриялық және қатаңдық параметрлері мен бес нүктелі массалары бар консоль өзегінің өзіндік тербелістеріне зерттеу жүргізілді (массалардың орналасуы арасындағы қашықтық, өзектер бөлімдерінің иілу қатаңдығының өзгеру коэффициенттері). Мора формуласына негізделген күштердің нақты әдісі ғасырлық теңдеудің негізгі бірлік коэффициенттерін есептеу үшін өрнектің жалпы түрінде алынады, бұл консольдік өзектердің физика-геометриялық күйінің бастапқы параметрлерінің өзгеруінің кең диапазонында табиғи тербелістердің негізгі жағдайын анықтауға есептеулер жүргізуге мүмкіндік береді.

Усынылған теориялық тәсілдерді суреттеу үшін сандық мысал келтірілген. Есептеу нәтижелерін салыстыру үшін еркіндік дәрежесі бірге тең ұқсас консольді өзекті (масса бойынша берілген) есептеу негізінде жүргізілді. Консольді өзектің иілу қатаңдығы мен өзек бойының ұзындығының өзгеруіне байланысты графикалық тәуелділік алынды. Жұмыста келтірілген теориялық ережелер мен қолданбалы нәтижелерді өзектік жүйелерді практикалық жобалау кезінде және деформацияланатын қатты дене механикасы саласындағы ғылыми зерттеулерде де кеңінен қолданылады.
Алынған теориялық және практикалық нәтижелер ғимараттар мен әртүрлі инженерлік құрылыстардағы тұтас арқалықтар құрылымын жобалау кезінде қолданыла алады.

**Түйін сөздер**: : Консольді өзек, нүктелік масса, айнымалы иілу қатаңдығы, негізгі бірлік коэффициенттер, өзіндік тербелістің негізгі жағдайы, негізгі жағдайдың графикалық тәуелділігі, келтірілген масса, есептеудің сенімділігі, есептеу номограммасы.

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**ОБОБЩЕННАЯ ФОРМУЛА ДЛЯ ОЦЕНКИ ОСНОВНОГО ТОНА КОНСОЛЬНОГО СТЕРЖНЯ С ТОЧЕЧНЫМИ МАССАМИ**

В данной работе выполнено исследование собственных колебаний консольного стержня с пятью точечными массами с переменными геометрическими и жесткостными параметрами (расстояния между местами расположения масс, коэффициентами переменности изгибных жесткостей участков стержней). Точным методом сил на основе формулы Мора получены в общем виде выражения для вычисления главных единичных коэффициентов векового уравнения, что позволяет производить расчеты на определение основного тона собственных колебаний при широком диапазоне изменения исходных параметров физико-геометрического состояния консольных стержней. Приведен численный пример для иллюстрации предлагаемых теоретических подходов. Выполнено сравнение результатов расчета на основе расчета аналогичного консольного стержня с одной (приведенной по массам) степенью свободы. Получена графическая зависимость величины основного тона от изменения значения изгибной жесткости по длине консольного стержня. Приведенные в работе теоретические положения и прикладные результаты найдут широкое применение как в практическом проектировании стержневых систем, так и в научных исследованиях в области механики деформируемого твердого тела.

**Ключевые слова**: Консольный стержень, точечные массы, переменная изгибная жесткость, главные единичные коэффициенты, основной тон собственных колебаний, графическая зависимость основного тона, приведенная масса, достоверность расчета, номограмма расчета.

## 1 Introduction

In the process of designing high-rise buildings (multi-storey structures) and tower-type structures (various support structures), in order to calculate them for pulsation from the dynamic effect of wind load, it is necessary to know the magnitude of the oscillation frequency response of free (natural) oscillations.

For this, various exact and approximate analytical and numerical methods are used.

In this paper, an approximate analytical method is proposed for calculating the oscillation frequency response of cantilever bars with step-variable bending stiffness along its length with point masses, which makes it possible to estimate the magnitude of the oscillation frequency response with sufficient engineering accuracy.

A certain number of works were dealing with the topic proposed by the authors of this article: for example, in work [1] the oscillatory processes of a statically determinate bar system with one degree of freedom are considered by the Runge-Kutta method in the MatCad program; there was compared the effect of the inelastic resistance of the material coefficient on the displacement of the concentrated mass.

Calculation for harmonic oscillations is also described in the works by Aizenberg Ya.N., Gvozdev A.A., Birbraer A.P., Shulman S.G., Rabinovich I.M., Barshtein M.F., Korenev B.G., Timoshenko S.P. and many others [2-8].

Study [9] considers special properties of the bending shapes of bars of constant bending stiffness to determine the value of the fundamental tone.

In work [10], nonlinear free oscillations of coating structures are considered based on studying the characteristic quadratic equation obtained by the matrix method.

In paper [11], a mixed form of the finite element method was used to calculate bar systems for free oscillations.

Papers [12, 13] consider the numerical implementation of the finite element method in calculations for free and forced oscillations; the matrices of stiffness, masses, examples of calculating beam systems of the Bernoulli-Euler and Timoshenko type are given, a new concept of "dynamic matrix"is introduced.

***The purpose and objective of this work*** is to study the stress-strain state of cantilever bars with several point masses for natural oscillations with determining the oscillation frequency response in a wide range of changes of geometric and physical and mechanical parameters of the system under study.

In this case, the problem of obtaining an analytical expression for calculating the main unit coefficients of the secular equation in general form has been solved, which makes it possible to operate mathematically with geometric dimensions and bending stiffness when calculating various cantilever bars.

One of the objectives of the study is to illustrate the generalized formula obtained by the authors using the example of calculating the oscillation frequency response of a cantilever bar.

A graphic dependence (nomogram) of the oscillation frequency response of natural oscillations has also been obtained with changing the bending stiffness along the length of the bar.

## 2 Theoretical provisions and calculation methods

Coefficients $k_1,\ k_2,\ k_3,\ k_4$,determining the variability of the step-variable bending stiffness along the height of the bar at five levels (floors) of the structure;

Coefficients $a_1,\ a_2,\ a_3,\ a_4$ determining the differences in the size of sections (floors) along the height of the cantilever rod;

Coefficients $b_1,\ b_2,\ b_3,\ b_4$ determining the differences in the values of concentrated masses.

By changing the values of the above coefficients over a wide range, it is possible to study free oscillations, that is, to determine the value of the oscillation frequency response of the five-step-variable bending stiffness of the cantilever bar with different lengths, and the bending stiffness of its five steps (floors) with different values of five concentrated point masses located at the joints of steps (floors) along the height of the structure.

Let us calculate the values of the main coefficients $\delta_{ii} (i = 1,\ 2,\ 3,\ 4,\ 5)$ according to the Vereshchagin rule, multiplying the corresponding single diagrams of the moments (Figure 1, b, c, d, e, f). Then, in generalized form, we obtain:

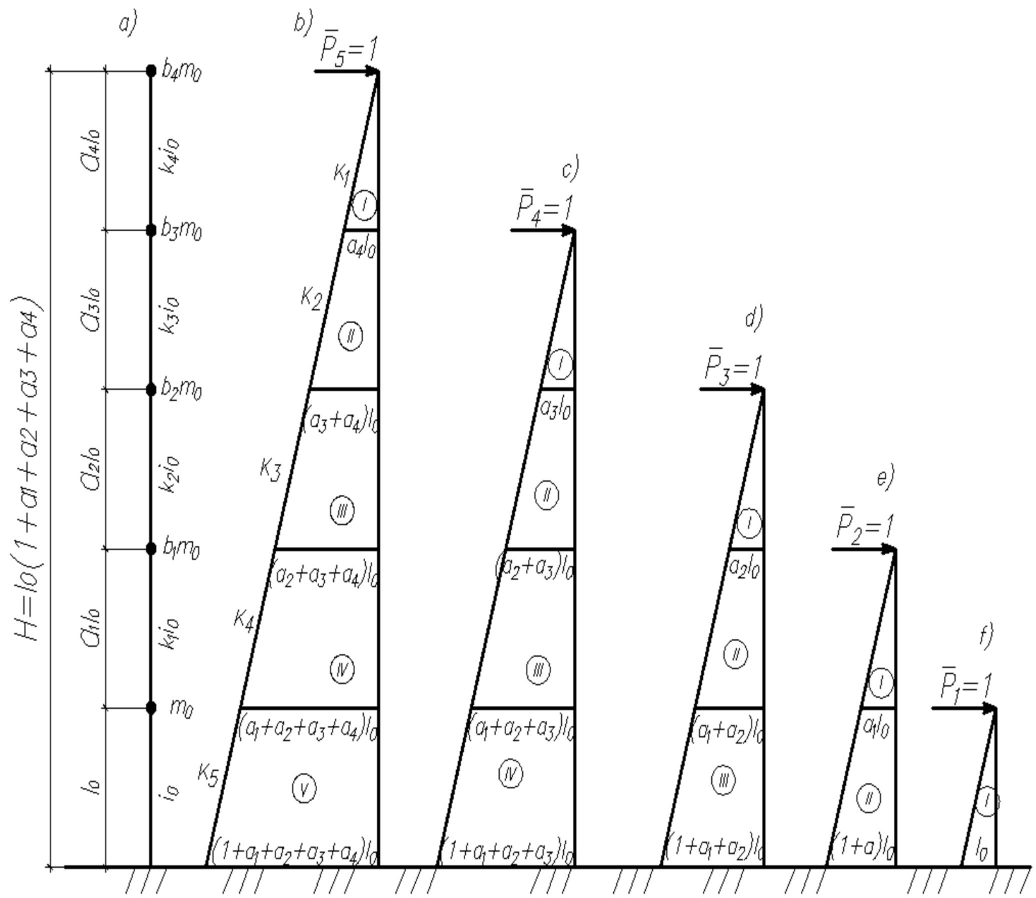


Figure 1: Towards the calculation of the cantilever bar for free oscillations: a) – the calculated scheme; b) – diagram $\overline{M}_5$; c) – diagram $\overline{M}_4$; d) – diagram $\overline{M}_3$; e) – diagram $\overline{M}_2$; f) – diagram $\overline{M}_1$

$$
\begin{aligned}
&\delta_{ii} = \tfrac{1}{E_iJ_i}(\overline{M}_i)\cdot(\overline{M}_i) = \tfrac{l_0^3}{l_0 i_0}\left[\tfrac{1}{k_{i-1}}\left(0.33a_{i-1}^3\right)\right] + \\
&+\tfrac{1}{k_{i-2}}\left[0.5a_{i-1}a_{i-2}(2a_{i-1}+a_{i-2}) + 0.1667(a_{i-2})^2(3a_{i-1}+2a_{i-2})\right] + \\
&+\tfrac{1}{k_{i-3}}\left[0.5(a_{i-2}+a_{i-1})a_{i-3}(2a_{i-2}+a_{i-1}) + a_{i-3} + 0.1667(a_{i-3})^2 3\left[(a_{i-2}+a_{i-1})+2a_{i-3}\right]\right] + \\
&+\tfrac{1}{k_{i-4}}\left[0.5(a_{i-2}+a_{i-1}+a_{i-3})\left[2(a_{i-2}+a_{i-1}+a_{i-3})+a_{i-4}\right] + \right.\\
&\left.+0.1667(a_{i-4})^2 3\left[(a_{i-2}+a_{i-1}+a_{i-3})+2a_{i-4}\right]\right] + \\
&+\left[0.5(a_{i-2}+a_{i-1}+a_{i-3}+a_{i-4})\left[2(a_{i-2}+a_{i-1}+a_{i-3}+a_{i-4})+1\right] + \right.\\
&\left.+0.1667\left[3(a_{i-2}+a_{i-1}+a_{i-3}+a_{i-4})+2\right]\right].
\end{aligned}
\tag{1}
$$

According to generalized formula (1) let's calculate the main coefficients:

a) $\delta_{55}(i=5)$

$$\delta_{55}=\frac{l_0^2}{i_0}\left[\frac{1}{k_4}\left(0.3333a_4^3\right)\right]+\frac{1}{k_3}\left[0.5a_3a_4(2a_4+a_3)+0.1667(a_3)^2(3a_4+2a_3)\right]+$$
$$+\frac{1}{k_2}\left[0.5(a_3+a_4)a_2\left[2(a_3+a_4+a_2\right]+0.1667(a_2)^2 3\left[(a_3+a_4)+2a_2\right]\right]+$$
$$+\frac{1}{k_1}\left[0.5(a_3+a_4+a_2)\left[2(a_3+a_4+a_2)+a_1\right]+0.1667(a_1)^2 3\left[(a_3+a_4+a_2)+2a_1\right]\right]+$$
$$+\left[0.5(a_3+a_4+a_2+a_1)+\left[2(a_3+a_4+a_2+a_1)+1\right]+0.1667\left[3(a_3+a_4+a_2+a_1)+2\right]\right]$$

b) $\delta_{44}(i=4)$

$$\delta_{44}=\frac{l_0^2}{i_0}\left[\frac{1}{k_3}\left(0.3333a_3^3\right)\right]+\frac{1}{k_2}\left[0.5a_3a_2(2a_3+a_2)+0.1667(a_2)^2(3a_3+2a_2)\right]+$$
$$+\frac{1}{k_1}\left[0.5(a_3+a_2)\left[2(a_3+a_2)+a_1\right]+0.1667(a_1)^2\left[3(a_3+a_2)+2a_2\right]\right]+$$
$$+\left[0.5(a_1+a_2+a_3)\left[2(a_1+a_2+a_3)+1\right]+0.1667\left[3(a_1+a_2+a_3)+2\right]\right]$$

c) $\delta_{33}(i=3)$

$$\delta_{33}=\frac{l_0^2}{i_0}\left[\frac{1}{k_2}\left(0.3333a_2^3\right)\right]+\frac{1}{k_1}\left[0.5a_2a_1(2a_2+a_1)+0.1667(a_1)^2(3a_2+2a_1)\right]+$$
$$+\left[0.5(a_3+a_2)\left[3(a_1+a_2)+2\right]+0.1667\left[3(a_1+a_2)+2\right]\right] \quad . \tag{2}$$

d) $\delta_{22}(i=2)$

$$\delta_{22}=\frac{l_0^2}{i_0}\left[\frac{1}{k_1}\left(0.3333a_1^3\right)\right]+\left[0.5(a_1)^2(2a_1+1)+0.1667\left[3(a_1+2)\right]\right];$$

e) $\delta_{11}(i=1)$

$$\delta_{11}=\frac{l_0^2}{i_0}\left(0.3333\right).$$

Let's calculate the point masses values (Figure 1, a).

$$m_1=m_0;\;\; m_2=b_1m_0;\;\; m_3=b_2m_0;\;\; m_4=b_3m_0;\;\; m_5=b_4m_0;$$

According to the formula presented in [1], let's calculate the approximate value of the oscillation frequency response for free oscillations of the cantilever bar:

$$\frac{1}{\omega_1^2}=m_1\delta_{11}+m_2\delta_{22}+m_3\delta_{33}+m_4\delta_{44}+m_5\delta_{55} \tag{3}$$

Let's substitute the values calculated in (1, 2) into expression (3).

Based on the proposed generalized formulas for the cantilever bar (Figure 1, a), let's calculate a numerical example with the following initial data (Figure 2, a):

$$a_1=a_2=a_3=a_4=1;\;\; m_0=43.3\,kg\cdot s^2/cm;$$

$$l_0=3.5\,m;\;\; b_1=1.0254;\;\; b_2=0.9931;\;\; b_3=0.8799;\;\; b_4=0.836;$$

$$i_0^*=8.06\cdot 10^8 kg/cm;\quad i_{AB}=3i_0^*=24.18\cdot 10^8=i_0;$$

$$k_1=0.8;\;\; k_2=0.6;\;\; k_3=0.4;\;\; k_4=0.2$$

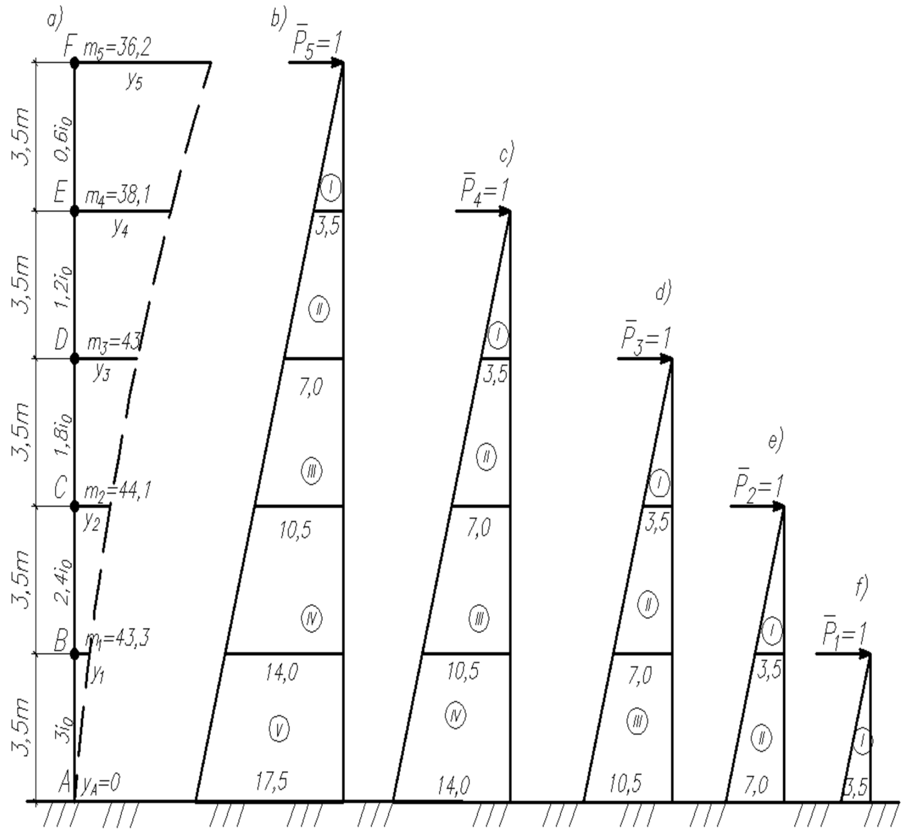


Figure 2: Towards the calculation of the cantilever bar (example): a) – the preset scheme; b) - diagram $\overline{M}_5$; c) - diagram $\overline{M}_4$; d) – diagram $\overline{M}_3$; e) – diagram $\overline{M}_2$; f) – diagram $\overline{M}_1$

## 3 Results

Based on the above theoretical calculations, we obtain the following results for which we calculate the values of the main coefficients $\delta_{ii}(i = 1,\ 2,\ 3,\ 4,\ 5)$ using formulas (2)

$$\delta_{55} = \frac{1}{li_0}\,(23.82 + 83.56 + 150.8676 + 220.371 + 290.57) = \frac{768.992 \cdot 10^4}{li_0} = 27.26 \cdot 10^{-4};$$

$$\delta_{44} = \frac{1}{li_0}\,(768.992 - 290.57) = \frac{478.422 \cdot 10^4}{li_0} = 16.96 \cdot 10^{-4};$$

$$\delta_{33} = \frac{1}{li_0}\,(478.4221 - 220.371) = \frac{258.051 \cdot 10^4}{li_0} = 9.1475 \cdot 10^{-4}; \qquad (4)$$

$$\delta_{22} = \frac{1}{li_0}\,(258..05 - 150.87) = \frac{107.18 \cdot 10^4}{li_0} = 3.8 \cdot 10^{-4};$$

$$\delta_{11} = \frac{1}{li_0}\,(107.18 - 83.363) = \frac{23.818 \cdot 10^4}{li_0} = 0.844 \cdot 10^{-4}.$$

Let's calculate the point masses values (Figure 2, a):

$$m_1 = \frac{P_1}{g} = \frac{43.3 \cdot 10^3}{981} = 44.14\; kg \cdot s^2/cm;\; m_2 = \frac{P_2}{g} = 44.954\;\; kgs^2/cm;$$

$$m_3 = \frac{P_3}{g} = 43.83\;\; kgs^2/cm;\; m_4 = \frac{P_4}{g} = 38.84\;\; kgs^2/cm;\; m_5 = \frac{P_5}{g} = 39.9\;\; kgs^2/cm$$

Let's calculate the approximate value of the oscillation frequency response for free oscillations of the cantilever bar (Figure 2, a) according to formula (3).

$$\frac{1}{\omega_1^2} = m_1\delta_{11} + m_2\delta_{22} + m_3\delta_{33} + m_4\delta_{44} + m_5\delta_{55} =$$
$$= 10^{-4}\,(37.25 + 170.83 + 400.93 + 658.73 + 1005.89)\;\; = 2273.63 \cdot 10^{-4}$$

$\omega_1 = 10^2 \cdot 0.020973 = 2.0973\; s^{-1}$ is the oscillation frequency response of the cantilever bar (Figure 1,a).

To estimate reliability of the result obtained, let's calculate the $(\omega_1)$ value through the reduced mass (M) (Figure 3) the coefficient of reduction of the distributed mass to the end of the cantilever bar [14].

$$M = \beta \frac{\left(\sum m_i\right) H}{H} = \beta \sum m_i H = 0.23\,(m_1 + m_2 + m_3 + m_4 + m_5) = 47.99$$

According to formula 7.70 [1]:

$$\omega_1^2 = \frac{1}{\delta_{55}} = \frac{1}{47.99 \cdot 27.26 \cdot 10^{-4}} = 10^2 \cdot 0.000764;\;\; \omega_1^* = 2.764\; s^{-1}$$

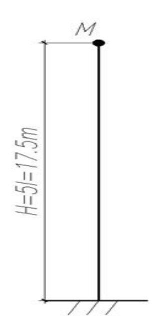


Figure 3: The bar reduced mass

The $\omega_1$ and $\omega_1^*$ values calculated by different methods (approaches) are sufficiently close which proves reliability of the proposed theory of calculating the cantilever bar of step-variable bending stiffness with point masses located along its length (height).

Let's study the effect of changing the scaling relative stiffness $i_0 = 8.06 \cdot 10^8\, kgcm$ on the oscillation frequency response $(\omega_1)$ of the cantilever bar (Figure 2, a) according to formulas (4):

$$\delta_{55} = \frac{768.992 \cdot 10^4}{3.5 i_0} = \frac{219.71 \cdot 10^4}{i_0}; \quad \delta_{44} = \frac{136.692 \cdot 10^4}{i_0};$$
$$\delta_{33} = \frac{73.73 \cdot 10^4}{i_0}; \quad \delta_{22} = \frac{30.62 \cdot 10^4}{i_0}; \quad \delta_{11} = \frac{60.81 \cdot 10^4}{i_0}. \tag{5}$$

According to formula (3) let's calculate taking into account expression (5) with $i_0 = (1, 3, 5, 7, 9, 11) \cdot 8.06 \cdot 10^8\, kgcm$ (Table 1).

Table 1 – Values of the oscillation frequency response for free oscillations of the cantilever bar

| $10^{-8}\, i_0$ | 1.0 | 3.0 | 5.0 | 7.0 | 9.0 | 11.0 |
|---|---|---|---|---|---|---|
| $\delta_{11}$ | 6.81x10-4 | 2.27x10-4 | 1.36x10-4 | 0.97x10-4 | 0.76x10-4 | 0.62x10-4 |
| $\delta_{22}$ | 30.62x10-4 | 10.21x10-4 | 6.12x10-4 | 4.37x10-4 | 3.4x10-4 | 2.78x10-4 |
| $\delta_{33}$ | 73.73x10-4 | 24.5x10-4 | 14.75x10-4 | 10.53x10-4 | 8.19x10-4 | 6.70x10-4 |
| $\delta_{44}$ | 136,69x10-4 | 45.5x10-4 | 23.34x10-4 | 19.52x10-4 | 45.19x10-4 | 12.42x10-4 |
| $\delta_{55}$ | 219.71x10-4 | 73.24x10-4 | 43.94x10-4 | 31.39x10-4 | 24.41x10-4 | 19.97x10-4 |
| $\omega_1, c^{-1}$ | 5.954 | 3.44 | 2.652 | 2.242 | 1.977 | 1.789 |

According to Table 1, let's build the graphs of the $\omega_{1,i} = f(i_{0,i})$ $(i = 1, 3, 5, 7, 9, 11)$ dependence. This graph is presented in Figure 4.

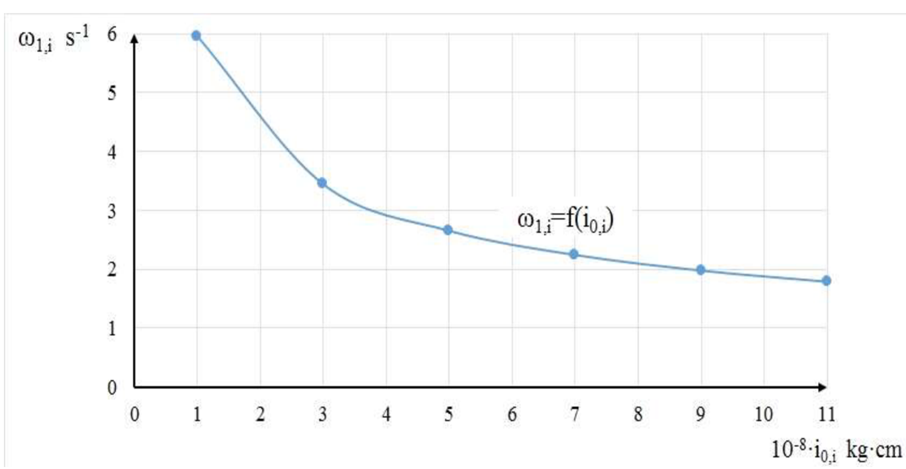


Figure 4: The oscillation frequency response dependence on the relative stiffness value of the cantilever bar

## 4 Conclusions

Based on the analytical operations, generalized formula (1) has been obtained for an approximate estimation of the oscillation frequency response with varying parameters of the section lengths, point mass values, and relative stiffness values of sections of a five-stage cantilever bar (Figure 1, a).

A numerical example (Figure 2, a) shows reliability of the proposed theoretical provisions; this is shown by the proximity of the oscillation frequency response values obtained by two independent methods of calculating $\omega_1 \approx \omega_1^*$.

The dependence of the oscillation frequency response value the cantilever bar (Figure 2, a) on changing the value of relative stiffness $i_0$ in the range from $1 \cdot 10^8\,kgcm$ to $11 \cdot 10^8\,kgcm$ has been studied, which is reflected graphically (Figure 4).

The dependence $\omega_{1,0} = f(i_{0,i})$ shown in Figure 4 can be used as a nomogram to determine the oscillation frequency response of various cantilever bars (Figure 1, a) at arbitrary values of the main relative stiffness of the lower section of the cantilever bar with step-variable bending stiffness.

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