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BLOW UP OF SOLUTION FOR A NONLINEAR VISCOELASTIC PROBLEM WITH INTERNAL DAMPING AND LOGARITHMIC SOURCE TERM

This paper is concerned with blow up of weak solutions of the following nonlinear viscoelastic problem with internal damping and logarithmic source term

$$|u_t|^\rho u_{tt} + M(\|u\|^2)(-\Delta u) - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)ds + u_t = u|u|_R^{p-2} \ln |u|_R^k$$

with Dirichlet boundary initial conditions in a bounded domain $\Omega \subset R^n$. In the physical point of view, this is a type of problems that usually arises in viscoelasticity. It has been considered with power source term first by Dafermos [3], in 1970, where the general decay was discussed. We establish conditions of p, ρ and the relaxation function g, for that the solutions blow up in finite time for positive and nonpositive initial energy. We extend the result in [15] where is considered $M = 1$ and external force type $|u|^{p-2}u$ in it. Further we estate and sketch the proof of a result of local existence of weak solution that is used in the proof of the theorem on blow up. The idea underlying the proof of local existence of solution is based on Faedo-Galerkin method combined with the Banach fixed point method.
Key words: Nonlinear Viscoelastic Equation, Logarithmic Source, Blow Up, Local existence.

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Ішкі демпферлік және логарифмдік көзді сызықты емес тұтқыр серпімді есеп шешімінің қирауы

Бұл жұмыс $\Omega \subset R^n$ шектелген облыста бастапқы және Дирихле шартымен қойылған тұтқыр-серпімді ішкі демпфірлік және логарифмдік сызықты емес мүшелері бар

$$|u_t|^\rho u_{tt} + M(\|u\|^2)(-\Delta u) - \Delta u_{tt} + \int_0^t g(ts)\Delta u(s)ds + u_t = u|u|_R^{p-2} \ln |u|_R^k$$

есебінің әлсіз шешімдерінің қирауын зерттеуге арналған. Физикалық тұрғыдан алғанда, бұл әдетте тұтқыр серпімділікте пайда болатын мәселелердің бір түрі. Оны қуат көзі терминімен алғаш рет 1970 жылы Дафермос [3] қарастырды, онда жалпы ыдырау талқыланған. Мұнда оң және теріс бастапқы энергия үшін шешімдердің ақырлы уақытта қирауы туралы p, ρ және g релаксация функциясына шарттар алынды . Нәтижені [15] үшін де кеңейттік, мұнда $M = 1$ алынды және оған сыртқы күштің түрі $|u|^{p-2}u$. Біз қирау теоремасын дәлелдеуінде қолданылатын әлсіз локалдік шешімнің шешімділігін дәлелін келтіреміз. Локалдік шешімнің болуын дәлелдейтін идея Фаедо-Галеркин әдісіне негізделген және Банахтың бекітілген нүкте әдісімен біріктірілген.
Түйін сөздер: Тұтқырсерпімді сызықты емес теңдеу, логарифмдік көз, шешімнің қирауы, локалдік бар болу.

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Разрушение решения нелинейной вязкоупругой задачи с внутренним затуханием и логарифмическим источником

Эта статья посвящена разрушению слабых решений следующих нелинейных вязкоупругая задача с внутренним демпфированием и логарифмическим исходным членом

$$|u_t|^\rho u_{tt} + M(\|u\|^2)(-\Delta u) - \Delta u_{tt} + \int_0^t g(ts)\Delta u(s)ds + u_t = u|u|_R^{p-2} \ln |u|_R^k$$

с граничными начальными условиями Дирихле в ограниченной области $\Omega \subset R^n$. С физической точки зрения это тип проблем, которые обычно возникают в вязкоупругости. Впервые он был рассмотрен с термином источника энергии Дафермосом [3] в 1970 году, где обсуждался общий распад энергии. Устанавливаются условия p, ρ и функции релаксации g, при которых решения разрушаются за конечное время при положительной и неположительной начальной энергии. Мы распространяем результат на [15], где рассматривается $M = 1$ и в нем внешняя сила типа $|u|^{p-2}u$. Далее мы сформулируем и набросаем доказательство результата локального существования слабого решения, используемого в доказательстве теоремы о разрушении. Идея, лежащая в основе доказательства локального существования решения, основана на сочетании метода Фаэдо-Галеркина с методом неподвижной точки банаха.
Ключевые слова: Нелинейное уравнение вязкоупругости, логарифмический источник, разрушение, локальное существование.

1 Introduction

In elasticity the existing theory accounts for materials which have a capacity to store mechanical energy with no dissipation (of the energy). On the other hand, a Newtonian viscous fluid in a nonhydrostatic stress state has a capacity for dissipating energy without storing it. Materials which are outside the scope of these two theories would be those for which some, but not all, of the work done to deform them, can be recovered. Such materials possess a capacity of storage and dissipation of mechanical energy. This is the case of viscoelastic materials.

Viscoelastic materials are those for which the behavior combines liquid-like and solid-like characteristics. Viscoelasticity is important in areas such as biomechanics; power industry or heavy construction; Synthetic polymers; Wood; Human tissue, cartilage; Metals at high temperature; Concrete.

Polymers, for instance, are viscoelastic materials since they exhibit an intermediate position between viscous liquids and elastic solids. The formulation of Boltzmann's superposition principle leads to a memory term involving a relaxation function of exponential type. But, it has been observed that relaxation functions of some viscoelastic materials are not necessarily of this type. See [13, 14]. In this work, we are concerned with the following initial boundary value problem:

$$\begin{cases} |u_t|^{\rho}_{\mathbb{R}} u_{tt} + M(\|u\|^2)(-\Delta u) - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)ds + u_t \\ = u|u|^{p-2}_{\mathbb{R}} \ln |u|^{k}_{\mathbb{R}} \quad \text{in} \quad \Omega \times (0,\infty) \\ u = 0 \quad \text{on} \quad \partial\Omega \times [0,\infty) \\ u(x,0) = u_0(x) \quad \text{in} \quad \Omega \\ u_t(x,0) = u_1(x) \quad \text{in} \quad \Omega. \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) is a bounded domain with a smooth boundary $\partial\Omega$, $p > 2$, $\rho > 0$ and $k > 0$ are constants and $g : \mathbb{R}^+ \to \mathbb{R}^+$ and $M : [0,\infty) \to \mathbb{R}$ are C^1 functions, respectively, left to be defined later.

As mentioned in [9], the logarithmic nonlinearity appears in several branches of physics such as inflationary cosmology, nuclear physics, optics, and geophysics. With all this specific underlying meaning in physics, the global-in-time well-posedness of solution to the problem of evolution equation with such logarithmic-type nonlinearity captures lots of attention. See [9] for the references related to each branch listed above.

The dispersive term Δu_{tt} arises in the study of extensional vibrations of thin rods, see Love [7], via the model

$$u_{tt} - \Delta u - \Delta u_{tt} = f$$

and was studied by one of the authors in [11]. The function $M(\lambda)$ in (1) has its motivation in the mathematical description of vibration of an elastic stretched string, modeled by the equation

$$u_{tt} - M\Big(\int_\Omega |\nabla u|^2 dx\Big)\Delta u = 0,$$

which for $M(\lambda) \geq m_0 > 0$ was studied in [2, 4, 5, 10, 12].

Concerning blow-up results, Messaoudi [8] considered the equation

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + a u_t |u_t|^{m-2} = b|u|^{r-2}u$$

and proved that any weak solution with negative initial energy blows up in finite time if $r < m$ and $\int_0^\infty g(s)ds \leq \frac{r-2}{r-2+\frac{1}{r}}$. Also, Liu [6] studied the equation

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds - \omega \Delta u_t + \mu u_t = |u|^{r-2}u$$

where he proved that the solution with nonpositive initial energy as well as positive initial energy blows up in finite time.

Our blow up result is motivated by the viscoelastic wave equation with delay considered by [15]

$$|u_t|^\rho u_{tt}\Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)ds + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = b|u|^{p-2}u.$$

We implemented the technique employed in it, in order to extend his/her problem to the case of logarithmic source term and M variable.

This work is divided as follows. The section 2 presents the notation and results underlying the methods used in this paper. In section 3 is stated and proved a result of blow up for locally defined solutions.

2 Preliminaries and assumptions

For simplicity of notations hereafter we denote by $|\cdot|$ the Lebesgue Space $L^2(\Omega)$-norm, $\|\cdot\| := \int_\Omega |\nabla(\cdot)|^2_{\mathbb{R}^n}dx$ the Sobolev space $H_0^1(\Omega)$-norm, $\|\cdot\|_r := \|\cdot\|_{L^r(\Omega)}$ and $|\cdot|_{\mathbb{R}}$ and $|\cdot|_{\mathbb{R}^n}$ for absolute value of a real number and the norm of a vetor in $\mathbb{R}^n$, respectively.

Lemma 1 *There exists $C > 0$ such that*

$$\|u\|_r^s \le C\Big(\|u\|^2 + \|u\|_r^r\Big)$$

for any $u \in H_0^1(\Omega)$ and $2 \le s \le r$.

We start setting some hypotheses for the problem (1). Firstly, we shall assume that

$$0 < \rho \le \frac{2}{n-2} \quad \text{if } n \ge 3, \quad \text{or } \rho > 0 \text{ if } n = 1, 2, \tag{2}$$

$$2 < p \le \frac{2(n-1)}{n-2} \quad \text{if } n \ge 3, \quad \text{or } p > 2 \text{ if } n = 1, 2. \tag{3}$$

Secondly, we assume:

(H.1) $M \in C^1([0,\infty),\mathbb{R})$ is such that $M(\lambda) \ge m_0$, $\forall \lambda \in [0,\infty)$, where $m_0 > 0$.

(H.2) $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue integrable and absolutely continuous function such that

$$1 - \int_0^\infty g(s)ds =: l > 0.$$

(H.3) There exist positive constants ξ_1 and ξ_2 verifying

$$-\xi_1 g(t) \le g'(t) \le -\xi_2 g(t) \quad \text{for almost all } t \ge 0.$$

We will need the very useful relation

$$\int_0^t g(t-\tau)(\nabla u(\tau), \nabla u_t(t))d\tau = \frac{1}{2}(g' \diamond \nabla u)(t) - \frac{1}{2}(g \diamond \nabla u)'(t) + \frac{d}{dt}\left\{\frac{1}{2}\left(\int_0^t g(s)ds\right)|\nabla u(t)|^2\right\} - \frac{1}{2}g(t)|\nabla u(t)|^2 \tag{4}$$

that can be checked directly, where

$$(g \diamond y)(t) = \int_0^t g(t-s)|y(t)-y(s)|^2 ds$$

Let us denote $\hat{M}(s) = \int_0^s M(\tau)d\tau$. If $u(t), u_t(t) \in H_0^1(\Omega)$, then we define the total energy functional of equation (1):

$$\begin{aligned}\mathcal{E}(t) :=& \frac{1}{\rho+2}\|u_t(t)\|_{\rho+2}^{\rho+2} + \frac{1}{2}\left(\hat{M}(\|u\|^2) - \int_0^t g(s)ds\|u\|^2\right) + \frac{1}{2}\|u_t\|^2 \\ &+ \frac{k}{p^2}\int_\Omega |u|^p dx + \frac{1}{2}(g\diamond \nabla u)(t) - \frac{1}{p}\int_\Omega |u|_{\mathbb{R}}^p \ln |u|_{\mathbb{R}}^k dx. \end{aligned} \tag{5}$$

From (4) and (H.3) one deduce that

$$\mathcal{E}'(t) = -|u_t(t)|^2 + \frac{1}{2}(g'\diamond \nabla u)(t) = \frac{1}{2}g(t)|\nabla u(t)|^2 \leq 0. \tag{6}$$

Using (H.1), (H.2), we infer

$$\begin{aligned}\mathcal{E}(t) &\geq \frac{m_0+l-1}{2}\|u\|^2 + \frac{1}{2}(g\diamond\nabla u)(t) - \frac{c_s^{p+1}}{p}\|u\|^{p+1} \\ &\geq F(\sqrt{(m_0+l-1)\|u\|^2 - (g\diamond\nabla u)(t)})\end{aligned}$$

where c_s is the constant obtained from Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$, and $F(x) = \frac{1}{2}x^2 - \frac{1}{p}B_1^{p+1}x^{p+1}$, with $B_1 = \frac{c_s}{(m_0+l-1)^{1/2}}$.

Remark 1 *As noticed in [15], F is increasing in $(0,\lambda_1)$, decreasing in (λ_1,∞), and F has a maximum at $\lambda_1 = B_1^{-\frac{p+1}{p-1}}$ with the maximum value $E_1 = F(\lambda_1) = \frac{p-1}{2(p+1)}\lambda_1^2$.*

Lemma 2 ([15]) *Supposing (2), (3), (H.1) and (H.2), and that $(m_0+l-1)\|u_0\|^2 > \lambda_1^2$ and $E(0) < E_1$, then there exists $\lambda_2 > \lambda_1$ such that, for all $t \in [0,T)$,*

$$(m_0+l-1)\|u\|^2 + (g\diamond\nabla u)(t) \geq \lambda_2^2 \tag{7}$$

and

$$\|u\|_{p+1}^{p+1} \geq \frac{B_1^p}{p}\lambda_2^{p+1}. \tag{8}$$

3 Blow up

Theorem 1 *Assume that (2), (3), (H.1) and (H.2), and that $m_0+l-1>0$. Let $f \in L^2(0,T;H^{-1}(\Omega))$ and $u_0, u_1 \in H_0^1(\Omega)$. Then there exists a unique weak solution u for the problem*

$$\begin{cases} M(\|u\|^2)(-\Delta u) - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)ds + u_t = f \\ u(0) = u_0, \quad u_t(0) = u_1. \end{cases} \tag{9}$$

Further, u_{tt} belongs to the class $L^\infty(0,T;H_0^1(\Omega))$.

Proof. Employ the Faedo-Galerkin method and Aubin-Lions Lemmas as in reference [1]. □

For our purposes hereafter, let us define

$$\mathbf{W} := \Big\{ w : w, w_t \in C(0,T; H_0^1(\Omega)), w_{tt} \in L^\infty(0,T; H_0^1(\Omega)) \Big\}$$

equipped with the norm

$$\|w\|_{\mathbf{W}}^2 := \alpha \|w\|^2_{L^\infty(0,T;H_0^1(\Omega))} + \delta \|w_t\|^2_{L^\infty(0,T;H_0^1(\Omega))} + \gamma \|w_{tt}\|^2_{L^2(0,T;H_0^1(\Omega))},$$

where $\alpha := \frac{m_0+l-1}{2}$, $\delta := \frac{1}{\sqrt{T}}$ and $\gamma := \frac{1}{\sqrt[4]{T}}$. □

It is easy to check that $\mathbf{W}$ is a Banach space with the norm $\|\cdot\|_{\mathbf{W}}$.

Theorem 2 *Let $u_0, u_1 \in H_0^1(\Omega)$ and assume that (H.1)-(H.3) and (2) and (3) are valid. Then the problem (1) has a local weak solution u in $\mathbf{W}$ for T small enough.*

Sketch of the proof. Let $M > 0$ and $T > 0$ and denote $\mathbf{Z}(M,T)$ the class of functions w belonging to $\mathbf{W}$, satisfying $w(0) = u_0$, $w_t(0) = u_1$ and $\|w\|_{\mathbf{W}} \leq M$. Let us consider the application $A : \mathbf{Z}(M,T) \to \mathbf{W}$ defined in the following way. For each $v \in \mathbf{Z}(M,T)$, take $u := A[v]$ as the unique solution of the problem (9) with $f = v|v|_{\mathbb{R}}^{p-2} \ln |v|_{\mathbb{R}}^k - |v_t|_{\mathbb{R}}^\rho v_{tt}$. One can prove that with the hypotheses for p and ρ, A is a contraction from $\mathbf{Z}(M,T)$ to itself if M is large and T small enough. Apply next the Banach fixed point Theorem. □

In order to establish our result, an extra assumption on g is required:

(H.4)

$$\int_0^\infty g(s)ds < \frac{m_0 \zeta}{1+\zeta},$$

with $\zeta := \Big((p-2) - \beta(p-1)\Big)\Big(p - \beta(p-1)\Big)$, where $0 < \beta < \frac{p-2}{p-1}$ is a fixed number.

Theorem 3 *Assume that (2), (3), (H.1) and (H.2), and that $(m_0 + l - 1)\|u_0\|^2 > \lambda_1^2$ and $E(0) < \beta E_1$ and $\rho < p - 2$. Also assume that $\hat{M}(\tau) \leq M(\tau)\tau$. Suppose that $u_0, u_1 \in H_0^1(\Omega)$. Then the solution u of (1) blows up in finite time.*

Proof. By contradiction we suppose there exists $K_1 > 0$ such that

$$\|u(t)\|^2 \leq K_1 \ , \forall t \geq 0.$$

Set

$$H(t) = E_2 - \mathcal{E}(t),$$

where $E_2 \in (E(0), \beta E_1)$. By Lemma 6, we obtain $H(t) > 0$ and $H'(t) \geq 0$, $\forall t \geq 0$. Also, since $E_1 = \frac{p-1}{2(p+1)}\lambda_1^2$, then

$$\begin{aligned} H(t) &\leq \beta E_1 - \frac{1}{2}\Big((m_0 + l - 1)\|u\|^2 + (g \diamond \nabla u)(t)\Big) + \frac{1}{p}\int_\Omega |u|_{\mathbb{R}}^p \ln |u|_{\mathbb{R}}^k dx \\ &\leq E_1 - \frac{1}{2}\lambda_1^2 + \frac{1}{p}\int_\Omega |u|_{\mathbb{R}}^p \ln |u|_{\mathbb{R}}^k dx \leq \frac{1}{p}\int_\Omega |u|_{\mathbb{R}}^p \ln |u|_{\mathbb{R}}^k dx. \end{aligned} \tag{10}$$

Define

$$L(t) = H^{1-\sigma}(t) + \frac{\varepsilon}{\rho+1}\int_\Omega |u_t|^\rho_{\mathbb{R}} u_t u dx + \varepsilon \int_\Omega \nabla u_t \nabla u dx + \frac{\varepsilon}{2}\int_\Omega u^2 dx, \tag{11}$$

where ε is chosen small enough for that $L(0) > 0$. Taking derivative of (11) and using (5), we get

$$\begin{aligned} L'(t) = (1+\sigma)L^{-\sigma}L' + \varepsilon\Big\{ &- M(\|u\|^2)\|u\|^2 + \int_0^t g(t-s)(\nabla u(s), \nabla u(t))ds \\ &+ \int_\Omega |u|^p_{\mathbb{R}} \ln |u|^k_{\mathbb{R}} dx\Big\} + \frac{\varepsilon}{\rho+1}\int_\Omega |u_t|^{\rho+2}_{\mathbb{R}} dx + \varepsilon \int_\Omega |\nabla u_t|^2_{\mathbb{R}^n} dx. \end{aligned} \tag{12}$$

It is easy to check the following inequality

$$\varepsilon \int_0^t g(t-s)(\nabla u(s), \nabla u(t))ds \geq \left(1 - \frac{1}{4\eta}\right)\int_0^t g(s)\|u\|^2 - \eta(g \diamond \nabla u)(t) \tag{13}$$

holds for all $\eta \geq 0$.

Employing the inequalities (13) into (12) we obtain

$$\begin{aligned} L'(t) &\geq (1-\sigma)H^{-\sigma}H' + \varepsilon\frac{1}{\rho+1}\|u_t\|^{\rho+2}_{\rho+2} - \varepsilon\eta(g \diamond \nabla u)(t) \\ &+ \varepsilon\Big[- M(\|u\|)\|u\|^2 + \left(1 - \frac{1}{4\eta}\right)\int_0^t g(s)ds\Big]\|u\|^2 \\ &+ \varepsilon \int_\Omega |u|^p_{\mathbb{R}} \ln |u|^k_{\mathbb{R}} dx + \varepsilon \int_\Omega |\nabla u_t|^2_{\mathbb{R}^n} dx. \end{aligned} \tag{14}$$

Adding $\varepsilon p(H(t) - E_2 + E(t))$ into (14), and regarding the equation of the total energy in (5) and that $\hat{M}(\tau) \geq M(\tau)\tau$, $\forall t \geq 0$, it follows

$$\begin{aligned} L'(t) &\geq (1-\sigma)H^{-\sigma}H' + \varepsilon\left(\frac{1}{\rho+1} + \frac{p}{\rho+2}\right)\|u_t\|^{\rho+2}_{\rho+2} + \varepsilon\left(\frac{p}{2} - \eta\right)(g \diamond \nabla u)(t) \\ &+ \varepsilon\Big[- M(\|u\|^2)\|u\|^2 + \frac{p}{2}\hat{M}(\|u\|^2) - \left(\frac{p-2}{2} + \frac{1}{4\eta}\right)\int_0^t g(s)ds\|u\|^2\Big] \\ &+ \frac{\varepsilon k}{p}\int_\Omega |u|^p_{\mathbb{R}} dx + \varepsilon\left(1 + \frac{1}{2}\right)\int_\Omega |\nabla u_t|^2_{\mathbb{R}^n} dx + \varepsilon p H(t) - \varepsilon p E_2 \\ &\geq (1-\sigma)H^{-\sigma}H' + \varepsilon\left(\frac{1}{\rho+1} + \frac{p}{\rho+2}\right)\|u_t\|^{\rho+2}_{\rho+2} + \varepsilon\left(\frac{p}{2} - \eta\right)(g \diamond \nabla u)(t) \\ &+ \varepsilon\Big[\frac{(p-2)m_0}{2}\|u\|^2 - \left(\frac{p-2}{2} + \frac{1}{4\eta}\right)\int_0^t g(s)ds\|u\|^2\Big] \\ &+ \frac{\varepsilon k}{p}\int_\Omega |u|^p_{\mathbb{R}} dx + \varepsilon\left(1 + \frac{1}{2}\right)\int_\Omega |\nabla u_t|^2_{\mathbb{R}^n} dx + \varepsilon p H(t) - \varepsilon(p+1)E_2 \end{aligned} \tag{15}$$

Taking now η to satisfy

$$\frac{1-l}{2\Big[(p-2)-\beta(p-1)\Big](m_0+l-1)} < \eta < \frac{p(1-\beta)}{2}+\beta, \tag{16}$$

which is possible by (H.4). Noticing that $\hat{M}(\tau) \leq M(\tau)\tau$ and that $(m_0+l-1)\|u\|^2 + (g \diamond \nabla u)(t) \geq \lambda_2^2$ (Lemma 2), we get

$$\begin{aligned}
&\frac{(p-2)m_0}{2}\|u\|^2 - \left(\frac{p-2}{2}+\frac{1}{4\eta}\right)\int_0^t g(s)ds\|u\|^2 + \left(\frac{p}{2}-\eta\right)(g \diamond \nabla u)(t) - (p+1)E_2 \\
&\geq \frac{\beta(p-1)}{2}\Big((m_0+l-1)\|u\|^2 + (g\diamond\nabla u)(t)\Big) - (p+1)E_2 \\
&= \frac{\beta(p-1)}{2}\frac{\lambda_1^2-\lambda_2}{\lambda_2^2}\Big((m_0+l-1)\|u\|^2 + (g\diamond\nabla u)(t)\Big) \\
&+ \frac{\beta(p-1)}{2}\frac{\lambda_1^2}{\lambda_2^2}\Big((m_0+l-1)\|u\|^2 + (g\diamond\nabla u)(t)\Big) - (p+1)E_2 \\
&\geq c_1\Big((m_0+l-1)\|u\|^2 + (g\diamond\nabla u)(t)\Big) + c_2
\end{aligned}$$

where $c_1 = \frac{\beta(p-1)}{2}\frac{\lambda_1^2-\lambda_2}{\lambda_2^2}$ and $c_2 = \frac{\beta(p-1)}{2}\lambda_1^2 - (p+1)E_2$. From $E_2 < \beta E_1$ and $E_1 = \frac{p-1}{2(p+1)}\lambda_1^2$, we have

$$c_2 = \frac{\beta(p-1)}{2}\lambda_1^2 - (p+1)E_2 > \beta\Big(\frac{(p-1)\lambda_1^2}{2} - (p+1)E_1\Big) = 0.$$

By the above estimates we deduce there exists $K > 0$ such that

$$L'(t) \geq K\Big(H(t) + \|u_t\|_{\rho+2}^{\rho+2} + \|u\|_p^p + \|u\|^2 + \|u_t\|^2\Big). \tag{17}$$

Next steps are aimed to estimate $L(t)^{\frac{1}{1-\sigma}}$. Let

$$0 < \sigma < \frac{1}{\rho+2} - \frac{1}{p}. \tag{18}$$

From Hölder inequality and Young's inequality we obtain:

$$\Big(\Big|\int_\Omega |u_t|_{\mathbb{R}}^\rho u_t u dx\Big|\Big)^{\frac{1}{1-\sigma}} \leq \|u_t\|_{\rho+2}^{\frac{\sigma+1}{1-\sigma}}\|u\|_{\rho+2}^{\frac{1}{1-\sigma}} \leq C_3\|u_t\|_{\rho+2}^{\frac{\sigma+1}{1-\sigma}}\|u\|_p^{\frac{1}{1-\sigma}} \tag{19}$$

$$\leq c_4\Big(\|u_t\|_{\rho+2}^{\frac{\sigma+1}{1-\sigma}\mu} + \|u\|_p^{\frac{1}{1-\sigma}\theta}\Big), \tag{20}$$

where $\frac{1}{\mu} + \frac{1}{\theta} = 1$. Choosing $\mu = \frac{(1-\sigma)(\rho+2)}{\rho+1} > 1$, it follows from (18) that $\frac{\theta}{1-\sigma} = \frac{\rho+2}{(1-\sigma)(\rho+2)-(\rho+1)} < p$. Thus, Lemma 1 implies

$$\Big(\Big|\int_\Omega |u_t|_{\mathbb{R}}^\rho u_t u dx\Big|_{\mathbb{R}}\Big)^{\frac{1}{1-\sigma}} \leq c_5\left(\|u_t\|_{\rho+2}^{\rho+2} + \|u\|^2 + \|u\|_p^p\right). \tag{21}$$

Similarly as derived in (20), we also obtain

$$\left(\int_\Omega |\nabla u_t \nabla u|_{\mathbb{R}} dx\right)^{\frac{1}{1-\sigma}} \leq c_6 \left(\|u_t\|^{2(1-\sigma)} + \|u\|^{\frac{2(1-\sigma)}{1-2\sigma}}\right)^{\frac{1}{1-\sigma}}$$
$$\leq c_7 \left(\|u_t\|^2 + \|u\|^{\frac{2}{1-2\sigma}}\right)^{\frac{1}{1-\sigma}}. \tag{22}$$

Notice that

$$\|u\|^{\frac{2}{1-2\sigma}} \leq K_1^{\frac{2}{1-2\sigma}} \leq K_1^{\frac{2}{1-2\sigma}} \frac{H(t)}{H(0)} = c_8 H(t). \tag{23}$$

Therefore, from (21), (22) and (23) we infer that

$$L(t)^{\frac{1}{1-\sigma}} \leq c_9 \Big(H(t) + \|u_t\|_{\rho+2}^{\rho+2} + \|u\|_p^p + \|u\|^2 + \|u_t\|^2 \Big). \tag{24}$$

Combining 24 with (17) it yelds

$$L'(t) \geq c_{10} L(t)^{\frac{1}{1-\sigma}}. \tag{25}$$

Integrating (25) from 0 to t, we have

$$L(t) \geq \left(L(0)^{\frac{-\sigma}{1-\sigma}} - \frac{c_{11}}{1-\sigma} t \right)^{-\frac{1-\sigma}{\sigma}}. \tag{26}$$

This is a contradiction with the supposition that $\|u\|$ is globally bounded in t. Hence, the proof is complete. □

4 Conclusion

This work deals with a nonlinear viscoelastic problem with internal damping and logarithmic source term, which is an improvement of the problem considered in [15] in the case of absence of the term involving delay. By admitting the initial energy to be even positive, the problem becomes slightly difficult, what makes necessary a study of the growth of the terms of the total energy separately (Lemma 2). This work also states and sketches the proof of local existence of solution assumed to exist in the Theorem 3.

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