Stability of differential systems on the first approach

T.M. Aldibekov, M.M. Aldazharova
al-Farabi Kazakh National University, Almaty, Kazakhstan
e-mail: tamash59@list.ru, a_maira77@mail.ru

Abstract

In the article we consider linear system of differential equations in critical cases of characteristic exponents of Lyapunov. Linear system is investigated at small disturbances. Exponential stability of the trivial solution of nonlinear system of the differential equations on the first approach relatively to some monotonously increasing function is proved .

Let's consider

$$\dot{x} = A(t)x \tag{1}$$

a linear system with a continuous real coefficients, where $t \geq t_0 > 1$, $x \in R^n$, $n \in \aleph$, $A(t) - n \times n$ matrix, $A(t) \in C[t_0, +\infty)$,

$$\|A(t)\| \leq K\varphi(t), K > 0, \varphi(t) \in C[t_0, +\infty), \varphi(t) > 0$$

$$q(t) = \int\limits_{t_0}^{t} \varphi(s)ds.$$

We assume, that the linear system (1) has the finite general exponents with respect to $q(t)$ and inequalities are carried out $\ln t < q(t) < t$.

Given linear system

$$\dot{x} = A(t)x + f(t,x) \tag{2}$$

where, $n-$ dimensional vector function $f(t,x)$, continuous on $t \geq t_0$ and continuous differentiable on $x \in R^n$, $f(t,0) = 0$.

Definition 1. *Trivial solution $x = 0$ of the system (2) is called exponential stable with respect to $q(t)$ for $t \to +\infty$, if for any solution $x(t) \equiv x(t;t_0,x_0)$ of this system in some area $t_0 \leq t < \infty, \|x\| \leq h$ inequality is valid*

$$\|x(t)\| \leq N\|x(t_0)\|e^{-\alpha(q(t)-q(t_0))}, t \geq t_0$$

where N and α – positive constants, which aren't depend from choice of the solution $x(t)$.

Notice, if $q(t) = t$, then we obtain usual definition of exponential stability of the trivial solution $x = 0$ of the system (2).

Theorem 1. *If following conditions are satisfied:*

1) linear system (1) generalized-regular with respect to $q(t)$,

2) higher generalized exponent with respect to $q(t)$ is negative i. e.,

$$\lambda_1(q) < 0$$

3) $n-$ dimensional vector function $f(t,x)$ satisfies inequality

$$\|f(t,x)\| \le \varphi(t)\|x\|^m$$

where

$$m > 1 + \frac{1}{|\lambda_1(q)|},$$

then trivial solution of nonlinear system (2) exponential stable with respect to $q(t)$ for $t \to +\infty$.

Proof. Let's take $\gamma > 0$ such, that

$$\frac{1}{m-1} < \gamma < |\lambda_1(q)|$$

On system (2) fulfill transformation

$$x = ye^{-\gamma[q(t)-q(t_0)]} \tag{3}$$

Then we obtain

$$\frac{dy}{dt} = B(t)y + g(t,y) \tag{4}$$

$$B(t) = A(t) + \gamma\frac{dq(t)}{dt}E,$$

where $E-$ $n \times n-$ dimensional individual matrix

$$g(t,y) = e^{\gamma[q(t)-q(t_0)]}f(t, ye^{-\gamma[q(t)-q(t_0)]})$$

and $y(t_0) = x(t_0)$. Moreover, linear system

$$\dot{y} = B(t)y \tag{5}$$

is generalized regular and has negative higher generalized exponent. Vector function $g(t,y)$ is continuous on $t \ge t_0$ and continuous differentiable on $y \in R^n$.

Let's transfer from the differential equation (4) with the entry condition

$$y(t_0) = x(t_0) = x_0$$

applying a method of arbitrary constants to the integral equation

$$y(t) = H(t)y(t_0) + \int_{t_0}^{t} K(t,\tau)g(t,y(\tau))d\tau \tag{6}$$

where H(t) – normed fundamental matrix of the linear system (5),

$$K(t,\tau) = H(t)H^{-1}(\tau), \|H(t)\| \le C_1, \quad (C_1 \ge 1), t \ge t_0$$
$$\|K(t,\tau)\| \le C_2 e^{\varepsilon[q(\tau)-q(t_0)]}$$

for $t_0 \leq \tau \leq t < +\infty$, where $C_2 \geq 1$ and $\varepsilon > 0$ are arbitrary.

Let's take ε such, that

$$0 < \varepsilon < min\{\frac{(m-1)\gamma - 1}{2}, \frac{1}{2}\}$$

Further, estimating vector function $g(t, y)$ we obtain

$$\|g(t,y)\| \leq C_3 e^{[\varepsilon-(m-1)\gamma][q(t)-q(t_0)]}\|y\|^m, (C_3 > 1)$$

Now, from the integral equation (6) estimating on norm, we obtain that

$$\|y(t)\| \leq C_3\|y(t_0)\| + \int_{t_0}^{t} C_2 C_3 e^{[2\varepsilon-(m-1)\gamma][q(\tau)-q(t_0)]}\|y\|^m d\tau \tag{7}$$

Notice that for enough small neighborhood of the point $y(t_0)$ owing to condition

$$\ln t < q(t)$$

Inequality is carried out

$$(m-1)C_1^{m-1}\|y(t_0)\|^{m-1}\int_{t_0}^{t} C_2 C_3 e^{-[(m-1)\gamma-2\varepsilon][q(\tau)-q(t_0)]}d\tau < 1.$$

From here using Lemma of Bichary we obtain

$$\|y(t)\| \leq \frac{C_1\|y(t_0)\|}{[1-(m-1)C_1^{m-1}\|y(t_0)\|^{m-1}\int_{t_0}^{t} C_2 C_3 e^{-[(m-1)\gamma-2\varepsilon][q(\tau)-q(t_0)]}d\tau]^{\frac{1}{m-1}}} \tag{8}$$

Hence, solution $y(t)$ isdefinedon $t \geq t_0 > 1$ and inequality is valid $\|y(t)\| \leq N\|y(t_0)\|$, where $N = N(t_0) > 0$

Coming back to the $x(t)$ we have

$$\|x(t)\| \leq N\|x(t_0)\|e^{-\gamma(q(t)-q(t_0))}$$

for $\|x(t_0)\| < \triangle$, where $\triangle$ is enough small. Thus, trivial solution of the nonlinear system (2) is exponential stable on Lyapunov with respect to $q(t)$ for $t \to +\infty$. The theorem is proved.

Reference

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[2]*Aldibekov T.M.* About stability on the first approach // Modern problems of science and education. – M, 2008. – No.– Page 133.

[3]*Demidovich B.P.* Lectures on the mathematical theory of stability. // – M, 1967. – 472 pages.

Т.М. Алдибеков, М.М. Алдажарова, Устойчивость дифференциальных систем по первому приближению, Вестник КазНУ, сер. мат., мех., инф. 2012, №1(72), 3 – 6

Рассматривается линейная система дифференциальных уравнении в критических случаях характеристических показателей Ляпунова. Исследуется линейная система при малых возмущениях. Доказана экспоненциальная устойчивость тривиального решения нелинейной системы дифференциальных уравнений по первому приближению относительно некоторой монотонно возрастающей функции.

Т.М. Алдибеков, М.М. Алдажарова, Бірінші жуықтау бойынша дифференциалдық жүйелердің орнықтылығы, ҚазҰУ хабаршысы, мат., мех., инф. сериясы 2012, №1(72), 3 – 6

Ляпунов сипаттауыш көрсеткіштерінің сын жағдайдағы дифференциалдық теңдеулер сызықты жүйесі қарастырылады. Аз ауытқулары бар сызықты жүйе зерттеледі. Сызықты емес дифференциалдық теңдеулер жүйесінің тривиал шешімінің қайсыбір монотонды өспелі функцияға қатысты бірінші жуықтау бойынша экспоненциалды орнықтылығы дәлелденген.