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MATHEMATICAL MODELING OF THE EPIDEMIC PROPAGATION WITH LIMITED TIME SPENT IN COMPARTMENTS AND VACCINATION

The paper proposes discrete and continuous mathematical models of epidemic development. A division of the population into nine compartments is suggested: susceptible, exposed, vaccinated, contact vaccinated, undetected patients, isolated patients, hospitalized patients, recovered and deceased. At the same time, the time spent in exposed and infected compartments is considered limited. According to the assumptions made in the models, a susceptible person can encounter the patient and go into the exposed compartment, and be vaccinated, and then also encounter the infection and go into the contact vaccinated compartment. Exposed people may become ill to any degree of severity or not, returning to the susceptible group. A contact vaccinated either does not become ill or becomes undetected or isolated patient. Every patient can recover. An undiagnosed patient may develop symptoms of the disease, because of which he moves into the isolated compartment. An isolated patient may be hospitalized, and a hospitalized patient may die. In the discrete model, discrete quantitative data for each day of the epidemic are considered, in the continuous one, these indicators are considered continuous functions. The article provides a qualitative and quantitative analysis of the proposed models. The influence of all parameters on the process under study is investigated.

Key words: mathematical model, epidemic, vaccination.

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ЭПИДЕМИЯНЫҢ ДАМУЫН ТОПТАРДА ОТЫРУ УАҚЫТЫ ШЕКТЕУЛІ БОЛУДЫ ЖӘНЕ ВАКЦИНАЦИЯЛАУДЫ ЕСКЕРЕ ОТЫРЫП МАТЕМАТИКАЛЫҚ МОДЕЛЬДЕУ

Эпидемия дамуының дискретті және үздіксіз математикалық модельдері ұсынылған. Олар халықты тоғыз топқа бөлуді ұсынады: сезімтал, контактілі, вакцинацияланған, вакцинацияланған контактілі, анықталмаған науқастар, оқшауланған науқастар, ауруханаға жатқызылған науқастар, сауығып кеткендер және қайтыс болғандар. Бұл модельдерде контактілі және ауру топтарында болу уақыты шектеулі болып саналады. Модельдерде жасалған болжамдарға сәйкес, сезімтал адам науқаспен байланыста болу арқылы байланыс тобына кіруі, және вакцинациялануы, содан кейін науқаспен байланыста болу арқылы вакцинацияланған контакт тобына өтуі мүмкін болады. Контактілі сезімтал топқа қайта оралуы, немесе кез келген ауру дәрежесімен ауруы мүмкін. Вакцинацияланған контактілі ауырмай сезімтал тобына қайта оралуы немесе анықталмай немесе оқшауланып ауыруы мүмкін. Әрбір науқас сауығып кете алады. Анықталмаған науқаста аурудың белгілері пайда болуы мүмкін, нәтижесінде ол оқшауланған топқа ауысады. Оқшауланған науқас ауруханаға жатқызылуы мүмкін, ал ауруханада жатқан науқас өлуі мүмкін. Дискретті модельде эпидемияның әрбір күні үшін дискретті сандық деректер қарастырылады, үздіксіз модельде бұл көрсеткіштер үздіксіз функциялар болып саналады. Мақалада ұсынылған модельдердің сапалық және сандық талдауы берілген. Барлық параметрлердің зерттелетін процеске әсері зерттеледі.

Түйін сөздер: математикалық модель, эпидемия, вакцинация.

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МАТЕМАТИЧЕСКОЕ МОДЕЛИРОВАНИЕ РАЗВИТИЯ ЭПИДЕМИИ С УЧЕТОМ ВАКЦИНАЦИИ И ОГРАНИЧЕННОГО ВРЕМЕНЕМ ПРЕБЫВАНИЯ В ГРУППАХ

Предлагаются дискретная и непрерывная математические модели развития эпидемии. Они предполагают разбиение популяции на девять групп: восприимчивые, контактные, вакцинированные, вакцинированные контактные, невыявленные больные, изолированные больные, госпитализированные больные, выздоровевшие и умершие. При этом время пребывания в группах контактных и больных считается ограниченным. Согласно допущениям, принятым в моделях, восприимчивый может войти в контакт с больным, перейдя в группу контактных, а также вакцинироваться, после чего также войти в контакт с больным, перейдя в группу контактных вакцинированных. Контактные могут заболеть в любой степени тяжести или не заболеть, вернувшись в группу восприимчивых. Контактный восприимчивый либо не заболевает, либо становится невыявленным или изолированным больным. Каждый больной может выздороветь. У невыявленного больного могут появиться симптомы болезни, в результате чего он переходит в группу изолированных. Изолированный больной может быть госпитализирован, а госпитализированный – умереть. В дискретной модули рассматриваются дискретные количественные данные по каждому дню эпидемии, в непрерывной, данные показатели считаются непрерывными функциями. В статье проводится качественный и количественный анализ предлагаемых моделей. Исследуется влияние всех параметров на исследуемый процесс.

Ключевые слова: математическая модель, эпидемия, вакцинация.

1 Introduction

The development of the COVID-19 pandemic has largely updated the development of mathematical models of epidemic development. The first application of mathematical methods in the analysis of epidemics is associated with the works of outstanding mathematicians of the second half of the 18th and early 19th centuries D. Bernoulli, I. Lambert, P.S. Laplace. Modern mathematical models of epidemiology go back to the work of R. Ross, published in 1911, on the study of the spread of malaria [1] and, to an even greater extent, to the SIR model proposed in 1927 by W. Kermack and A. McKendrick [2]. This model is based on the division of the entire population into three compartments of susceptible, infected and recovered. The model is a system of non-linear differential equations and describes the change in the number of these population compartments over time.

The main drawback of the SIR model is that it does not take into account the presence of an incubation period, i.e. it assumes that a person who has had contact with a sick person immediately falls ill. To eliminate it the SEIR model was proposed, in which a compartment of exposed was added, see, for example, [3]. Thus, in the process of infection, a person susceptible to the disease first becomes exposed and only then becomes infected. There are a significant number of SEIR model modifications. Thus, the SEIRD model additionally includes a compartment of deceased [4, 5]. In the MSEIR model in addition to the compartments of the SEIR model, people endowed with immunity from birth (maternally derived immunity) are added [6]. In [7], a model which additionally takes into account patients in whom the disease

proceeds in an asymptomatic form (asymptomatic) is considered. The SEIRHCD model also has compartments of hospitalized and critical patients [8, 9]. Along with continuous models, discrete models, in which time is an integer variable, are also considered, see, for example, [10].

These models do not take into account the limited stay in exposed and infected compartments. In particular, any person who has been in contact with a sick person, after some time, will most likely either get sick or not get sick, which means that they will certainly leave the exposed compartment. Anyone who falls ill after some time will surely either recover or die, i.e. will definitely leave the compartment of infected. This shortcoming is overwhelmed in [8, 9, 11] for continuous systems and in [11–13] for discrete systems. There are also models that take into account the vaccination of the population [14–22]. In this case, vaccination is considered at certain points in time (impulsive vaccination), as well as vaccination of newborns. Here, vaccinated susceptible people go directly into the compartment of recovered, see [15–18]. In the SIRV model [19], the vaccinated are treated as an independent compartment. The SEIRV model also uses a separate compartment of vaccinated people, some of whom may become infected in the future, and birth and natural mortality are also taken into account [20]. In [21], a model is proposed in which there is an additional compartment of people in quarantine. [22] explores the SUIHTER model, which also includes compartment of asymptomatic and hospitalized patients, and separately considers people received one and two doses of the vaccine.

This paper proposes discrete and continuous models for the development of the epidemic, providing for vaccination and limited time spent in compartments, which are a generalization of the models described in [11] for the case of vaccination. They assume the division of the entire population into nine compartments: susceptible, exposed, vaccinated, contact vaccinated, undetected, isolated and hospitalized patients, as well as recovered and deceased. A qualitative and quantitative analysis of the models is carried out. The influence of various parameters of the system on the process is investigated.

2 Description of models

An isolated population under the conditions of an epidemic is considered. The entire population is divided into the following compartments:

S: susceptible (healthy, but potentially sick);

V: vaccinated (healthy vaccinated);

E: exposed (healthy, in contact with sick);

C: contact vaccinated (vaccinated, who were in contact with patients);

U: undetected (infected with an asymptomatic course of the disease and mildly ill with an undiagnosed disease);

I: isolated (patients in a mild form, undergoing treatment at home);

H: hospitalized (seriously ill, hospitalized);

R: recovered (recovered from illness, who do not have any signs of illness);

D: died.

The sum of N numbers of people in all compartments is considered unchanged, i.e. natural births and deaths are not taken into account in the model.

The change in the number of people in each compartment is carried out due to intercompartment transitions, see Fig. 1.

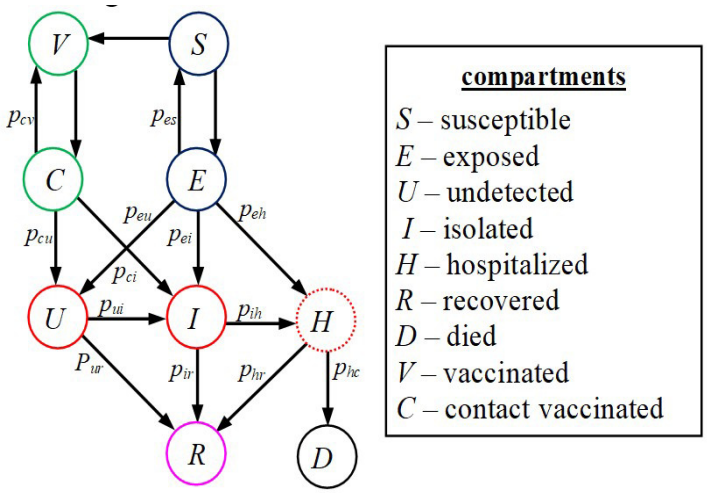

Figure 1: Graph of intercompartment transitions.

According to the accepted assumptions, a susceptible person can come into contact with the patient by moving to the exposed compartment, and also be vaccinated. A vaccinated person can also encounter a sick person and move into compartment of contact vaccinated people. The exposed may become ill in any degree of severity or not get ill, returning to the susceptible compartment. A contact vaccinated either does not become ill or becomes undetected or isolated sick. Every patient can recover. An undiagnosed patient may develop symptoms of the disease moving into the isolated compartment in result. An isolated patient may be hospitalized, and a hospitalized patient may die.

The number of days spent in all compartments of contact and patients is considered fixed and is indicated as follows n_e, n_c, n_u, n_i and n_h, where the index corresponds to the name of the compartment (the first letter of the compartment name). For vaccinated contacts, the time spent in the compartment is assumed to be the same as for unvaccinated contacts, i.e. $n_c = n_e$. At the end of the time spent in the compartment, each person in it goes into one of the possible compartments in accordance with the above figure. In this case, $p_{\alpha\beta}$ denotes the proportion of people in the compartment indicated as α passing into the compartment β. In this case, the conditions

$$\sum_{\beta} p_{\alpha}\beta = 1 \forall \alpha,$$

where the sum is taken over all compartments β, to which you can go from the compartment α.

Sources of infection are people in undetected (to a greater extent) and isolated (to a lesser extent) compartments, but not hospitalized. The degree of infectivity is described by the coefficients of contagiousness k_u and k_i undetected and isolated patients, and $k_u > k_i$. Vaccination of the population is characterized by the rate of vaccination v.

The mathematical model of the process is a system of equations for the number of people in each compartment that changes over time. In this case, the number of people in each compartment is indicated by the first letter of the compartment name, i.e. S, V, etc. These quantities are functions of a continuous argument t or an integer argument n, written as an index. Thus, S_k, V_k, etc. characterize the number of susceptible, vaccinated, etc. at the k-th time step (on the k-th day from the beginning of the study). In the continuous model, the values of $S(t)$, $V(t)$, etc. characterize the number of susceptible, vaccinated, etc. at time t (after t time from the start of the study).

Let us formulate a description of the discrete model. The number of all categories of contacts and patients at a given point in time is the sum of their numbers by the days they were in the compartment, i.e. following equalities are true

$$Z_k = \sum_{j=1}^{n_z} z_k^j, Z = E, C, U, I, H, \tag{1}$$

where z_k^j denotes the number of people in compartment Z at time k on the j-th day of being in this compartment. Here, any compartment of exposed and patients is chosen as Z, i.e. Z can take the values E, C, U, I, H. In this case, each member of Z of the j-th day of being in this compartment passes to the category of the $j+1$st day of being in the compartment every day, if this was not the last day of being in the compartment, which corresponds to the equalities

$$z_{k+1}^{j+1} = z_k^j, j = 2, ..., n_z - 1, z = e, c, u, i, h. \tag{2}$$

The susceptible number on the following day is equal to the susceptible number on the previous day minus the number of those vaccinated on that day, minus the susceptible number who contacted infection on that day, plus the number of contacts of the last day of stay in the exposed compartment who did not get infected. At the same time, the vaccinated number is directly proportional to the susceptible number, and the susceptible number contacted with infection is directly proportional to the susceptible number, as well as the number of undetected and isolated patients who are sources of infection. As a result, we obtain the equality

$$S_{k+1} = S_k - vS_k - \frac{k_u U_k + k_i I_k}{N} S_k + p_{es} e_k^{n_e}. \tag{3}$$

The division by the size of the entire population is carried out for reasons of normalization (otherwise, the numbers of two compartments, which are sufficiently large values, are multiplied).

The vaccinated number on the following day is equal to the vaccinated number on the previous day plus the number of new susceptible people who were vaccinated that day minus the number of vaccinated people contacted with infection on that day plus the number of people on the last day of stay in the contact vaccinated compartment who did not get infected. The corresponding quantities are determined in the same way as in the previous formula. As a result, we obtain the equality

$$V_{k+1} = V_k + vS_k - \frac{k_u U_k + k_i I_k}{N} V_k + p_{cv} c_k^{n_c}. \tag{4}$$

The number of all people in compartments of exposed and infected patients on the next day is equal to their number on the previous day plus the number of people who entered this compartment this day, minus the number of people who left the compartment the previous day

$$Z_{k+1} = Z_k + z_{k+1}^1 - z_k^{n_z},\ Z = E, C, U, I, H. \tag{5}$$

The recovery number of at the next time point is equal to their number on the previous day plus the number of patients of all compartments who recovered on the previous day.

$$R_{k+1} = R_k + p_{ur} u_k^{n_a} + p_{ir} i_k^{n_i} + p_{hr} h_k^{n_h}. \tag{6}$$

The death number at a subsequent point in time is equal to their number on the previous day plus the number of people that died on this day

$$D_{k+1} = R_k + p_{hd} h_k^{n_h}. \tag{7}$$

The number of new exposed (exposed of the first day of being in the compartment), both unvaccinated and vaccinated, is exactly equal to the number, respectively, susceptible and vaccinated, who had contact with patients on the previous day

$$e_{k+1}^1 = (k_u U_k + k_i I_k)\frac{S_k}{N},\ c_{k+1}^1 = (k_u U_k + k_i I_k)\frac{V_k}{N}, \tag{8}$$

The number of new undetected is the sum of both exposed compartments of the last day of being in the compartment, who fell ill with an undetected form of the disease

$$u_{k+1}^1 = p_{eu} e_k^{n_e} + p_{cu} c_k^{n_c}. \tag{9}$$

The number of new isolated patients is the sum of the number of both exposed compartments of the last day being in the compartment who fell ill with an isolated form of the disease, and the number of undetected contacts of the last day being in the compartment in whom the disease was detected

$$i_{k+1}^1 = p_{ei} e_k^{n_e} + p_{ci} c_k^{n_c} + p_{ui} u_k^{n_u}. \tag{10}$$

The number of new hospitalized patients is the sum of exposed and isolated patients of the last day of being in the compartment, in which the disease turned into a severe form, as a result of which they were hospitalized

$$h_{k+1}^1 = p_{eh} e_k^{n_e} + p_{ih} i_k^{n_i}. \tag{11}$$

The initial states of the system S_0, E_0, U_0, V_0, C_0, I_0, H_0, R_0, D_0 are known, and the distribution of all forms of exposed and patients at the initial moment of time by days of being in compartments is considered uniform, i.e. taken according equalities

$$z_0^j = Z_0/n_z, \quad j = 1, ..., n_z, \ \ z = e, c, u, i, h. \tag{12}$$

Relations (1) - (12) constitute a discrete model of the process under study.

Let us proceed to the description of the corresponding continuous model. The change in the number of susceptible people is its decrease due to vaccination and the fact that a certain number of susceptible people contacted with infection, and an increase, since some of the exposed do not get sick. At the same time, the new vaccinated number is directly proportional to the susceptible number and the number of susceptible who became exposed is directly proportional to the susceptible number, as well as the number of undetected and isolated patients. The number of non-diseased exposed is proportional to the exposed number and inversely proportional to the number of days spent in the exposed compartment. As a result, we obtain the equation

$$\frac{dS(t)}{dt} = -vS(t) - \frac{k_u U(t) + k_i I(t)}{N} S(t) + p_{es} \frac{E(t)}{n_e}. \tag{13}$$

The change in the vaccinated number is its decrease due to the fact that some part of the vaccinated who contacted with patients, and the increase due to vaccination and the fact that part of the contact vaccinated people does not get sick. The corresponding quantities are determined in the same way as in the previous formula. As a result, we obtain the equality

$$\frac{dV(t)}{dt} = vS(t) - \frac{k_u U(t) + k_i I(t)}{N} V(t) + p_{cv} \frac{C(t)}{n_v}. \tag{14}$$

The change in the number of contacts, both unvaccinated and vaccinated, increases due to, respectively, susceptible and vaccinated, who had contact with patients, and decreases due to the limited time spent in these compartments. Thus, we have the equalities

$$\frac{dE(t)}{dt} = \frac{k_u U(t) + k_i I(t)}{N} S(t) - \frac{E(t)}{n_e}, \tag{15}$$

$$\frac{dC(t)}{dt} = \frac{k_u U(t) + k_i I(t)}{N} V(t) - \frac{C(t)}{n_c}. \tag{16}$$

The number of undetected patients increases due to the disease of both exposed compartments and decreases due to the limited time spent in this compartment:

$$\frac{dU(t)}{dt} = p_{eu}\frac{E(t)}{n_e} + p_{cu}\frac{C(t)}{n_c} - \frac{U(t)}{n_u}. \tag{17}$$

The number of isolated patients increases due to the disease of both exposed compartments and the detection of the disease in some of the undetected and decreases due to the limited time spent in this compartment:

$$\frac{dI(t)}{dt} = p_{ei}\frac{E(t)}{n_e} + p_{ci}\frac{C(t)}{n_c} + p_{ui}\frac{U(t)}{n_u} - \frac{I(t)}{n_i}. \tag{18}$$

The number of hospitalized increases due to infection of people in exposed compartment in a severe form and the hospitalization of a part of the isolated ones and decreases due to the limited time spent in this compartment:

$$\frac{dH(t)}{dt} = p_{eh}\frac{E(t)}{n_e} + p_{ih}\frac{I(t)}{n_i} - \frac{H(t)}{n_h}. \tag{19}$$

The number of recovered patients is increasing due to the recovery of patients of all categories:

$$\frac{dR(t)}{dt} = p_{ur}\frac{U(t)}{n_u} + p_{ir}\frac{I(t)}{n_i} + p_{hr}\frac{H(t)}{n_h}. \tag{20}$$

The number of deaths increases due to the death of a part of the hospitalized:

$$\frac{dD(t)}{dt} = p_{hd}\frac{H(t)}{n_h}. \tag{21}$$

The initial states of the system S_0, E_0, U_0, V_0, C_0, I_0, H_0, R_0, D_0 are known, i.e. the following equalities hold

$$Z(0) = Z_0. \tag{22}$$

where $Z = S, E, U, V, C, I, H, R, D$. The system of differential equations (13) – (21) with initial conditions (22) constitutes a continuous model of the system.

3 Analysis of mathematical models

Let us establish the simplest qualitative properties of the models under consideration. The discrete model is characterized by the following statement.

Theorem 1. *For any values of the parameters, the system has a unique equilibrium position, and the limiting values of the numbers of all categories of exposed and infected people are equal to zero, and the functions R and D are increasing.*

To prove it, it suffices to pass to the limit in recurrence relations (3) – (7), taking into account that the sequences $\{Z_k\}$ and $\{Z_{k+1}\}$ have the same limit. At the same time, the zero limit values of the numbers of all categories of exposed and patients indicate the end of the epidemic. The monotonicity of the functions R and D (growth in the number of recovered and deceased) is due to the negativity of all expressions on the right side of equalities (6) and (7).

Theorem 2. *For any values of the system parameters, problem (13) - (22) has a unique equilibrium position, and the limiting values of the numbers of all categories of contact and patients is equal to zero, and the functions R and D are increasing.*

To prove it, it suffices to equate all derivatives to zero in differential equations (13) - (21). The results obtained indicate that the qualitative properties of the continuous and discrete models generally coincide.

The quantitative analysis of both models was carried out at the same parameter values, and the continuous model was implemented using the 4th order Runge - Kutta method. In doing so, the following numbers of days spent in compartments have been taken:$n_e = 14, n_u = 3, n_i = 5, n_h = 7, n_c = n_e = 7$. The coefficients of the equations take the following values: $k_u = 3.180, k_i = 0.171, p_{es} = 0.679, p_{eu} = 0.154, p_{ei} = 0.145, p_{eh} = 0.022, p_{cv} = 0.9, p_{cu} = 0.05, p_{ci} = 0.05, p_{ui} = 0.03, p_{ur} = 0.97, p_{ih} = 0.021, p_{ir} = 0.979, p_{hr} = 0.982, p_{hd} = 0.018, v = 0.0005$. The calculations were carried out at the initial stage of the epidemic, and $N = 18699640$, which corresponded to the population of Kazakhstan at the time of the start of the COVID-19 epidemic. In addition, it was assumed that at the initial moment of time there are 140 contact people, and all the rest are susceptible. Graphs of the obtained solutions are shown in Fig. 2, where the red curves correspond to the discrete model, and the blue curves to the continuous one.

Based on the results obtained, the following conclusions can be made. The qualitative properties of the solutions of both models are almost the same, and the corresponding functions for the continuous model are smoother. For some time, the number of exposed and patients has been growing. Then the epidemic reaches its peak, after which the incidence decreases. Over time, the system is observed to reach a position of equilibrium, and the number of all compartments of exposed and patients tends to zero, which corresponds to the end of the epidemic. The susceptible number decreases monotonously as more and more people get sick or get vaccinated over time. The number of vaccinated, recovered and dead people is gradually increasing, which is quite natural, since the vaccinated people will no longer become usually susceptible, the recovered acquire immunity.

Table. 1 shows the most important quantitative characteristics corresponding to the selected computation variant. According to the results obtained, the general characteristics of the discrete and continuous models are approximately the same. However, for the discrete model, the epidemic proceeds somewhat less intensively than for the continuous model. In

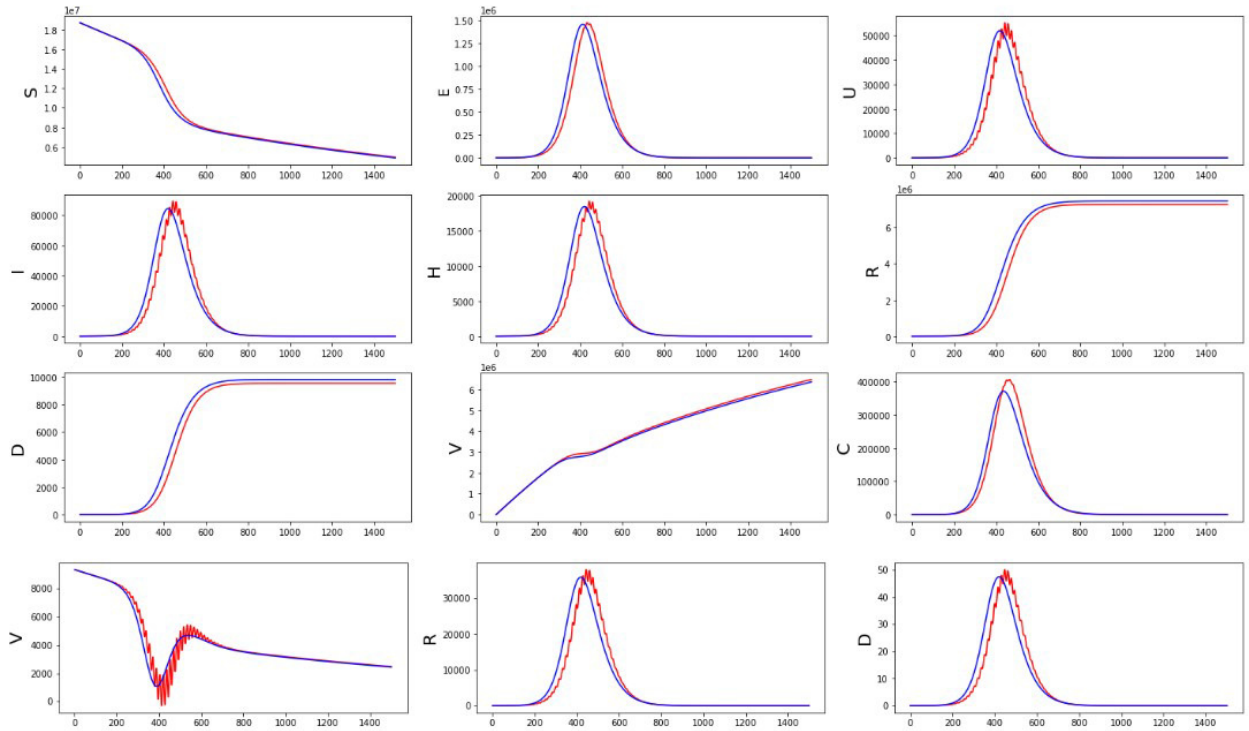

Figure 2: System states for discrete (red) and continuous (blue) models.

particular, the duration of the epidemic is shorter (by about two months), the time of the peak of the epidemic comes later (almost a month), the total number of cases and deaths is slightly less. However, the observed difference is insignificant, as a result of which we can conclude that the considered models are equivalent.

Let us now estimate the influence of system parameters on the considered process. Each of the tables below shows the values of the most important characteristics of the system for three counting options. The first of them corresponds to the main variant of the computation given above, and the next two correspond to the specified parameter, increased and decreased by some value.

Table. 2 evaluates the impact of the coefficient of contagiousness of undiagnosed patients. It turns out to be about the same for both models. In particular, an increase in the contagiousness coefficient leads to a reduction in the duration of the epidemic and the time it takes to reach its peak, as well as an increase in the number of simultaneously ill people, the total number of ill people and deaths. Such changes indicate a greater intensity of the epidemic development, which seems quite logical. At the same time, the percentage of recovered and dead people remains unchanged, since these characteristics are determined by the transition coefficients in the compartments of patients. Comparing the degree of influence of the parameter on the models under consideration, we note, for example, that an increase (respectively, a decrease) in the coefficient by 10% leads to a decrease in the duration of the epidemic by 11.8% for the discrete model and 11.6% for the continuous model (respectively,

Table 1: The most important quantitative characteristics of the system

	Discrete model	**continuous model**
Epidemic end time	1101	1154
Peak time of the epidemic	442	419
Total number of cases and % of the total population	7284183 (38.95%)	
The total number of recovered and % of the total number of cases	7274363 (99.87%)	7456303 (39.93%)
The total number of deaths and % of the total number of cases	9546 (0.13%)	9822 (0.13%)
The maximum number of patients at the same time	163412 (0.87%)	154790 (0.83%)

an increase of 20.5% for the discrete model and 20.4% for the continuous model). At the same time, the total number of cases increases by 21.3% for the discrete model and 21.2% for the continuous model (respectively, it decreases by 32.2% for the discrete model and by 32.4% for the continuous model). Thus, the degree of influence of the parameter on both models is almost the same.

Table 2: Influence of the coefficient of contagiousness of undiagnosed patients

Parameter ku	**Epidemic end time**		**Peak time of the epidemic**		**Total number of cases and % of the population**	
	Discrete	continuous	Discrete	continuous	Discrete	continuous
3.18	1101	1154	442	419	7284183 (38.95%)	7466126 (39.93%)
3.48	972	1020	377	344	8833599 (47.24%)	9047409 (48.38%)
2.88	1327	1389	571	547	4939864 (26.42%)	5049614 (27%)

Table 3 shows the results of assessing the impact of the contagiousness coefficient of isolated patients with an increase and decrease in this parameter by 58.8%. With its increase, there is a decrease in the duration of the epidemic and the time it takes to reach its peak, with an increase in the total number of cases of simultaneously infected, the percentage of recovered and dead remains unchanged. However, with the indicated increase (respectively, decrease) in the contagiousness coefficient of isolated patients, there is a reduction in the duration of the epidemic by 6.8% for the discrete model and by 6.5% for the continuous model (respectively, it increases by 9.1% for the discrete model and by 8.8% for the continuous model). Under the same conditions, there is an increase in the death number by 13.1% for the discrete model and by 13.0% for the continuous model (respectively, a decrease of 16.1% for the discrete model and 16.0% for continuous model). The weaker effect on the process of the contagiousness coefficient of isolated patients compared to the similar coefficient for unidentified patients is explained by the fact that isolated patients are a significantly less important source of infection compared to unidentified ones.

Table. 4 examines the effect of the recovering proportion of hospitalized patients when it changes by 1.5%. This parameter does not affect the duration of the epidemic, the time of its

Table 3: Influence of the contagiousness coefficient of isolated patients

Parameter ku	**Epidemic end time**		**Peak time of the epidemic**		**Total number of cases and % of the population**	
	Discrete	continuous	Discrete	continuous	Discrete	continuous
0.171	1101	1154	442	419	7284183 (38.95%)	7466126 (39.93%)
0.271	1026	1079	410	376	8187581 (43.78%)	8384067 (44.84%)
0.071	1201	1255	505	474	6160891 (32.95%)	6322669 (33.81%)

peak, the total number of cases and the maximum number of cases at a time, since it only applies to those patients who have already been hospitalized. Thus, it can only influence the ratio between the recovered and the dead. In particular, an increase (respectively, a decrease) in this parameter leads to a decrease (respectively, an increase) in the number of deaths by 83.3% for both models. It is clear that a reduction in the death number by a certain amount means an increase in the recovered number by the same amount.

Table 4: Influence of recovering rate of hospitalized patients

Parameter p_{hr}	**The total number of recovered and % of the total number of cases**		**The total number of deaths and % of the total number of cases**	
	Discrete	continuous	Discrete	continuous
0.982	7274636 (99.87%)	7456303 (99.87%)	9546 (0.13%)	9822 (0.13%)
0.997	7282592 (99.98%)	7464489 (99.98%)	1591 (0.02%)	1637 (0.02%)
0.967	7266681 (99.76%)	7448118 (99.76%)	17501 (0.24%)	18007 (0.24%)

Table. 5 examines the impact of the proportion of isolated patients who were hospitalized, with a change of 71.4%. This parameter does not affect the duration of the epidemic and the time of its peak, as well as the total number of cases, however, it affects the further fate of the patient. In particular, an increase (respectively, a decrease) in this parameter indicates a more severe (respectively, milder) course of the epidemic. This is reflected in the fact that the number of deaths increased by 9.7% for the discrete model and 9.6% for the continuous model (respectively, it decreased by 9.7% for both models).

Table. 6 assesses the impact of the proportion of undetected patients who subsequently developed symptoms of the disease and were isolated. A change in this parameter slightly affects the duration of the epidemic, the maximum number of patients at a time, as well as the proportion of recovered and dead. With an increase (respectively, decrease) of this parameter by 66.7%, there is an increase (respectively, a decrease) in the total number of cases by 0.5% for both models. At the same time, the number of deaths increases (respectively, decreases) by 0.8% for both models.

Table. 7 examines the effect of the proportion of contact vaccinated pcv who do not

Table 5: Influence of isolated patients proportion who were hospitalized

Parameter p_{ih}	**The total number of recovered and % of the total number of cases**		**The total number of deaths and % of the total number of cases**	
	Discrete	continuous	Discrete	continuous
0.021	7274636 (99.87%)	7456303 (99.87%)	9546 (0.13%)	9822 (0.13%)
0.036	7273712 (99.86%)	7455356 (99.86%)	10471 (0.14%)	10769 (0.14%)
0.006	7275561 (99.88%)	7457251 (99.88%)	8621 (0.12%)	8874 (0.12%)

Table 6: Impact of the proportion of undetected patients who were isolated

Parameter p_{ui}	**Total number of cases and % of the population**		**The total number of recovered and % of the total number of cases**		**The total number of deaths and % of the total number of cases**	
	Discrete	continuous	Discrete	continuous	Discrete	continuous
0.03	7284183 (38.95%)	7466126 (39.93%)	7274636 (99.87%)	7456303 (99.87%)	9546 (0.13%)	9822 (0.13%)
0.05	7318481 (39.14%)	7500949 (40.11%)	7308861 (99.87%)	7491052 (99.87%)	9620 (0.13%)	9897 (0.13%)
0.01	7249636 (38.77%)	7431047 (39.74%)	7240163 (99.87%)	7421300 (99.87%)	9472 (0.13%)	9746 (0.13%)

become ill. This value does not affect the temporal characteristics of the epidemic, as well as the percentage of recovered and dead, but affects their number. In particular, with an increase (respectively, decrease) in this value by 3.3%, the number of cases decreases by 2.9% for the discrete model and 2.8% for the continuous model (respectively, an increase of 2.8% for the discrete model and 2.7% for the continuous model). This is explained by the fact that with such a change, the number of cases among those who have been vaccinated decreases (respectively, increases). As a result, the number of recovered and dead people also decreases (respectively, increases).

Table 7: Impact of the proportion of contact vaccinated who did not infected

Parameter p_{cv}	**Total number of cases and % of the population**		**The total number of recovered and % of the total number of cases**		**The total number of deaths and % of the total number of cases**	
	Discrete	continuous	Discrete	continuous	Discrete	continuous
0.90	7284183 (38.95%)	7466126 (39.93%)	7274636 (99.87%)	7456303 (99.87%)	9546 (0.13%)	9822 (0.13%)
0.93	7071716 (37.82%)	7260647 (38.83%)	7062295 (99.87%)	7250946 (99.87%)	9421 (0.13%)	9701 (0.13%)
0.87	7486743 (40.04%)	7661702 (40.97%)	7477078 (99.87%)	7651766 (99.87%)	9665 (0.13%)	9936 (0.13%)

Table. 8 evaluates the impact of the exposed compartment proportion who did not infected. An increase in this parameter leads to an increase in the duration of the epidemic and the time it takes to reach its peak and a decrease in the number of ill, and therefore recovered and died. This suggests that with less infection, the epidemic becomes less intense, i.e. its terms are stretched, and fewer people get infected overall. In particular, with an increase (respectively, decrease) of this parameter by 1.5%, the duration of the epidemic increases by 12.5% for the discrete model and 13.3% for the continuous model (respectively, a decrease of

9.6% for the discrete model and 9.4% for the continuous model). Under the same conditions, there is a decrease in the number of cases by 19.2% for the discrete model and 19.3% for the continuous model (respectively, an increase of 15.1% for both models). Roughly the same effect has the proportion of exposed that get infected and isolated.

Table 8: Impact of the proportion of exposed who do not get sick

Parameter p_{es}	**Epidemic end time**		**Peak time of the epidemic**		**Total number of cases and % of the population**	
	Discrete	continuous	Discrete	continuous	Discrete	continuous
0.679	1101	1154	442	419	5887261 (31.48%)	6024809 (32.22%)
0.689	1249	1307	522	497	8385774 (44.84%)	8596170 (45.97%)
0.669	996	1045	393	363	5887261 (31.48%)	6024809 (32.22%)

Table. 9 evaluates the impact of the rate of vaccination on the overall process. An increase in this parameter leads to an increase in the duration of the epidemic and the time of its peak and a significant reduction in the total number of cases and those who are simultaneously ill, with a slight decrease in mortality. In particular, with an increase (respectively, a decrease) of this parameter by 80%, it leads to an increase in the duration of the epidemic by 5% for a discrete model and by 4% for a continuous model (respectively, a decrease in the duration of an epidemic by 1.6% for a discrete model and by 0.3% for a continuous model). At the same time, the total number of cases decreases by 36.8% for the discrete model and by 34.9% for the continuous one (respectively, the total number of cases increases by 32.4% for the discrete model and by 30.3% for the continuous model). The number of deaths is reduced by 41.2% for the discrete model and 39.3% for the continuous model (respectively, increases by 40.6% for the discrete model and 37.9% for the continuous model). The results obtained indicate the extreme importance of maintaining a high rate of vaccination of the population.

Table 9: Impact of vaccination rate

Parameter v	**Epidemic end time**		**The total number of deaths and % of the total number of cases**		**Total number of cases and % of the population**	
	Discrete	continuous	Discrete	continuous	Discrete	continuous
0.0005	1101	1154	7284183 (38.95%)	7466126 (39.93%)	9546 (0.13%)	9822 (0.13%)
0.0009	1157	1201	4601617 (24.61%)	4857401 (25.98%)	5610 (0.12%)	5961 (0.12%)
0.0001	1083	1151	9645972 (51.58%)	9730947 (52.04%)	13420 (0.14%)	13548 (0.14%)

4 Conclusion

The results obtained indicate a fairly high efficiency of the proposed models and can be used to predict the development of epidemics. In this case, in each specific case, the system is first

identified based on the available statistical information, after which the forecasting problem is solved. For models of epidemic development in the absence of vaccination, this procedure is implemented in [5, 9, 11].

Further refinement of the models can be carried out by considering the possibility of re-infection of those who have been ill due to the mutation of the virus and the gradual decrease of immunity in recovered people, as well as the limited duration of the vaccine. In this case all considered population compartments are preserved, but intercompartment transitions are added, taking into account the possibility of transition from the compartments of recovered and vaccinated to the susceptible compartment.

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