1-бөлім **Раздел 1** **Section 1**

Математика **Математика** **Mathematics**

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ON GREEN'S FUNCTION OF SECOND DARBOUX PROBLEM FOR HYPERBOLIC EQUATION

A definition and justify a method for constructing the Green's function of the second Darboux problem for a two-dimensional linear hyperbolic equation of the second order in a characteristic triangle is given. In contrast to the (well-developed) theory of the Green's function for self-adjoint elliptic problems, this theory has not yet been developed. And for the case of asymmetric boundary value problems such studies have not been carried out. It is shown that the Green's function for a hyperbolic equation of the general form can be constructed using the Riemann-Green function for some auxiliary hyperbolic equation. The notion of the Green's function is more completely developed for Sturm-Liouville problems for an ordinary differential equation, for Dirichlet boundary value problems for Poisson equation, for initial boundary value problems for a heat equation. For many particular cases, the Greens' function has been constructed explicitly. However, many more problems require their consideration. In this paper, the problem of constructing the Green's function of the second Darboux problem for a hyperbolic equation was investigated. The Green's function for the hyperbolic problems differs significantly from the Green's function of problems for equations of elliptic and parabolic types.
Key words: Hyperbolic equation, initial-boundary value problem, second Darboux problem, boundary condition, Green function, a characteristic triangle, Riemann–Green function.

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ГИПЕРБОЛАЛЫҚ ТЕҢДЕУ ҮШІН ЕКІНШІ ДАРБУ ЕСЕБІНІҢ ГРИН ФУНКЦИЯСЫ

Сипаттамалық үшбұрышта қарастырылатын екінші ретті екі өлшемді сызықтық гиперболалық теңдеу үшін Грин функциясын құру әдістемесі анықталды және негізделді. Өз-өзіне түйіндес эллиптикалық есептер үшін Грин функциясының (жақсы дамыған) теориясынан айырмашылығы, сипаттамалық шекаралық есептер үшін бұл теория әлі жетік әзірленбегендігінде. Ал симметриялық емес шекаралық есептер жағдайында мұндай зерттеулер жүргізілмеген. Жалпы түрдегі гиперболалық теңдеуге арналған Грин функциясын кейбір (арнайы жолмен құрылған) көмекші гиперболалық теңдеу үшін Риман-Грин функциясын қолдана отырып құруға болатындығы көрсетілді. Грин функциясының толығырақ тұжырымдамасы қарапайым дифференциалдық теңдеу үшін Штурм-Лиувиль есептері үшін, Пуассон теңдеуі үшін Дирихле шеткі есептері үшін, жылуөткізгіштік теңдеуі үшін бастапқы шекаралық есептер үшін жасалған. Көптеген дербес жағдайларда Грин функциясы айқын түрде құрылған. Алайда, басқа да көптеген есептер оларды қарастыруды талап етеді. Бұл мақалада гиперболалық теңдеу үшін екінші Дарбу есебінің Грин функциясын құру мәселесі зерттелді. Гиперболалық есептер үшін құрылған Грин функциясы эллиптикалық және параболалық есептер үшін құрылған Грин функциясынан айтарлықтай ерекшеленеді.

Түйін сөздер: Гиперболалық теңдеу, бастапқы-шекаралық есеп, екінші Дарбу есебі, шекаралық шарт, Грин функциясы, характеристикалық үшбұрыш, Риман–Грин функциясы.

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О ФУНКЦИИ ГРИНА ВТОРОЙ ЗАДАЧИ ДАРБУ ДЛЯ ГИПЕРБОЛИЧЕСКОГО УРАВНЕНИЯ

Дано определение и обоснована методика построения функции Грина для второй задачи Дарбу для двумерного линейного гиперболического уравнения второго порядка, рассматриваемого в характеристическом треугольнике. В отличие от (хорошо разработанной) теории функции Грина для самосопряженных эллиптических задач, для характеристических граничных задач эта теория еще не подробно разработана. А для случая несимметрических граничных задач таких исследований не проводилось. Показано, что функция Грина для гиперболического уравнения общего вида может быть построена с использованием функции Римана-Грина для некоторого (специальным образом построенного) вспомогательного гиперболического уравнения. Наиболее полно понятие функции Грина разработано для задач Штурма-Лиувилля для обыкновенного дифференциального уравнения, для краевых задач Дирихле для уравнения Пуассона, для начально-краевых задач для уравнения теплопроводности. Для многих частных случаев функция Грина была построена в явном виде. Однако, еще многие задачи требуют своего рассмотрения. В настоящей статье исследована задача о построении функции Грина для второй задачи Дарбу для гиперболического уравнения. Функция Грина для гиперболических задач существенно отличается от функций Грина задач для уравнений эллиптического и параболического типа.

Ключевые слова: Гиперболическое уравнение, начально-краевая задача, вторая задача Дарбу, граничное условие, функция Грина, характеристический треугольник, функция Римана–Грина.

1 Introduction

In $S \subset \mathbb{R}^n$ let us consider some a linear differential equation

$$Lu(x) = f(x), \ x \in S, \tag{1}$$

with homogeneous boundary conditions

$$Qu(x) = 0, \ x \in S. \tag{2}$$

If a solution of this problem exists, is unique and can be represented in the integral form

$$u(x) = \int\limits_S G_Q(x,y) f(y) dy, \tag{3}$$

then the kernel of this integral operator (3), that is, the function $G_Q(x,y)$, is called the Green's function of problem (1), (2).

It is also said that the Green's function for each fixed $y \in S$ satisfies the equation

$$LG_Q(x,y) = \delta(x-y), \ x \in S, \tag{4}$$

and the boundary conditions (2). Here $\delta(x-y)$ is the Dirac delta function. Equation (4) should be understood in the sense of generalized functions.

It is known that if the operator of problem (1), (2) has eigenfunctions $\{u_k(x)\}_{k=1}^{\infty}$ forming the Riesz basis in $L_2(S)$, then the solution of the problem can be represented as

$$u(x) = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \langle f, v_k \rangle_{L_2(S)} u_k(x), \tag{5}$$

where $\langle \cdot, \cdot \rangle_{L_2(S)}$ is a scalar product in $L_2(S)$, λ_k are eigenvalues of the operator, $\{v_k(x)\}_{k=1}^{\infty}$ is a biorthogonal system to $\{u_k(x)\}_{k=1}^{\infty}$. Formula (5) is called the spectral representation of the solution or the spectral representation of the inverse operator.

Representing the scalar product as an integral, we obtain the integral representation (3) of the solution of the problem, where

$$G_Q(x, y) = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} u_k(x) \overline{v_k(y)} \tag{6}$$

is the Green's function of problem (1), (2). In the case when problem (1), (2) is self-adjoint, the system of its eigenfunctions forms an orthogonal basis. Therefore, we can choose $v_k(x) = u_k(x)$. In this case, it is easy to see from (6) that the Green's function is the symmetric function: $G_Q(x, y) = G_Q(y, x)$.

For series of characteristic problems for a wave equation and a wave equation with potential (despite the fact that these problems are solved by the method of separation of variables) all eigenvalues and eigenfunctions are constructed in the works of T. Sh. Kal'menov [1], [2] and M. A. Sadybekov [3]- [5]. Therefore, for these problems the Green's function can be constructed in the form of series (6). Although the presence of the Green's function is guaranteed for any self-adjoint problem, and it can be constructed in the form of series (6), the use of infinite series for constructing a solution of the problem is not very convenient. Therefore, the construction of the Green's function in the form of finite sums is actual.

We are interested in the integral representation of Green's function of the second Darboux problem for a general hyperbolic equation of the second order, since all the properties of Green's function of this problem come from the integral representation of Green's function.

The main difference between this paper and others, that in contrast to the previous works of other authors ([6]- [15] and others), we conduct the investigation and construction of the Green's function without the assumption of its symmetry. Also, unlike other authors, in this paper we will give a definition of the Green's function and a method for constructing it for the case of general coefficients.

2 Formulation of the problem

Let $\Omega = \{(\xi, \eta) : \ 0 \le \xi \le 1, \ \xi \le \eta \le 1\}$. The following hyperbolic equation is considered in Ω:

$$\frac{\partial^2 u}{\partial \xi \partial \eta} + a(\xi, \eta) \frac{\partial u}{\partial \xi} + b(\xi, \eta) \frac{\partial u}{\partial \eta} + c(\xi, \eta) u = f(\xi, \eta), \quad (\xi, \eta) \in \Omega, \tag{7}$$

with the initial condition

$$(u_\xi - u_\eta)(\xi, \xi) = \nu(\xi), \ 0 \le \xi \le 1, \tag{8}$$

and the boundary condition

$$u(0,\xi) = \tau(\xi),\ 0 \le \xi \le 1. \tag{9}$$

We will assume that $a, b, a_\xi, b_\eta, c, f \in C\left(\overline{\Omega}\right);\ \nu, \tau \in C^1\left([0,1]\right)$ and

$$a(\xi,\xi) = b(\xi,\xi),\ 0 \le \xi \le 1. \tag{10}$$

In [16] it was shown that equality (10) we can always get.

Also, we assume

$$a_\xi(\xi,\xi) = b_\eta(\xi,\xi),\ 0 \le \xi \le 1. \tag{11}$$

3 Green's function of the problem (7)-(9)

Definition 1 *Green's function of the problem* (1)-(3) *let us call the function* $G(\xi,\eta;\xi_1,\eta_1)$*, which for every fixed* $(\xi_1,\eta_1) \in \Omega$*, satisfies the homogeneous equation*

$$L_{(\xi,\eta)}G(\xi,\eta;\xi_1,\eta_1) = 0,\ (\xi,\eta) \in \Omega,\ at\, \xi \ne \xi_1,\ \eta \ne \eta_1,\ \eta \ne \xi_1,\ \xi \ne \eta_1; \tag{12}$$

and the next boundary conditions

$$(G_\xi - G_\eta)(\xi,\xi;\xi_1,\eta_1) = 0,\ 0 \le \xi \le 1,\ (\xi_1,\eta_1) \in \Omega; \tag{13}$$

$$G(0,\xi;\xi_1,\eta_1) = 0,\ 0 \le \xi \le 1,\ (\xi_1,\eta_1) \in \Omega; \tag{14}$$

and on the above characteristic lines, the following conditions must be met: the values of the derivatives of the Green function in directions parallel to these characteristics must coincide in adjacent regions; i.e.,

$$\frac{\partial G(\xi_1+0,\eta;\xi_1,\eta_1)}{\partial \eta} + a(\xi_1,\eta)G(\xi_1+0,\eta;\xi_1,\eta_1)$$

$$= \frac{\partial G(\xi_1-0,\eta;\xi_1,\eta_1)}{\partial \eta} + a(\xi_1,\eta)G(\xi_1-0,\eta;\xi_1,\eta_1),\ at\ \eta \ne \eta_1,\ \eta \ne \xi_1; \tag{15}$$

$$\frac{\partial G(\eta_1+0,\eta;\xi_1,\eta_1)}{\partial \eta} + a(\eta_1,\eta)G(\eta_1+0,\eta;\xi_1,\eta_1)$$

$$= \frac{\partial G(\eta_1-0,\eta;\xi_1,\eta_1)}{\partial \eta} + a(\eta_1,\eta)G(\xi_1-0,\eta;\xi_1,\eta_1),\ at\ \eta \ne \eta_1,\ \eta \ne \xi_1; \tag{16}$$

$$\frac{\partial G(\xi,\eta_1+0;\xi_1,\eta_1)}{\partial \xi} + b(\xi,\eta_1)G(\xi,\eta_1+0;\xi_1,\eta_1)$$

$$= \frac{\partial G(\xi,\eta_1-0;\xi_1,\eta_1)}{\partial \xi} + b(\xi,\eta_1)G(\xi,\eta_1-0;\xi_1,\eta_1),\ at\ \xi \ne \xi_1\ \xi \ne \eta_1; \tag{17}$$

$$\frac{\partial G(\xi,\xi_1+0;\xi_1,\eta_1)}{\partial \xi} + b(\xi,\xi_1)G(\xi,\xi_1+0;\xi_1,\eta_1)$$

$$= \frac{\partial G(\xi, \xi_1 - 0; \xi_1, \eta_1)}{\partial \xi} + b(\xi, \xi_1) G(\xi, \xi_1 - 0; \xi_1, \eta_1) \text{ at } \xi \neq \xi_1 \ \xi \neq \eta_1; \tag{18}$$

and the "corner condition"

$$G(\xi_1 - 0, \eta_1 - 0; \xi_1, \eta_1) - G(\xi_1 + 0, \eta_1 - 0; \xi_1, \eta_1)$$

$$+G(\xi_1 + 0, \eta_1 + 0; \xi_1, \eta_1) - G(\xi_1 - 0, \eta_1 + 0; \xi_1, \eta_1) = 1. \tag{19}$$

must be satisfied as the regions meet at $(\xi, \eta) = (\xi_1, \eta_1)$.

4 Existence and uniqueness of the Green's function of the problem (7)-(9)

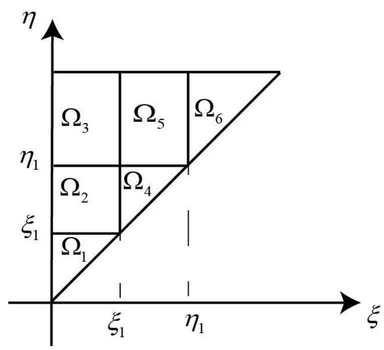

Figure 1: Splitting the domain Ω.

Theorem 1 *The function* $G(\xi, \eta; \xi_1, \eta_1)$ *that satisfies the conditions* (12)-(19) *exists and is unique.*

Proof. To show that a function $G(\xi, \eta; \xi_1, \eta_1)$, which satisfies the conditions (12)-(19) exists and unique, we divide the domain Ω into several subdomains (see Figure (1)) and consider the following problems sequentially. Let (ξ_1, η_1) be an arbitrary point of the domain Ω.

In the domain $\Omega_1 = \{(\xi, \eta) : 0 < \xi < \xi_1, \xi < \eta < \xi_1\}$ we consider the problem

$$L_{(\xi,\eta)} G = 0, \ (\xi, \eta) \in \Omega_1; \tag{20}$$

$$(G_\xi - G_\eta)(\xi, \xi; \xi_1, \eta_1) = 0, \ 0 \leq \xi \leq \xi_1; \tag{21}$$

$$G(0, \xi; \xi_1, \eta_1) = 0, \ 0 \leq \xi \leq \xi_1, \ (\xi_1, \eta_1) \in \Omega_2. \tag{22}$$

The problem (20)-(22) is a second Darboux problem and has a unique solution

$$G(\xi, \eta; \xi_1, \eta_1) \equiv 0, \ (\xi, \eta) \in \Omega_1. \tag{23}$$

In the domain $\Omega_2 = \{(\xi,\eta) : 0 \le \xi \le \xi_1, \xi_1 \le \eta \le \eta\}$ let us consider the problem

$$L_{(\xi,\eta)}G = 0,\ (\xi,\eta) \in \Omega_2; \tag{24}$$

$$G(0,\xi;\xi_1,\eta_1) = 0,\ \xi_1 \le \xi \le \eta_1,\ (\xi_1,\eta_1) \in \Omega_2. \tag{25}$$

From (23) we have the next equality

$$\frac{\partial G(\xi,\xi_1+0;\xi_1,\eta_1)}{\partial \xi} + b(\xi,\xi_1)G(\xi,\xi_1+0;\xi_1,\eta_1) = 0,\ 0 \le \xi \le \xi_1. \tag{26}$$

Integrating (26) by ξ we have

$$G(\xi,\xi_1+0;\xi_1,\eta_1) = \exp\left(-\int_0^\xi B(t,\xi_1)dt\right) C_1(\xi_1,\eta_1),\ 0 \le \xi \le \xi_1. \tag{27}$$

Substituting $\xi = 0$ in (27), using condition (14) we have that $C_1(\xi_1,\eta_1) \equiv 0$ and

$$G(\xi,\xi_1+0;\xi_1,\eta_1) = 0,\ 0 \le \xi \le \xi_1. \tag{28}$$

The problem (24),(25),(28) is a Goursat problem and has a unique solution

$$G(\xi,\eta;\xi_1,\eta_1) \equiv 0,\ (\xi,\eta) \in \Omega_2. \tag{29}$$

Therefore from (29) in the domain $\Omega_3 = \{(\xi,\eta) : 0 \le \xi \le \xi_1, \eta_1 \le \eta \le 1\}$, we get the problem

$$L_{(\xi,\eta)}G = 0,\ (\xi,\eta) \in \Omega_3; \tag{30}$$

$$G(0,\xi;\xi_1,\eta_1) = 0,\ \eta_1 \le \xi \le 1,\ (\xi_1,\eta_1) \in \Omega_3; \tag{31}$$

$$\frac{\partial G(\xi,\eta_1+0;\xi_1,\eta_1)}{\partial \xi} + b(\xi,\eta_1) \cdot G(\xi,\eta_1+0;\xi_1,\eta_1) = 0,\ 0 \le \xi \le \xi_1. \tag{32}$$

Integrating (32) by ξ we have

$$G(\xi,\eta_1+0;\xi_1,\eta_1) = \exp\left(-\int_0^\xi b(t,\eta_1)dt\right) C_2(\xi_1,\eta_1),\ 0 \le \xi \le \xi_1. \tag{33}$$

Substituting $\xi = 0$ in (33), using condition (14) we have that $C_2(\xi_1,\eta_1) \equiv 0$ and

$$G(\xi,\eta_1+0;\xi_1,\eta_1) = 0,\ 0 \le \xi \le \xi_1. \tag{34}$$

Therefore, the problem (30),(31),(34) is a Goursat problem and has a unique solution

$$G(\xi,\eta;\xi_1,\eta_1) \equiv 0,\ (\xi,\eta) \in \Omega_3. \tag{35}$$

In the domain $\Omega_4 = \{(\xi,\eta) : 0 \le \xi \le \xi_1, \xi \le \eta \le \eta_1\}$ we get the problem

$$L_{(\xi,\eta)}G = 0,\ (\xi,\eta) \in \Omega_4; \tag{36}$$

$$(G_\xi - G_\eta)(\xi,\xi;\xi_1,\eta_1) = 0,\ \xi_1 \le \xi \le \eta_1. \tag{37}$$

From (29) we have

$$\frac{\partial G(\xi_1+0,\eta;\xi_1,\eta_1)}{\partial\eta}+a(\xi_1,\eta)G(\xi_1+0,\eta;\xi_1,\eta_1)=0,\ \xi_1\le\eta\le\eta_1. \tag{38}$$

Integrating (38) by η we get

$$G(\xi_1+0,\eta;\xi_1,\eta_1)=\exp\left(-\int_{\xi_1}^{\eta}a(\xi_1,t)dt\right)C_3(\xi_1,\eta_1),\ \xi_1\le\eta\le\eta_1. \tag{39}$$

Substituting $\eta=\xi_1$ in (39), using condition (14) we have that $C_3(\xi_1,\eta_1)\equiv 0$ and

$$G(\xi_1+0,\eta;\xi_1,\eta_1)=0,\ \xi_1\le\eta\le\eta_1. \tag{40}$$

This problem (36),(37),(40) is a second Darboux problem and has a unique solution

$$G(\xi,\eta;\xi_1,\eta_1)\equiv 0,\ (\xi,\eta)\in\Omega_4. \tag{41}$$

Therefore, from (35), (41) in the domain $\Omega_5=\{(\xi,\eta):\xi_1\le\xi\le\eta_1,\eta_1\le\eta\le 1\}$ our problem is a Goursat problem

$$L_{(\xi,\eta)}G=0,\ (\xi,\eta)\in\Omega_5; \tag{42}$$

$$\frac{\partial G(\xi_1+0,\eta;\xi_1,\eta_1)}{\partial\eta}+a(\xi_1,\eta)G(\xi_1+0,\eta;\xi_1,\eta_1)=0,\ \eta_1\le\eta\le 1; \tag{43}$$

$$\frac{\partial G(\xi,\eta_1+0;\xi_1,\eta_1)}{\partial\xi}+b(\xi,\eta_1)G(\xi,\eta_1+0;\xi_1,\eta_1)=0,\ \xi_1\le\xi\le\eta_1; \tag{44}$$

$$G(\xi_1+0,\eta_1+0;\xi_1,\eta_1)=1. \tag{45}$$

The problem (41)-(45) has a unique solution, and it is easy to see that its solution coincides with the Riemann-Green function, that is,

$$G(\xi,\eta;\xi_1,\eta_1)=R(\xi,\eta;\xi_1,\eta_1),\ (\xi,\eta)\in\Omega_5. \tag{46}$$

Therefore from (46) in the domain $\Omega_6=\{(\xi,\eta):\eta_1\le\xi\le 1,\xi\le\eta\le 1\}$ we get the problem

$$L_{(\xi,\eta)}G=0,\ (\xi,\eta)\in\Omega_6; \tag{47}$$

$$(G_\xi-G_\eta)(\xi,\xi;\xi_1,\eta_1)=0,\ \eta_1\le\xi\le 1; \tag{48}$$

$$\frac{\partial G(\eta_1+0,\eta;\xi_1,\eta_1)}{\partial\eta}+b(\eta_1,\eta)G(\eta_1+0,\eta;\xi_1,\eta_1)$$
$$=\frac{\partial R(\eta_1,\eta;\xi_1,\eta_1)}{\partial\eta}+b(\eta_1,\eta)R(\eta_1,\eta;\xi_1,\eta_1),\ \eta_1\le\eta\le\xi_1. \tag{49}$$

The problem (47)-(49) is a second Darboux problem and has a unique solution.

Thus, we have shown that for any $(\xi_1,\eta_1)\in\Omega$ and $(\xi,\eta)\in\Omega$ the Green's function that satisfies the conditions (12)-(19) exists and unique. The theorem is proved.

5 Construction of the Green's function of the problem (7)-(9)

As can be seen from the proof of Theorem (1), the Green's function $G(\xi,\eta;\xi_1,\eta_1)=0$ in the domains Ω_1, Ω_2, Ω_3, Ω_4. And in the domain Ω_5 it coincides with the Riemann function (46).

Let us find a representation of the Green's function in the domain Ω_6. To construct the Green's functions, we will continue the coefficients of equation (47) in $\Omega_6^*=\{(\xi,\eta):\eta_1\le\xi\le 1,\ \eta_1\le\eta\le\xi\}$ such a way that the following conditions

$$A(\xi,\eta)=\begin{cases}a(\xi,\eta), & (\xi,\eta)\in\Omega_6,\\ b(\eta,\xi), & (\xi,\eta)\in\Omega_6^*,\end{cases}$$

$$B(\xi,\eta)=\begin{cases}b(\xi,\eta), & (\xi,\eta)\in\Omega_6,\\ a(\eta,\xi), & (\xi,\eta)\in\Omega_6^*,\end{cases}$$

$$C(\xi,\eta)=\begin{cases}c(\xi,\eta), & (\xi,\eta)\in\Omega_6,\\ c(\eta,\xi). & (\xi,\eta)\in\Omega_6^*\end{cases}$$

are met. Actually, show that coefficients of (47) have the following symmetry:

$$A(\xi,\eta)=B(\eta,\xi),\ C(\xi,\eta)=C(\eta,\xi),\ (\xi,\eta)\in\Omega_6. \tag{50}$$

From (50) we have

$$A(\eta,\xi)=\begin{cases}a(\eta,\xi), & (\eta,\xi)\in\Omega_6,\\ b(\xi,\eta), & (\eta,\xi)\in\Omega_6^*,\end{cases}=\begin{cases}b(\xi,\eta), & (\xi,\eta)\in\Omega_6,\\ a(\eta,\xi), & (\xi,\eta)\in\Omega_6^*,\end{cases}=B(\xi,\eta).$$

If we have chosen (ξ,η) from Ω_6, then (η,ξ) will be from Ω_6^*.

From (4) and (5) we get

$$A(\xi,\xi)=B(\xi,\xi),\ A_\xi(\xi,\xi)=B_\eta(\xi,\xi),\ \eta_1\le\xi\le 1.$$

If the coefficients $a,b,a_\xi,b_\eta,c\in C\left(\overline{\Omega}\right)$ then in virtue of (50) coefficients $A(\xi,\eta),\ B(\xi,\eta),\ C(\xi,\eta)$ in the domain $\widetilde{\Omega_6}=\Omega_6\cup\Omega_6^*=\{(\xi,\eta):\eta_1\le\xi\le 1,\ \eta_1\le\eta\le 1\}$ have the following smoothness

$$A,B,A_\xi,B_\eta,C\in C\left(\overline{\widetilde{\Omega_6}}\right). \tag{51}$$

Let (ξ_1,η_1) be an arbitrary point of the domain Ω. In order to construct the Green function in the domain Ω_6, consider the problem:

$$\frac{\partial^2 G_1}{\partial\xi\partial\eta}+A(\xi,\eta)\frac{\partial G_1}{\partial\xi}+B(\xi,\eta)\frac{\partial G_1}{\partial\eta}+C(\xi,\eta)G_1=0,\ (\xi,\eta)\in\widetilde{\Omega}_6; \tag{52}$$

$$\begin{aligned}&\frac{\partial G_1(\eta_1+0,\eta;\xi_1,\eta_1)}{\partial\eta}+b(\eta_1,\eta)G_1(\eta_1+0,\eta;\xi_1,\eta_1)\\ &=\frac{\partial R(\eta_1,\eta;\xi_1,\eta_1)}{\partial\eta}+b(\eta_1,\eta)R(\eta_1,\eta;\xi_1,\eta_1),\ \eta_1\le\eta\le\xi_1;\end{aligned} \tag{53}$$

$$\frac{\partial G_1(\xi,\eta_1+0;\xi_1,\eta_1)}{\partial \xi}+a(\xi,\eta_1)G_1(\xi,\eta_1+0;\xi_1,\eta_1)$$
$$=\frac{\partial R(\xi,\eta_1;\xi_1,\eta_1)}{\partial \xi}+a(\xi,\eta_1)R(\xi,\eta_1;\xi_1,\eta_1),\ \eta_1\le\xi\le\xi_1. \tag{54}$$

The problem (52)-(54) is a Goursat problem. Its solution exists and unique. We are interested in the representation of the function $G_1(\xi,\eta;\xi_1,\eta_1)$.

Lemma 1 *If the function* $G_1(\xi,\eta;\xi_1,\eta_1)$ *is the solution to the problem* (52)-(54), *then for any* $(\xi,\eta)\in\widetilde{\Omega}_6$ *we have* $G_1(\xi,\eta;\xi_1,\eta_1)=G_1(\eta,\xi;\xi_1,\eta_1)$.

To show that the function $G_1(\eta,\xi;\xi_1,\eta_1)$ satisfies the equation (52), in (52) replace $\xi=\eta_2,\ \eta=\xi_2,\ (\eta_2,\xi_2)\in\Omega_6^*$ and after using the symmetry conditions of coefficients, we get that $G_1(\eta,\xi;\xi_1,\eta_1)$ satisfies the equation (52).

Also doing the substitution of $\xi=\eta_2$ in (53) and using the symmetry conditions of coefficients, we get the condition (54). Similarly, by replacing $\eta=\xi_2$ in (54) and using the symmetry conditions of coefficients, we get the condition (53).

Thus, we have shown that the function $G_1(\eta,\xi;\xi_1,\eta_1)$ is also a solution to the problem (52)-(54). Since the solution to problem (52)-(54) is unique, then

$$G_1(\xi,\eta;\xi_1,\eta_1)=G_1(\eta,\xi;\xi_1,\eta_1),\ (\xi,\eta)\in\widetilde{\Omega}_6.$$

Solution of the problem (52)-(54) we search in the following form

$$G_1(\xi,\eta;\xi_1,\eta_1)=R(\xi,\eta;\xi_1,\eta_1)+g(\xi,\eta;\xi_1,\eta_1),(\xi,\eta)\in\widetilde{\Omega}_6.$$

Then we get the following problem

$$\frac{\partial^2 g}{\partial\xi\partial\eta}+A(\xi,\eta)\frac{\partial g}{\partial\xi}+B(\xi,\eta)\frac{\partial g}{\partial\eta}+C(\xi,\eta)g=0,\ (\xi,\eta)\in\widetilde{\Omega}_6; \tag{55}$$

$$\frac{\partial g(\eta_1,\eta;\xi_1,\eta_1)}{\partial\eta}+b(\eta_1,\eta)g(\eta_1,\eta;\xi_1,\eta_1)=0,\ \eta_1\le\eta\le\xi_1; \tag{56}$$

$$\frac{\partial g(\xi,\eta_1;\xi_1,\eta_1)}{\partial\xi}+a(\xi,\eta_1)g(\xi,\eta_1;\xi_1,\eta_1)=0,\ \eta_1\le\xi\le\xi_1. \tag{57}$$

It is easy to see that the solution to the problem (55)-(57) has the form

$$g(\xi,\eta;\xi_1,\eta_1)=R(\eta,\xi;\xi_1,\eta_1),\ (\xi,\eta)\in\widetilde{\Omega}_6.$$

Lemma 2 *Let* (ξ,η) *be an arbitrary point of the domain* Ω. *By internal variables* (ξ_1,η_1) *the Green's function of the problem* (7)-(9) *has the following properties:*

$$L^*_{(\xi_1,\eta_1)}G(\xi,\eta;\xi_1,\eta_1)=0,\ (\xi_1,\eta_1)\in\Omega,\ \textit{at}\ \xi_1\ne\xi,\ \xi_1\ne\eta,\ \eta_1\ne\xi; \tag{58}$$

$$(G_{\xi_1}-G_{\eta_1})(\xi,\eta;\xi_1,\xi_1)+(a-b)(\xi_1,\xi_1)G(\xi,\eta;\xi_1,\xi_1)=0,\ 0\le\xi_1\le1; \tag{59}$$

$$G(\xi,\eta;0,\eta_1)=0,\ 0\le\eta_1\le1; \tag{60}$$

$$\frac{\partial G(\xi,\eta;\xi-0,\eta_1)}{\partial\eta_1}-a(\xi,\eta_1)G(\xi,\eta;\xi-0,\eta_1)=0,\ at\ \eta_1\neq\eta,\ \ \eta_1\neq\xi; \tag{61}$$

$$\frac{\partial G(\xi,\eta;\xi_1,\eta-0)}{\partial\xi_1}-b(\xi_1,\eta)G(\xi,\eta;\xi_1,\eta-0)=0,\ at\ \xi_1\neq\xi; \tag{62}$$

$$\frac{\partial G(\xi,\eta;\xi_1,\xi-0)}{\partial\xi_1}-b(\xi_1,\xi)\ G(\xi,\eta;\xi_1,\xi-0)$$

$$=\frac{\partial G(\xi,\eta;\xi_1,\xi+0)}{\partial\xi_1}-b(\xi_1,\xi)G(\xi,\eta;\xi_1,\xi+0); \tag{63}$$

$$G(\xi,\eta;\xi-0,\eta-0)-G(\xi,\eta;\xi+0,\eta-0)$$

$$+G(\xi,\eta;\xi+0,\eta+0)-G(\xi,\eta;\xi-0,\eta+0)=1; \tag{64}$$

$$G(\xi,\eta;\xi,\xi-0)-G(\xi,\eta;\xi,\xi+0)-G(\xi,\eta;\xi-0,\xi)=0. \tag{65}$$

Properties (58)-(65) are easy to get out of the construction of the Green's function of problem (7)-(9). Under these conditions (58)-(65) it is possible to uniquely restore the Green's function of problem (7)-(9).

Using properties (58)-(65) we can use it to write the integral representation of the solution to problem (7)-(9). To do this, we consider the following integral

$$\iint_{\Omega_{(\xi\eta)}} G(\xi,\eta;\xi_1,\eta_1)f(\xi_1,\eta_1)d\xi_1 d\eta_1$$

$$=\iint_{\Omega_{(\xi\eta)}} G(\xi,\eta;\xi_1,\eta_1)\left(\frac{\partial^2 u}{\partial\xi_1\partial\eta_1}+a\frac{\partial u}{\partial\xi_1}+b\frac{\partial u}{\partial\eta_1}+cu\right)d\xi_1 d\eta_1. \tag{66}$$

Applying Green's theorem in a plane [17] and using the conditions (8), (9) properties of Green's function (58)-(65), from (66) we get the following representation of the solution to problem (7)-(9) in the domain $\Omega_{(\xi\eta)}=\Omega_5\cup\Omega_6$:

$$u(\xi,\eta)=\frac{1}{2}\left(G(\xi,\eta;0,\xi+0)-G(\xi,\eta;0,\xi-0)\right)\tau(\xi)+\frac{1}{2}G(\xi,\eta;0,\eta-0)\tau(\eta)$$

$$+\frac{1}{2}\int_0^{\xi} G(\xi,\eta;\xi_1,\eta_1)\nu(\xi_1)d\xi_1+\iint_{\Omega_{(\xi\eta)}} G(\xi,\eta;\xi_1,\eta_1)f(\xi_1,\eta_1)d\xi_1 d\eta_1.$$

6 Conclusion

In this paper, an integral representation of the Green function for a general second-order hyperbolic equation for the second Darboux problem is constructed, since all the properties of the Green function of this problem follow from the integral representation of the Green function. It is shown that the main difference between this work and other previous works by other authors, we conduct research and build a function Green's solution of this problem without using the symmetry conditions of the lower coefficients. In addition, unlike other authors, it is in the article that we will give a definition of the Green function and a method for constructing it for cases of general coefficients.

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