1-бөлім **Раздел 1** **Section 1**

Математика **Математика** **Mathematics**

IRSTI 27.31.21 DOI: https://doi.org/10.26577/JMMCS.2022.v115.i3.01

S.E. Aitzhanov[1*], **J. Ferreira**[2], **K.A. Zhalgassova**[3]
[1]Al-Farabi Kazakh National University, Kazakhstan, Almaty
[2]Federal University of Fluminense, Brazil, Volta Redonda
[3] M.Auezov South Kazakhstan University, Kazakhstan, Shymkent
*e-mail: aitzhanovserik81@gmail.com

SOLVABILITY OF THE INVERSE PROBLEM FOR THE PSEUDOHYPERBOLIC EQUATION

This paper investigates the solvability of the inverse problem of finding a solution and an unknown coefficient in a pseudohyperbolic equation known as the Klein-Gordon equation. A distinctive feature of the given problem is that the unknown coefficient is a function that depends only on the time variable. The problem is considered in the cylinder, the conditions of the usual initial-boundary value problem are set. The integral overdetermination condition is used as an additional condition. In this paper, the inverse problem is reduced to an equivalent problem for the loaded nonlinear pseudohyperbolic equation. Such equations belong to the class of partial differential equations, not resolved with respect to the highest time derivative, and they are also called composite type equations. The proof uses the Galerkin method and the compactness method (using the obtained a priori estimates). For the problem under study, the authors prove existence and uniqueness theorems for the solution in appropriate classes.
Key words: Pseudohyperbolic equation, inverse problem, Klein-Gordon equation, Galerkin method, compactness method, existence, uniqueness.

С.Е. Айтжанов[1*], Ж. Феррейра[2], К.А. Жалғасова[3]
[1]Әл-Фараби атындағы Қазақ ұлттық университеті, Қазақстан, Алматы қ.
[2]Флюминенс Федерал университеті, Бразилия, Вольта-Редонда қ.
[3]М.Әуезов атындағы Оңтүстік Қазақстан университеті, Қазақстан, Шымкент қ.
*e-mail: aitzhanovserik81@gmail.com

Псевдогиперболалық теңдеу үшін кері есептің шешімділігі

Мақалада Клейн-Гордон теңдеуі деген атпен белгілі псевдогиперболалық теңдеудің шешімін және оң жақ коэффициентін табу кері есебі зерттеледі. Бұл есеп ізделінді коэффициенттің тек уақыттан тәуелді функция болуымен ерекшеленеді. Есеп цилиндрлік аймақта қарастырылады, әдеттегідей бастапқы-шеттік есептің шарттары қойылады. Қосымша шарт ретінде интегралдық түрдегі артық анықталған шарт берілген. Бұл жұмыста кері есеп жүктелген сызықтық емес псевдогиперболалық теңдеу үшін қойылған эквивалентті есепке келтіріледі. Мұндай теңдеулер уақыт бойынша ең жоғары туындыға қатысты шешілмеген дербес туындылы дифференциалдық теңдеулер класына жатады және оларды құрама типті теңдеулер деп те атайды. Дәлелдеуде Галеркин әдісі және компакт әдісі (априорлық бағалаулар алу арқылы) қолданылады. Жұмыста зерттеліп отырған есептің сәйкес кластардағы шешімнің бар болу және жалғыздық теоремалары дәлелденеді.
Түйін сөздер: Псевдогиперболалық теңдеу, кері есеп, Клейн-Гордон теңдеуі, Галеркин әдісі, компакт әдісі, шешімнің бар болуы және жалғыздығы.

С.Е. Айтжанов[1*], Ж. Феррейра[2], К.А. Жалгасова[3]
[1]Казахский национальный университет имени аль-Фараби, Казахстан, г. Алматы
[2]Федеральный университет Флуминенсе, Бразилия, г. Вольта-Редонда
[3]Южно-Казахстанский университет имени М.Ауезова, Казахстан, г. Чимкент
*e-mail: aitzhanovserik81@gmail.com

Разрешимость обратной задачи для псевдогиперболического уравнения

Исследуется разрешимость обратной задачи нахождения решения и неизвестного коэффициента в псевдогиперболическом уравнении, известного как уравнение Клейна-Гордона. Отличительной особенностью изучаемой задачи является то, что неизвестный коэффициент является функцией, зависящей лишь от временной переменной. Задача рассматривается в цилиндрической области, задаются условия обычной начально-краевой задачи. В качестве дополнительного условия используется условие интегрального переопределения. В работе обратная задача сводится к эквивалентной задаче для нагруженного нелинейного псевдогиперболического уравнения. Подобные уравнения относятся к классу дифференциальных уравнений в частных производных, не разрешенные относительно старшей производной по времени и они также называются уравнениями составного типа. При доказательстве применяются метод Галеркина и метод компактности (с использованием полученных априорных оценок). Для изучаемой задачи авторы доказывают теоремы существования и единственности решения в рассматриваемых классах.

Ключевые слова: Псевдогиперболическое уравнение, обратная задача, уравнение Клейна-Гордона, метод Галеркина, метод компактности, существование, единственность.

1 Introduction

The work is devoted to the study of the solvability of the inverse problem of reCЃovering an external influence in the pseudohyperbolic equation known as the Klein-Gordon equation. Nowadays, inverse problems have become a powerful and rapidly developing field of knowledge, penetrating almost all areas of mathematics. Similar inverse problems arise naturally in the mathematical modeling of certain processes occurring in the media with unknown characteristics. Since it is the characteristics of the medium that determine the coefficients of the corresponding differential equation or the coefficients of the external influence. The Klein-Gordon equation plays an important role in mathematical physics. This equation is used in modeling various phenomena of relativistic quantum mechanics [1] and nonlinear optics, in studying the behavior of elementary particles and dislocation propagation in crystals, as well as in studying nonlinear wave equations [2]. For such equations, many problems have been investigated in different formulations by various methods [3]-[14].

In this paper, the inverse problem under study is reduced to an equivalent problem for the loaded nonlinear pseudohyperbolic equation. Pseudohyperbolic equations belong to the class of partial differential equations, not solved with respect to the highest time derivative, and they are also known as composite type equations. Initial-boundary value problems for linear and nonlinear pseudohyperbolic equations were studied in various works [15]-[20]. Moreover, it is necessary to note the works [21]-[25], where studied the qualitative properties of solutions of inverse problems for hyperbolic type equations.

In the cylinder $Q_T = \{(x,t):\ x \in \Omega,\ 0 < t < T\}$ we consider the inverse problem of reCЃovering the right-hand side of the Klein-Gordon equation

$$u_{tt}-\chi\Delta u_t-\left(a_0+a_1\left\|\nabla u\right\|_{2,\Omega}^{2r}\right)\Delta u+|u_t|^{q-2}u_t=b(x,t)|u|^{p-2}u+f(t)h(x,t),\ \ (x,t)\in Q_T,\ (1)$$

with initial conditions

$$u(x,0)=u_0(x),\ \ u_t(x,0)=u_1(x),\ \ x\in\Omega, \qquad (2)$$

the boundary condition

$$u|_S=0, \qquad (3)$$

and the overdetermination condition

$$\int\limits_\Omega u(x,t)\omega(x)dx=\varphi(t),\ t\in(0,T). \qquad (4)$$

Here $\Omega\subset R^N,\ N\geq 1$ is bounded area, $\partial\Omega$ is sufficiently smooth boundary, $b(x,t),\ h(x,t),\ u_0(x),\ u_1(x),\ \omega(x),\ \varphi(t)$ are the given functions, $\chi,\ a_0,\ a_1,\ p,\ q$ and r are positive constants. Let the given functions of the problem (1)-(4) satisfy the conditions

$$\begin{aligned}&\omega\in L_2(\Omega)\bigcap\overset{0}{W}{}_2^2(\Omega),\\&h(x,t)\in C^1(Q_T),\quad h_1(t)\equiv\int\limits_\Omega h(x,t)\omega(x)dx\neq 0,\ \ \forall t\in[0,T],\end{aligned} \qquad (5)$$

$$\begin{aligned}&\varphi(t)\in W_2^2(0,T),\\&\int\limits_\Omega u_0(x)\omega(x)dx=\varphi(0),\ \ \int\limits_\Omega u_1(x)\omega(x)dx=\varphi'(0),\\&u_0\in\overset{0}{W}{}_2^2(\Omega),\quad u_1\in\overset{0}{W}{}_2^1(\Omega).\end{aligned} \qquad (6)$$

2 Materials and methods

2.1 The Equivalent Problem

Lemma 1. The problem (1)-(4) is equivalent to the next problem for nonlinear pseudoparabolic equation containing nonlinear nonlocal operator from function $u(x,t)$

$$u_{tt}-\chi\Delta u_t-\left(a_0+a_1\left\|\nabla u\right\|_{2,\Omega}^{2r}\right)\Delta u+|u_t|^{q-2}u_t=b(x,t)|u|^{p-2}u+F(t,u)h(x,t),\ \ x\in\Omega,\ \ \ t>0, \qquad (7)$$

$$u(x,0)=u_0(x),\ \ x\in\Omega,\quad u|_S=0. \qquad (8)$$

Here

$$\begin{aligned}F(t,u)=\tfrac{1}{h_1(t)}\Bigg(&\varphi''(t)+\chi\int\limits_\Omega\nabla u_t\nabla\omega dx+\left(a_0+a_1\left\|\nabla u\right\|_{2,\Omega}^{2r}\right)\int\limits_\Omega\nabla u\nabla\omega dx+\\&+\int\limits_\Omega|u_t|^{q-2}u_t\omega dx-\int\limits_\Omega b(x,t)|u|^{p-2}u\omega dx\Bigg).\end{aligned} \qquad (9)$$

Proof. Indeed, it follows from equation (1) that

$$\begin{aligned}&\int_\Omega (u_{tt}-\chi\Delta u_t)\omega dx-\left(a_0+a_1\|\nabla u\|_{2,\Omega}^{2r}\right)\int_\Omega \Delta u\omega dx-\\&-\int_\Omega b(x,t)|u|^{p-2}u\omega dx=\int_\Omega f(t)h(x,t)\omega dx,\end{aligned}\tag{10}$$

next, if conditions (4) and (5) are performed, then

$$\begin{aligned}&F(t,u)=\tfrac{1}{h_1(t)}\left(\varphi''(t)+\chi\int_\Omega \nabla u_t\nabla\omega dx+\left(a_0+a_1\|\nabla u\|_{2,\Omega}^{2r}\right)\int_\Omega \nabla u\nabla\omega dx+\right.\\&\left.+\int_\Omega |u_t|^{q-2}u_t\omega dx-\int_\Omega b(x,t)|u|^{p-2}u\omega dx\right).\end{aligned}\tag{11}$$

Therefore, the relation (9) is satisfied.

Now let us consider the problem (7)-(8). If the relation (9) is satisfied, then equality (11) obviously follows from it. Then

$$\begin{aligned}&F(t,u)=\tfrac{1}{h_1(t)}\left(\varphi''(t)+\chi\int_\Omega \nabla u_t\nabla\omega dx+\left(a_0+a_1\|\nabla u\|_{2,\Omega}^{2r}\right)\int_\Omega \nabla u\nabla\omega dx+\right.\\&\left.+\int_\Omega |u_t|^{q-2}u_t\omega dx-\int_\Omega b(x,t)|u|^{p-2}u\omega dx\right)=\\&=\tfrac{1}{h_1(t)}\left(\varphi''(t)-\chi\int_\Omega \Delta u_t\omega dx-\left(a_0+a_1\|\nabla u\|_{2,\Omega}^{2r}\right)\int_\Omega \Delta u\omega dx+\right.\\&\left.+\int_\Omega |u_t|^{q-2}u_t\omega dx-\int_\Omega b(x,t)|u|^{p-2}u\omega dx\right).\end{aligned}$$

By virtue of (10), we obtain that

$$\begin{aligned}&F(t,u)=\tfrac{1}{h_1(t)}\left(\varphi''(t)+\chi\int_\Omega \nabla u_t\nabla\omega dx+\left(a_0+a_1\|\nabla u\|_{2,\Omega}^{2r}\right)\int_\Omega \nabla u\nabla\omega dx-\int_\Omega b(x,t)|u|^{p-2}u\omega dx\right)=\\&=\tfrac{1}{h_1(t)}\left(\varphi''(t)-\chi\int_\Omega \Delta u_t\omega dx-\left(a_0+a_1\|\nabla u\|_{2,\Omega}^{2r}\right)\int_\Omega \Delta u\omega dx-\int_\Omega b(x,t)|u|^{p-2}u\omega dx\right)\\&=\tfrac{1}{h_1(t)}\left(\varphi''(t)-\int_\Omega u_{tt}\omega dx+\int_\Omega b(x,t)|u|^{p-2}u\omega dx+\int_\Omega f(t)h(x,t)\omega dx-\int_\Omega b(x,t)|u|^{p-2}u\omega dx\right).\end{aligned}$$

$$\varphi''(t)-\int_\Omega u_{tt}\omega dx=0.$$

In this way, $\frac{d^2}{dt^2}\left(\varphi(t)-\int_\Omega u\omega dx\right)=0$. Denote by $v(t)=\varphi(t)-\int_\Omega u\omega dx$. Then the function $v(t)$ can be found as a solution of the Cauchy problem: $v''(t)=0,\ v(0)=0,\ v'(0)=0$. ($v(0)=0,\ v'(0)=0$ follows from the matching condition (5)). The unique solution of the problem is the function $v(t)=0$, consequently, $\int_\Omega u(x,t)\omega(x)dx=\varphi(t)$.

3 Existence of the solution. Galerkin approximations

Theorem 1. *Let the conditions* (5), (6) *and* $2 \le p < \frac{2n-2}{n-2}$, $n \ge 3$, $q \ge 2$, $r > 1$ *are performed. Then there exists the generalized solution* Δu, Δu_t, $u_{tt} \in L_2(Q_T)$ *of the problem* (7)-(8).

Proof. Let us choose in $\overset{0}{W}{}_2^1(\Omega)$ some system of functions $\{\Psi_j(x)\}$ forming a basis in the given space. As a basis, we can take the eigenfunctions of the Sturm-Liouville problem

$$\Delta\Psi + \lambda\Psi = 0, \quad \Psi|_{\partial\Omega} = 0.$$

We will look for an approximate solution of the problem (7)-(8) in the form

$$u_m(x,t) = \sum_{k=1}^{m} C_{mk}(t)\Psi_k(x) \tag{12}$$

where coefficients $C_{mk}(t)$ are searched out from the relations

$$\begin{gathered}\sum_{k=1}^{m} C''_{mk}(t)\int_\Omega \Psi_k\Psi_j dx + \chi\sum_{k=1}^{m} C'_{mk}(t)\int_\Omega \nabla\Psi_k\nabla\Psi_j dx + \left(a_0 + a_1\|\nabla u_m\|_{2,\Omega}^{2r}\right)\int_\Omega \nabla u_m\nabla\Psi_j dx + \\ + \sum_{k=1}^{m} C'_{mk}(t)\int_\Omega |\partial_t u_m|^{p-2}\Psi_k\Psi_j dx - \int_\Omega b(x,t)\,|u_m|^{p-2}u_m\Psi_j dx = \int_\Omega F(t,u_m)\Psi_j dx.\end{gathered} \tag{13}$$

$$\begin{aligned} u_{m0} &= u_m(0) = \sum_{k=1}^{m} C_{mk}(0)\Psi_k = \sum_{k=1}^{m}\alpha_{0k}\Psi_k, \\ u_{m1} &= u'_m(0) = \sum_{k=1}^{m} C'_{mk}(0)\Psi_k = \sum_{k=1}^{m}\alpha_{1k}\Psi_k \end{aligned} \tag{14}$$

and besides

$$\begin{aligned} &u_{m0} \to u_0 \text{ strongly in } \overset{0}{W}{}_2^2(\Omega) \text{ at } m\to\infty \\ &u_{m1} \to u_1 \text{ strongly in } \overset{0}{W}{}_2^1(\Omega) \text{ at } m\to\infty \end{aligned} \tag{15}$$

Let us introduce denotations

$$\vec{C}_m \equiv \{C_{1m}(t), ..., C_{mm}(t)\}^T, \vec{\alpha} \equiv \{\alpha_1, ..., \alpha_m\}^T, a_{kj} = \int_\Omega \Psi_k\Psi_j dx, b_{kj} = \chi\int_\Omega (\nabla\Psi_k, \nabla\Psi_j)\, dx,$$

$$\begin{gathered} f_{kj} = \chi\int_\Omega (\nabla\Psi_k, \nabla\Psi_j)\, dx + \left(a_0 + a_1\|\nabla u_m\|_{2,\Omega}^{2r}\right)\int_\Omega \nabla\Psi_k\nabla\Psi_j dx + \\ + \sum_{k=1}^{m} C'_{mk}(t)\int_\Omega |\partial_t u_m|^{p-2}\Psi_k\Psi_j dx - \int_\Omega b(x,t)\,|u_m|^{p-2}\Psi_k\Psi_j dx + \int_\Omega F(t,u_m)\Psi_j dx, \\ A_m\left(\vec{C}_m\right) \equiv \left\{a_{jk}\left(\vec{C}_m\right)\right\}, \vec{F}_m\left(\vec{C}_m, \vec{C}'_m\right) \equiv \left\{f_{jk}\left(\vec{C}_m, \vec{C}'_m\right)\right\}\vec{C}_m. \end{gathered}$$

Then the system of equations (13) and condition (14) take the matrix form

$$\begin{aligned} & A_m\vec{C}''_m \equiv \vec{F}_m\left(\vec{C}_m,\, \vec{C}'_m\right), \\ & \vec{C}_m(0)=\vec{\alpha}_0,\ \vec{C}'_m(0)=\vec{\alpha}_1. \end{aligned} \tag{16}$$

Relations (16) represent the Cauchy problem for the system of ordinary differential equations, which is solvable on the segment $[0,\, T_m]$. In order to verify the existence of the solution on $[0,\, T]$, we obtain a priori estimates.

Lemma 2. If $u \in \overset{0}{W}{}_2^1(\Omega)$,$1<\sigma\le 2$, then the following inequality is performed

$$\int\limits_\Omega |u_m|^\sigma dx \le \left(\int\limits_\Omega |u|^2dx\right)^{\frac{\sigma}{2}} |\Omega|^{\frac{2-\sigma}{2}} \le C_0\left(\int\limits_\Omega |u|^2dx + \chi\int\limits_\Omega |\nabla u|^2dx\right).$$

Lemma 3. If $u \in \overset{0}{W}{}_2^1(\Omega), 2<\beta<\frac{2N}{N-2},\ N\ge 3,$, then the following inequality is performed

$$\|u\|^2_{\beta,\Omega} \le C_0^2\,\|\nabla u\|^{2\alpha}_{2,\Omega}\,\|u\|^{2(1-\alpha)}_{2,\Omega} \le \chi\,\|\nabla u\|^2_{2,\Omega} + \frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}C_0^{\frac{2}{1-\alpha}}}{\chi^{\frac{\alpha}{1-\alpha}}}\,\|u\|^2_{2,\Omega}\,,$$

where $C_0=\left(\frac{2(N-1)}{N-2}\right)^\alpha,\ \ \alpha=\frac{(\beta-2)N}{2\beta},\ \ 0<\alpha<1.$

We multiply the equality (13) by $C'_{mj}(t)$ and summarize over $j=\overline{1,m}$. As a result, we take

$$\begin{aligned} & \tfrac{1}{2}\tfrac{d}{dt}\int\limits_\Omega |\partial_t u_m(t)|^2dx + \chi\int\limits_\Omega |\partial_t\nabla u_m|^2dx + \tfrac{a_0}{2}\tfrac{d}{dt}\,\|\nabla u_m\|^2_{2,\Omega} + \\ & +\tfrac{a_1}{2r+2}\tfrac{d}{dt}\,\|\nabla u_m\|^{2r+2}_{2,\Omega} + \int\limits_\Omega |\partial_t u_m(t)|^q dx = \\ & = \int\limits_\Omega b(x,t)|u_m|^{p-2}u_m\partial_t u_m dx + \int\limits_\Omega F(t,u_m)h\partial_t u_m dx. \end{aligned} \tag{17}$$

We integrate with respect to τ from 0 to t, then we get the relation

$$\begin{aligned} & \tfrac{1}{2}\int\limits_\Omega |\partial_t u_m(t)|^2dx + \chi\int\limits_0^t\int\limits_\Omega |\partial_\tau\nabla u_m|^2dxd\tau + \tfrac{a_0}{2}\,\|\nabla u_m\|^2_{2,\Omega} + \\ & +\tfrac{a_1}{2r+2}\,\|\nabla u_m\|^{2r+2}_{2,\Omega} + \int\limits_0^t\int\limits_\Omega |\partial_\tau u_m|^q dxd\tau = \\ & = \tfrac{1}{2}\int\limits_\Omega |\partial_t u_m(x,0)|^2dx + \tfrac{a_0}{2}\,\|\nabla u_m(x,0)\|^2_{2,\Omega} + \tfrac{a_1}{2r+2}\,\|\nabla u_m(x,0)\|^{2r+2}_{2,\Omega} + \\ & + \int\limits_0^t\int\limits_\Omega b(x,\tau)|u_m|^{p-2}u_m\partial_\tau u_m dxd\tau + \int\limits_0^t\int\limits_\Omega F(t,u_m)h\partial_\tau u_m dxd\tau. \end{aligned} \tag{18}$$

Denote by

$$y(t) = \frac{1}{2}\int\limits_{\Omega} |\partial_t u_m(t)|^2 dx + \frac{a_0}{2}\left\|\nabla u_m\right\|_{2,\Omega}^2 + \frac{a_1}{2r+2}\left\|\nabla u_m\right\|_{2,\Omega}^{2r+2}.$$

Estimating the right-hand side of (18) using Lemma 2 and 3, as well as the Hölder and Young inequality, we obtain

$$\begin{aligned} &\left|\int\limits_0^t\int\limits_\Omega b(x,\tau)|u_m|^{p-2}u_m\partial_\tau u_m dxd\tau\right| \le b_0\int\limits_0^t\int\limits_\Omega |u_m|^{p-1}\partial_\tau u_m dxd\tau \le \\ &\le \left\|\partial_\tau u_m\right\|_{2,Q_t}\left(\int\limits_0^t\int\limits_\Omega |u_m|^{\frac{2n}{n-2}}dxd\tau\right)^{\frac{n-2}{2n}}\left(\int\limits_0^t\int\limits_\Omega |u_m|^{(p-2)n}dxd\tau\right)^{\frac{1}{n}} \le \\ &\le \left\|\partial_\tau u_m\right\|_{2,Q_t}^2 + C_1\left\|\nabla u_m\right\|_{2,Q_t}^{\frac{2n}{n-2}}. \end{aligned} \tag{19}$$

$$\begin{aligned} &\left|\chi\int\limits_0^t\int\limits_\Omega \tfrac{1}{h_1(\tau)}\int\limits_\Omega \partial_\tau\nabla u_m\nabla\omega dx\, h\partial_\tau u_m dxd\tau\right| \le \\ &\le \chi\int\limits_0^t \tfrac{1}{h_1(\tau)}\left\|\partial_\tau\nabla u_m\right\|_{2,\Omega}\left\|\nabla\omega\right\|_{2,\Omega}\left\|h\right\|_{2,\Omega}\left\|\partial_\tau u_m\right\|_{2,\Omega}d\tau \le \\ &\le \tfrac{\chi}{2}\int\limits_0^t \left\|\partial_\tau\nabla u_m\right\|_{2,\Omega}^2 d\tau + \tfrac{\chi}{2}\left\|\nabla\omega\right\|_{2,\Omega}^2\sup_{0\le t\le T}\tfrac{\|h(x,t)\|_{2,\Omega}^2}{|h_1(t)|^2}\int\limits_0^t\left\|\partial_\tau u_m\right\|_{2,\Omega}^2 d\tau. \end{aligned}$$

$$\begin{aligned} &\left|\int\limits_0^t\int\limits_\Omega \tfrac{1}{h_1(\tau)}\left(a_0 + a_1\left\|\nabla u_m\right\|_{2,\Omega}^{2r}\right)\int_\Omega \nabla u_m\nabla\omega dx\, h\partial_\tau u_m dxd\tau\right| \le \\ &\le \int\limits_0^t \tfrac{1}{h_1(\tau)}\left(a_0 + a_1\left\|\nabla u_m\right\|_{2,\Omega}^{2r}\right)\left\|\nabla u_m\right\|_{2,\Omega}\left\|\nabla\omega\right\|_{2,\Omega}\left\|h\right\|_{2,\Omega}\left\|\partial_\tau u_m\right\|_{2,\Omega}d\tau \le \\ &\le a_0\int\limits_0^t \left\|\nabla u_m\right\|_{2,\Omega}^2 d\tau + a_0\left\|\nabla\omega\right\|_{2,\Omega}^2\int\limits_0^t \tfrac{1}{h_1^2(\tau)}\left\|h\right\|_{2,\Omega}^2\left\|\partial_\tau u_m\right\|_{2,\Omega}^2 d\tau + \\ &+ a_1\int\limits_0^t \left\|\nabla u_m\right\|_{2,\Omega}^{2r+2} d\tau + C_2\left\|\nabla\omega\right\|_{2,\Omega}^{2r+2}\sup_{0\le t\le T}\tfrac{\|h(x,t)\|_{2,\Omega}^{2r+2}}{|h_1(t)|^{2r+2}}\int\limits_0^t\left\|\partial_\tau u_m\right\|_{2,\Omega}^{2r+2} d\tau, \end{aligned}$$

$$C_2 = \frac{a_1(2r+1)^{2r+1}}{(2r+2)^{2r+2}}.$$

$$\begin{aligned} &\left|\int\limits_0^t\int\limits_\Omega \tfrac{1}{h_1(\tau)}\int\limits_\Omega |\partial_\tau u_m|^{q-2}\partial_\tau u_m\omega dx\, h\partial_\tau u_m dxd\tau\right| \le \\ &\le \int\limits_0^t \tfrac{1}{h_1(\tau)}\left\|\partial_\tau u_m\right\|_{q,\Omega}^{q-1}\left\|\omega\right\|_{q,\Omega}\left\|h\right\|_{2,\Omega}\left\|\partial_\tau u_m\right\|_{2,\Omega}d\tau \le \\ &\le \tfrac{1}{2}\int\limits_0^t \left\|\partial_\tau u_m\right\|_{q,\Omega}^q d\tau + C_3\left\|\omega\right\|_{q,\Omega}^q\sup_{0\le t\le T}\tfrac{\|h(x,t)\|_{2,\Omega}^q}{|h_1(t)|^q}\int\limits_0^t\left\|\partial_\tau u_m\right\|_{2,\Omega}^q d\tau, \end{aligned}$$

$$C_3 = \frac{q-1}{q\left(\frac{q}{2}\right)^{\frac{1}{q-1}}}.$$

$$\left|\int\limits_0^t\int\limits_\Omega \frac{1}{h_1(\tau)}\int\limits_\Omega b(x,t)|u_m|^{p-2}u_m\omega dx\, h\partial_\tau u_m dx d\tau\right| \le$$
$$\le b_0\|\omega\|_{p,\Omega}\sup_{0\le t\le T}\frac{\|h(x,t)\|_{2,\Omega}}{h_1(t)}\int\limits_0^t\|\partial_\tau u_m\|_{2,\Omega}\|u_m\|_{p,\Omega}^{p-1}d\tau \le$$
$$\le b_0\|\omega\|_{p,\Omega}\sup_{0\le t\le T}\frac{\|h(x,t)\|_{2,\Omega}}{h_1(t)}\left(\int\limits_0^t\|u_m\|_{p,\Omega}^p d\tau+\int\limits_0^t\|\partial_\tau u_m\|_{2,\Omega}^p d\tau\right)\le$$
$$\le b_0\|\omega\|_{p,\Omega}\sup_{0\le t\le T}\frac{\|h(x,t)\|_{2,\Omega}}{h_1(t)}\int\limits_0^t\left(\|u_m\|_{2,\Omega}^2+\chi\|\nabla u_m\|_{2,\Omega}^2\right)^{\frac{p}{2}}d\tau+$$
$$+b_0\|\omega\|_{p,\Omega}\sup_{0\le t\le T}\frac{\|h(x,t)\|_{2,\Omega}}{h_1(t)}\int\limits_0^t\|\partial_\tau u_m\|_{2,\Omega}^p d\tau.$$

Denote by

$$y(t)=\frac{1}{2}\int\limits_\Omega|\partial_t u_m(t)|^2dx+\frac{a_0}{4}C(\Omega)\|u_m\|_{2,\Omega}^2+\frac{a_0}{4}\|\nabla u_m\|_{2,\Omega}^2+\frac{a_1}{2r+2}\|\nabla u_m\|_{2,\Omega}^{2r+2}.$$

$$d=\max\left\{\frac{n}{n-2},\ \frac{p}{2},\ \frac{q}{2},\ r+1\right\}.$$

Then from the relation (18), we get

$$y(t)\le C_4+C_5\int\limits_0^t[y(\tau)]^d d\tau.$$

Applying for this the generalized Bihari lemma, then the next inequality is true

$$y(t)\le\frac{C_4}{\left[1-(d-1)C_5C_4^{d-1}t\right]^{\frac{1}{d-1}}},$$

i.Pμ. $\frac{1}{2}\int\limits_\Omega|\partial_t u_m(t)|^2dx+\frac{a_0}{4}C(\Omega)\|u_m\|_{2,\Omega}^2+\frac{a_0}{4}\|\nabla u_m\|_{2,\Omega}^2+\frac{a_1}{2r+2}\|\nabla u_m\|_{2,\Omega}^{2r+2}\le\frac{!_4}{\left[1-(d-1)C_5C_4^{d-1}t\right]^{\frac{1}{d-1}}}.$

From this estimate we can conclude that there exists $T_0>0$ such that

$$\begin{aligned}&\int\limits_\Omega|\partial_t u_m(t)|^2dx+\|u_m\|_{2,\Omega}^2+\|\nabla u_m\|_{2,\Omega}^2+\\&+\|\nabla u_m\|_{2,\Omega}^{2r+2}+\int\limits_0^T\int\limits_\Omega|\partial_\tau\nabla u_m|^2dxd\tau+\int\limits_0^T\int\limits_\Omega|\partial_\tau u_m|^q dxd\tau\le C_6,\end{aligned}\qquad(20)$$

for all $t\in[0,T],\ T<T_0$, where C_6 is constant which does not depend on $m\in N$.

We multiply the relation (13) by $\lambda_j C_{mj}(t)$ and $C''_{mj}(t)$, then summarize over $j=\overline{1,m}$. As a result, we get the next relations

$$\begin{aligned}&-\frac{d}{dt}\int\limits_\Omega\partial_t u_m(t)\Delta u_m(t)dx-\|\partial_t\nabla u_m\|_{2,\Omega}^2+\frac{\chi}{2}\frac{d}{dt}\|\Delta u_m\|_{2,\Omega}^2+\\&+\left(a_0+a_1\|\nabla u_m\|_{2,\Omega}^{2r}\right)\|\Delta u_m\|_{2,\Omega}^2-\int\limits_\Omega|\partial_t u_m(t)|^{q-2}\partial_t u_m(t)\Delta u_m dx=\\&=-\int\limits_\Omega b(x,t)|u_m|^{p-2}u_m\Delta u_m dx-\int\limits_\Omega F(t,u_m)h\Delta u_m dx.\end{aligned}$$

$$\int\limits_\Omega |\partial_t^2 u_m(t)|^2 dx + \tfrac{\chi}{2}\tfrac{d}{dt}\int\limits_\Omega |\partial_t \nabla u_m|^2 dx - \left(a_0 + a_1 \|\nabla u_m\|_{2,\Omega}^{2r}\right)\int\limits_\Omega \Delta u_m \partial_t^2 u_m dx +$$
$$+\tfrac{1}{q}\tfrac{d}{dt}\int\limits_\Omega |\partial_t u_m|^q dx = \int\limits_\Omega b\,(x,t)\,|u_m|^{p-2}u_m\partial_t^2 u_m dx + \int\limits_\Omega F(t,u_m)\partial_t^2 u_m dx.$$

By integrating these relations from 0 to t, we get

$$\tfrac{\chi}{2}\|\Delta u_m\|_{2,\Omega}^2 + \int\limits_0^t \left(a_0 + a_1\|\nabla u_m\|_{2,\Omega}^{2r}\right)\|\Delta u_m\|_{2,\Omega}^2 d\tau = \tfrac{\chi}{2}\|\Delta u_m(0)\|_{2,\Omega}^2 + \int\limits_\Omega \partial_t u_m(t)\Delta u_m(t)dx -$$
$$-\int\limits_\Omega \partial_t u_m(0)\Delta u_m(0)dx + \int\limits_0^t \|\partial_\tau \nabla u_m\|_{2,\Omega}^2 d\tau + \int\limits_0^t\int\limits_\Omega |\partial_\tau u_m(\tau)|^{q-2}\partial_\tau u_m(\tau)\Delta u_m dx d\tau -$$
$$-\int\limits_0^t\int\limits_\Omega b(x,\tau)|u_m|^{p-2}u_m\Delta u_m dx d\tau - \int\limits_0^t\int\limits_\Omega F(\tau,u_m)h\Delta u_m dx d\tau. \tag{21}$$

$$\tfrac{\chi}{2}\int\limits_\Omega |\partial_t\nabla u_m(t)|^2 dx + \tfrac{1}{q}\int\limits_\Omega |\partial_t u_m(t)|^q dx + \int\limits_0^t\int\limits_\Omega |\partial_\tau^2 u_m(x,\tau)|^2 dx d\tau =$$
$$= \int\limits_0^t \left(a_0 + a_1\|\nabla u_m\|_{2,\Omega}^{2r}\right)\int\limits_\Omega \Delta u_m \partial_\tau^2 u_m dx d\tau + \tag{22}$$
$$+\int_0^t\int\limits_\Omega b\,(x,\tau)\,|u_m|^{p-2}u_m\partial_\tau^2 u_m dx d\tau + \int\limits_0^t\int\limits_\Omega F(\tau,u_m)\partial_\tau^2 u_m dx d\tau.$$

Analogically, we estimate the right-hand side of (21) and (22), applying lemmas 2 and 3, Hölder and Young inequalities, Bihari's lemma and a priori estimate (20), as a result we obtain

$$\|\Delta u_m\|_{2,\Omega}^2 + \int\limits_0^T \left(a_0 + a_1\|\nabla u_m\|_{2,\Omega}^{2r}\right)\|\Delta u_m\|_{2,\Omega}^2 dt \le C_7, \text{ for all } t\in[0,T],\ T<T_0, \tag{23}$$

$$\int\limits_\Omega |\partial_t\nabla u_m(t)|^2 dx + \int\limits_\Omega |\partial_t u_m(t)|^q dx + \int\limits_0^T\int\limits_\Omega |\partial_\tau^2 u_m(x,\tau)|^2 dx dt \le C_8, \text{ for all } t\in[0,T],\ T<T_0, \tag{24}$$

where C_7 and C_8 are constants which does not depend on $m \in N$.

From the obtained estimates (20), (23) and (3) follows the estimate

$$\int\limits_0^T \|\partial_t \Delta u_m\|_{2,\Omega}^2 dt \le C_9, \text{ for all } t\in[0,T],\ T<T_0,\ m\in N. \tag{25}$$

Then by using (20), (23), (3) and (25), considering the conditions of the theorem, we can show the existence of the derivative $u_{xx} \in L_2(Q_T)$. In this way, $\Delta u,\ \Delta u_t,\ u_{tt} \in L_2(Q_T)$.

4 Uniqueness of the generalized solution

Theorem 2. *Let the conditions* (5), $r > 2,\ q > 2,\ 2 < p \leq 2+\frac{1}{N-2},\quad N \geq 3$, *are performed. Then the generalized solution of the problem* (1)-(3) *on the segment* $(0,T)$ *is unique.*

Proof. Assume that the problem (7)-(8) has two generalized solutions: $u_1(x,t)$ and $u_2(x,t)$. Let us put $u(x,t) = u_1(x,t) - u_2(x,t)$. Then there are the following equalities

$$\begin{gathered} u_{tt} - \chi\Delta u_t - a_0\Delta u - a_1\left(\|\nabla u_1\|_{2,\Omega}^{2r}\Delta u_1 - \|\nabla u_2\|_{2,\Omega}^{2r}\Delta u_2\right) + \\ +|u_{1t}|^{q-2}u_{1t} - |u_{2t}|^{q-2}u_{2t} = b(x,t)\left(|u_1|^{p-2}u_1 - |u_2|^{p-2}u_2\right) + \\ +h(x,t)\left(F(t,u_1) - F(t,u_2)\right),\ \ x\in\Omega,\quad t>0, \end{gathered} \tag{26}$$

$$u(x,0) = 0,\ \ u_t(x,0) = 0,\ \ x\in\Omega,\quad u|_S = 0. \tag{27}$$

We consider the equality

$$\begin{gathered} \int_0^t\int_\Omega \left[u_{\tau\tau} - \chi\Delta u_\tau - a_0\Delta u - a_1\left(\|\nabla u_1\|_{2,\Omega}^{2r}\Delta u_1 - \|\nabla u_2\|_{2,\Omega}^{2r}\Delta u_2\right) + \right. \\ \left. +|u_{1\tau}|^{q-2}u_{1\tau} - |u_{2\tau}|^{q-2}u_{2\tau}\right]u_\tau dxd\tau = \int_0^t\int_\Omega \left[b(x,\tau)\left(|u_1|^{p-2}u_1 - |u_2|^{p-2}u_2\right) + \right. \\ \left. +h(x,\tau)\left(F(\tau,u_1) - F(\tau,u_2)\right)\right]u_\tau dxd\tau. \end{gathered}$$

By applying the next inequalities

$$||u_1|^q u_1 - |u_2|^q u_2| \leq (q+1)\left(|u_1|^q + |u_2|^q\right)|u_1 - u_2| \text{ at } q>0,$$

$$|\left(|u_1|^q u_1 - |u_2|^q u_2\right)(u_1 - u_2)| \geq |u_1 - u_2|^{q+2} \text{ at } q>0.$$

As a result, we obtain the inequality

$$\begin{gathered} \frac{1}{2}\int_\Omega u_t^2(t)dx + \chi\int_0^t\int_\Omega |\nabla u_\tau|^2 dxd\tau + \frac{a_0}{2}\int_\Omega |\nabla u|^2dx + \int_0^t\int_\Omega |u_\tau|^q dxd\tau \leq \\ \leq -a_1\int_0^t\left(\|\nabla u_1\|_{2,\Omega}^{2r}\int_\Omega \nabla u\nabla u_\tau dx - \left(\|\nabla u_1\|_{2,\Omega}^{2r} - \|\nabla u_2\|_{2,\Omega}^{2r}\right)\int_\Omega \nabla u_2\nabla u_\tau dx\right)d\tau + \\ +\int_0^t\int_\Omega b(x,\tau)\left(|u_1|^{p-2} + |u_2|^{p-2}\right)u_\tau dxd\tau + \int_0^t\int_\Omega h(x,\tau)\left(F(\tau,u_1) - F(\tau,u_2)\right)u_\tau dxd\tau. \end{gathered} \tag{28}$$

We estimate the right-hand side of the inequality (28), applying the Hölder's inequality

$$\begin{gathered} \left|\int_0^t\int_\Omega b(x,\tau)\left(|u_1|^{p-2}u_1 - |u_2|^{p-2}u_2\right)u_\tau dxd\tau\right| \leq b_1(p-1)\int_0^t\int_\Omega \left(|u_1|^{p-2} + |u_2|^{p-2}\right)uu_\tau dxd\tau \leq \\ \leq b_1(p-1)\left(\left(\int_0^t\int_\Omega |u_1|^{\frac{2r(p-2)}{r-2}}dxd\tau\right)^{\frac{r-2}{2r}} + \left(\int_0^t\int_\Omega |u_2|^{\frac{2r(p-2)}{r-2}}dxd\tau\right)^{\frac{r-2}{2r}}\right)\times \\ \times\left(\int_0^t\int_\Omega u^r dxd\tau\right)^{\frac{1}{r}}\left(\int_0^t\int_\Omega u_\tau^2 dxd\tau\right)^{\frac{1}{2}}. \end{gathered}$$

Let us put $r = \frac{2N}{N-2}$, $p \le 2 + \frac{1}{N-2}$, $N \ge 3$. Then by the Sobolev embedding theorem $H^1(\Omega) \to\to L_r(\Omega)$ and $H^1(\Omega) \to\to L_{2r(p-2)/(r-2)}(\Omega)$. In this case, taking into account the smoothness class of solutions $u_1(x,t)$ and $u_2(x,t)$, we come to the estimate

$$\left| \int_0^t \int_\Omega b(x,\tau) \left(|u_1|^{p-2}u_1 - |u_2|^{p-2}u_2\right) u_\tau dx d\tau \right| \le C_1 \int_0^t \left(\|u_\tau\|^2_{2,\Omega} + \|\nabla u\|^2_{2,\Omega} + \|u\|^2_{2,\Omega} \right) d\tau. \quad (29)$$

Let us estimate the first term

$$\left| a_1 \int_0^t \left(\|\nabla u_1\|^{2r}_{2,\Omega} \int_\Omega \nabla u \nabla u_\tau dx - \left(\|\nabla u_1\|^{2r}_{2,\Omega} - \|\nabla u_2\|^{2r}_{2,\Omega} \right) \int_\Omega \nabla u_2 \nabla u_\tau dx \right) d\tau \right| \le$$
$$\le a_1 \int_0^t \|\nabla u_1\|^{2r}_{2,\Omega} \|\nabla u\|_{2,\Omega} \|\nabla u_\tau\|_{2,\Omega} d\tau + a_1 \int_0^t \left(\|\nabla u_1\|^{2r-2}_{2,\Omega} + \|\nabla u_2\|^{2r-2}_{2,\Omega} \right) \|\nabla u_2\|_{2,\Omega} \|\nabla u_\tau\|_{2,\Omega} \times$$
$$\times \int_\Omega \left(|\nabla u_1|^2 - |\nabla u_2|^2\right) dx d\tau \le a_1 \int_0^t \|\nabla u_1\|^{2r}_{2,\Omega} \|\nabla u\|_{2,\Omega} \|\nabla u_\tau\|_{2,\Omega} d\tau +$$
$$+ a_1 C_2' \int_0^t \| |\nabla u_1| + |\nabla u_2| \|_{2,\Omega} \|\nabla u\|_{2,\Omega} \|\nabla u_\tau\|_{2,\Omega} \le$$
$$\le \frac{\chi}{4} \int_0^t \|\nabla u_\tau\|^2_{2,\Omega} d\tau + C_2 \int_0^t \left(\|u_\tau\|^2_{2,\Omega} + \|\nabla u\|^2_{2,\Omega} + \|u\|^2_{2,\Omega} \right) d\tau.$$

The third term is estimated in a similar way. From the obtained estimates, we get

$$\int_\Omega u_t^2(t) dx + C_0 \int_\Omega |u|^2 dx + a_0 \int_\Omega |\nabla u|^2 dx + \chi \int_0^t \int_\Omega |\nabla u_\tau|^2 dx d\tau + \int_0^t \int_\Omega |u_\tau|^q dx d\tau \le$$
$$\le C_4 \int_0^t \left(\|u_\tau\|^2_{2,\Omega} + a_0 \|\nabla u\|^2_{2,\Omega} + C_0 \|u\|^2_{2,\Omega} \right) d\tau + C_5 \int_0^t \left(\|u_\tau\|^2_{2,\Omega} + a_0 \|\nabla u\|^2_{2,\Omega} + C_0 \|u\|^2_{2,\Omega} \right)^d d\tau,$$

where $d > 1$.

From the last inequality follows that

$$\int_\Omega u_t^2(t) dx + C_0 \int_\Omega |u|^2 dx + a_0 \int_\Omega |\nabla u|^2 dx \le C_4 \int_0^t \left(\|u_\tau\|^2_{2,\Omega} + a_0 \|\nabla u\|^2_{2,\Omega} + C_0 \|u\|^2_{2,\Omega} \right) d\tau +$$
$$+ C_5 \int_0^t \left(\|u_\tau\|^2_{2,\Omega} + a_0 \|\nabla u\|^2_{2,\Omega} + C_0 \|u\|^2_{2,\Omega} \right)^d d\tau,$$

where by Bihari's lemma, implies $\int_\Omega u_t^2(t) dx + C_0 \int_\Omega |u|^2 dx + a_0 \int_\Omega |\nabla u|^2 dx = 0$ almost everywhere on the time interval $(0,T)$, which means that the generalized solution is unique.

5 Conclusion

In the paper, we investigated the solvability of the inverse problem of determining the solution of the pseudohyperbolic equation, also an unknown coefficient of a special form which identifies the external source. The methods used are based on the transition from the original problem to the equivalent problem for the loaded nonlinear pseudohyperbolic equation. For this problem we use Galerkin's method to prove the existence of a strong generalized solution. The obtained results on the solvability of the inverse problem are new and can be useful to study another problems in the given area.

6 Acknowledgment

This research has been funded by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan, Grant No. AP08052425.

References

[1] Greiner W., Relativistic Quantum Mechanics - Wave Equations, 3rd ed. Berlin: Springer; 2000.

[2] Dodd R. K. Solitons and nonlinear wave equations / R. K. Dodd, I. C. Eilbeck, J. D. Gibbon,H. C. Morris. L.: Acad. Press, 1982.

[3] Strauss W, Vazquez L. Numerical solution of a nonlinear Klein-Gordon equation.JComput Phys. 1978;28(2):271-278.

[4] JimГ©nez S, Vazquez L. Analysis of four numerical schemes for a nonlinear Klein-Gordon equation.Appl Math Comput. 1990;35(1):61-94.

[5] Wong YS, Qianshun C, Lianger G. An initial-boundary value problem of a nonlinear Klein-Gordon equation.Appl Math Comput.1997;84(1):77-93.

[6] Wazwaz A. M. Compactons, solitons and periodic solutions for some forms of nonlinear Klein-Gordon equations // Chaos Solitons Fractals. 2006. V. 28. P. 1005вЂ“1013.

[7] Sassaman R., Biswas A. Soliton perturbation theory four phi-four model and nonlinear Klein-Gordon equations // Comm. Nonlinear Sci. Numer. Simulat. 2009. V. 14. P. 3239вЂ“3249.

[8] Sassaman R., Biswas A. Topological and non-topological solitons of the generalized Klein-Gordon equations // Appl. Math. Comput. 2009. V. 215. P. 212вЂ“220.

[9] Bratsos A. G., Petrakis L. A. A modified predictor-corrector scheme for the Klein-Gordonequation // Intern. J. Comput. Math. 2010. V. 87, N 8. P. 1892вЂ“1904.

[10] Böhme C, Michael R. A scale-invariant Klein-Gordon model with time-dependent potential.Ann Univ Ferrara. 2012;58(2):229-250.

[11] BГ¶hme C, Michael R. Energy bounds for Klein-Gordon equations with time-dependent potential.Ann Univ Ferrara. 2013;59(1):31-55.

[12] Hao Cheng, Xiyu Mu, GreenвЂ™s function for the boundary value problem of the static Klein-Gordon equation stated on a rectangular region and its convergence analysis. Boundary Value Problems (2017) 2017:72.

[13] Tekin I, Mehraliyev YT, Ismailov MI. Existence and uniqueness of an inverse problemfor nonlinear Klein-Gordon equation.Math Meth Appl Sci. 2019;1-15.https://doi.org/10.1002/mma.5609

[14] Sobolev, S. L.; On certain new problem of mathematical physics. Izvestia Acad. Nauk.Mathem. 1954, no. 18, p. 3-50.

[15] Kozhanov, A.I., Comparison Theorems and Solvability of Boundary Value Problems for Some Classes of Evolution Equations Like Pseudoparabolic and Pseudohyperbolic Ones, Preprint Inst. Mat. Acad. Sci., Novosibirsk, 1990, no. 17

[16] Demidenko, G., The Cauchy problem for pseudohyperbolic equations, Selcuk J Appl Math, Vol. 1,No.1, pp.47-62, 2000.

[17] Sveshnikov, A. G.; Alshin, A. B.; Korpusov, O. M.; Pletner, Yu. D.; Linear and nonlinearSobolev type equations. Moscow.,Phizmatlit. 2007.

[18] Korpusov, O. M., Blow-up in nonclassical wave equations. Moscow, URSS. 2010.

[19] Pulkina, L.S. Solution to nonlocal problems of pseudohyperbolic equations // Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 116, pp. 1вЂ“9.

[20] Yamamoto, M., Zhang, X.: Global uniqueness and stability for a class of multidimensional inverse hyperbolic problems with two unknowns. Appl. Math. Optim. 48, 211вЂ“228 (2003)

[21] Oussaeif,T.E., Bouziani, A.: Inverse problem of hyperbolic equationwith an integral overdetermination condition. Electron J. Diff. Equ. 2016(138), 1вЂ“7 (2016)

[22] Shahrouzi, M.: On behavior of solutions to a class of nonlinear hyperbolic inverse source problem. Acta. Math. Sin-English Ser. 32(6), 683вЂ“698 (2016)

[23] Shahrouzi, M.: Blow up of solutions to a class of damped viscoelastic inverse source problem. Differ. Equ. Dyn. Syst. 28, 889вЂ“899 (2020)

[24] Shahrouzi, M.: General decay and blow up of solutions for a class of inverse problem with elasticity term and variable-exponent nonlinearities. Math Meth Appl Sci, (2021), Early view, https://doi.org/10.1002/mma.7891