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A ROBUST NUMERICAL METHOD FOR SINGULARLY PERTURBED SOBOLEV PERIODIC PROBLEMS ON B-MESH

This article examines periodic Sobolev reports with a singular deviation, which causes significant difficulties in numerical approximation due to the presence of sharp or boundary layers. A stable quantitative method for the effective solution of such problems in the Bakhvalov lattice, a special grid for the deviant action of the solution, is proposed. Singularly perturbed periodic Sobolev problems create significant difficulties in numerical approximation due to the presence of sharp layers or boundary layers. Our proposed reliable numerical method for efficiently solving such problems on the Bakhvalov grid, a specialized grid, is designed to account for the singular behavior of the solution. First, an asymptotic analysis of the exact solution is performed. Then a finite difference scheme is created by applying quadrature interpolation rules to an adaptive network. The stability and convergence of the presented algorithm in a discrete maximum norm is analyzed. The results show that the proposed approach provides an accurate approximation of the solution for singular problems while maintaining computational efficiency.

Key words: Difference scheme, error estimate, periodic boundary value problem, singular perturbation, Sobolev differential equation.

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B-тордағы сингулярлы ауытқыған Соболев периодты проблемалары үшін тұрақты сандық әдісі

Бұл мақалада Соболевтің сингулярлық ауытқуы бар мерзімді есептері қарастырылады, бұл өткір немесе шекаралық қабаттардың болуына байланысты сандық жуықтауда айтарлықтай қиындықтар тудырады. Бахвалов торында мұндай мәселелерді тиімді шешудің тұрақты сандық әдісі, шешімнің ауытқу әрекеті үшін арнайы тор ұсынылған. Соболевтің ерекше ашуланған мерзімді міндеттері өткір қабаттардың немесе шекаралық қабаттардың болуына байланысты сандық жуықтауда айтарлықтай қиындықтар туғызады. Бахвал торында, мамандандырылған торда осындай мәселелерді тиімді шешу үшін біз ұсынатын сенімді сандық әдіс шешімнің сингулярлық мінез-құлқын есепке алуға арналған. Алдымен нақты шешімге асимптотикалық талдау жасалады. Содан кейін адаптивті желіге квадратуралық интерполяция ережелерін қолдану арқылы ақырлы айырмашылық схемасы жасалады. ұсынылған алгоритмнің тұрақтылығы мен конвергенциясы дискретті максималды нормада талданады. Нәтижелер ұсынылған тәсіл есептеу тиімділігін сақтай отырып, сингулярлық есептер үшін шешімнің дәл жуықтауын қамтамасыз ететінін көрсетеді.

Түйін сөздер: Айырмашылық схемасы, қатені бағалау, периодты шекаралық есеп, сингулярлық бұзылыс, Соболевтің дифференциалдық теңдеуі.

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Устойчивый численный метод для сингулярных возмущенных периодических проблем Соболева на B-сетке

В данной статье рассматриваются периодические отчеты Соболева с сингулярным отклонением, что вызывает значительные трудности в численном приближении из-за наличия острых или пограничных слоев. Предложен устойчивый количественный метод эффективного решения таких задач в решетке Бахвалова, специальная сетка для отклоняющегося действия решения. Сингулярно возмущенные периодические задачи Соболева создают значительные трудности при численной аппроксимации из-за наличия резких слоев или пограничных слоев. Предлагаемый нами надежный численный метод для эффективного решения таких задач на Бахваловской сетке, специализированной сетке, предназначен для учета сингулярного поведения решения. Сначала проводится асимптотический анализ точного решения. Затем создается конечно-разностная схема путем применения квадратурных правил интерполяции к адаптивной сети. Анализируется устойчивость и сходимость представленного алгоритма в дискретной максимальной норме. Результаты показывают, что предложенный подход обеспечивает точное приближение решения для сингулярных задач при сохранении вычислительной эффективности.
Ключевые слова: Разностная схема, оценка погрешности, периодическая краевая задача, сингулярное возмущение, дифференциальное уравнение Соболева.

1 Introduction

In this study, we consider the following singularly perturbed initial-periodic boundary value problem in the domain $\overline{D} = \overline{\Omega} \times [0,T]$; $\overline{\Omega} = [0,l]$, $\Omega = (0,l)$, $D = \Omega \times (0,T]$:

$$Lu \equiv L_1[u_{tt}] + L_2 u = f(x,t), \quad (x,t) \in D, \tag{1}$$

$$u(x,0) = \varphi(x), \quad x \in \overline{\Omega}, \tag{2}$$

$$u_t(x,0) = \psi(x), \quad x \in \overline{\Omega} \tag{3}$$

$$u(0,t) = u(l,t), \quad t \in (0,T], \tag{4}$$

$$u_t(0,t) = u_t(l,t), \quad t \in (0,T], \tag{5}$$

where

$$L_1[u_{tt}] \equiv -\varepsilon u_{xxtt} + a(x)u_{tt},$$

$$L_2[u(x,t)] \equiv -\varepsilon u_{xx} + b(x,t)u(x,t),$$

and $0 < \varepsilon \ll 1$ perturbation parameter; the functions $a,\ b,\ f$ and φ are sufficiently smooth, l-periodic, and $a(x) \geq \alpha > 0$, $b^* \geq b(x,t) > 0$.

This study presents numerical solutions for partial differential equations with a second derivative with respect to time in the highest order term and small parameters in that term. These equations are commonly found in mathematical physics and fluid mechanics and are used in various fields such as transmission lines, electron plasma waves, and ion-acoustic waves in plasmas [14].

Previous research has looked at problems similar to the one we are investigating in regular classical difference schemes. In this study, we are focusing on the singular-perturbed version of the problem, where small parameters affect the coefficient of the higher-order derivative. One unique aspect of this issue is that when ? is small, the solution changes quickly around boundary points in x and the derivatives of the solution become unbounded. As a result, the traditional difference scheme is not effective with a uniform mesh, as the approximate solution deviates from the exact solution as the steps in the schema decrease [12]. In this study, a three-level difference scheme is introduced for the problem under investigation. This scheme was developed utilizing linear basis functions, interpolation quadrature formulas with integral terms, and the weight function as outlined by Duru and Gunes (2022). The stability and convergence criteria of the proposed difference scheme were analyzed, and the convergence speed was assessed for each scenario.The research also delves into the existence, uniqueness, and smoothness of the exact solutions to similar problems. Moreover, numerous mathematicians have investigated the presence, distinctiveness, and regularity of the precise solution to such issues [17]. Our main objective is to develop a reliable and stable finite difference scheme for addressing problems (1)–(5), incorporating interpolating quadrature rules and linear basis functions in the construction process.

2 Asymptotic Estimates

Lemma 1 *The following estimation is true for the solution* $u(x,t)$ *of the problems* (1)–(5)

$$\left\|\frac{\partial^{k+s}u}{\partial t^k\partial x^s}\right\| \leq C\{\varepsilon^{-\frac{s}{2}}\left[\|f\|_{L_2(D)}+\|\varphi\|+\varepsilon\|\varphi'\|+\|\psi\|+\varepsilon\|\psi'\|\right]+$$

$$+s(s-1)\left[\|\varphi''\|+\|\psi''\|\right]\},\quad k,s=0,1,2. \tag{6}$$

Proof. Multiplying both sides of the differential equation (1) as a scaler with $\frac{\partial u}{\partial t}$, we have

$$\left\|\frac{\partial u}{\partial t}\right\|^2 \leq \Psi_* e^{c_* t}+\int\limits_0^t\left\{\alpha^{-1}(b^*\|u\|^2+\|f\|^2)e^{c_*(t-s)}\right\}ds, \tag{7}$$

where $\Psi_*=\alpha^{-1}\left[\varepsilon\|\psi'\|^2+(a(x)\psi,\psi)+\varepsilon\|\varphi'\|^2\right]$, $c_*=\alpha^{-1}(1+b^*)$.

In (7), using the following inequality for the arbitrary function $\vartheta(t)\in C^1$

$$\frac{1}{2T}\vartheta^2(t)-\frac{1}{T}\vartheta^2(0)\leq\int\limits_0^t|\vartheta'(s)|^2ds \tag{8}$$

and by taking $\xi\in(0,t)$ instead of t by integrating we get

$$\|u\|^2\leq C_1\left[\|\varphi\|^2+\varepsilon\|\varphi'\|^2+\|\psi\|^2+\varepsilon\|\psi'\|^2\right]+C_2\int\limits_0^t\left[\|u(s)\|^2+\|f(s)\|^2\right]ds.$$

From integral inequality

$$\|u\|^2 \leq C\left(\|f\|^2_{L_2(D)} + \|\varphi\|^2 + \varepsilon\|\varphi'\|^2 + \|\psi\|^2 + \varepsilon\|\psi'\|^2\right).$$

This is proof of Lemma 1 for the case $k = s = 0$. Lemma 1 is true for the case $k = 1,\ s = 0$. From this inequality and (7), the estimate (6) is

$$\varepsilon\left\|\frac{\partial^2 u}{\partial t \partial x}\right\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2 + \varepsilon\left\|\frac{\partial u}{\partial x}\right\|^2 \leq$$

$$\leq C\left\{\|f\|^2_{L_2(D)} + \|\phi\|^2 + \varepsilon\|\phi'\|^2 + \|\Psi\|^2 + \varepsilon\|\Psi'\|^2\right\}.$$

Thus, the lemma is proved for the cases $k = s = 1$ and $k = 0,\ s = 1$.

Now, if both sides of equation (1) are multiplied by $\dfrac{\partial^2 u}{\partial t^2}$ as the scalar, the following inequality is obtained

$$\varepsilon\left\|\frac{\partial^3 u}{\partial t^2 \partial x}\right\|^2 + \alpha\left\|\frac{\partial^2 u}{\partial t^2}\right\|^2 \leq C\left(\varepsilon\left\|\frac{\partial u}{\partial x}\right\|^2 + \|u\|^2 + \|f\|^2\right).$$

From the results for $\|u\|$ and $\left\|\dfrac{\partial u}{\partial x}\right\|$, Lemma 1 is proved for the cases $k = 2,\ s = 0$, and $k = 2,\ s = 1$.

Similarly, the other cases are proved by multiplying the differential equation (1) by $\dfrac{\partial^4 u}{\partial x^2 \partial t^2}$ as a scalar.

3 The Finite Difference Scheme

In this section the finite difference scheme is constructed by using interpolating quadrature rules We use the interpolation quadrature rules when constructing the difference scheme [2].Now we give the node points of Bakhvalov mesh.

3.1 Bakhvalov Mesh

In this subsection, adaptive mesh points are presented. For these points, the mesh generation function that Bakhvalov 1969, mentioned in his paper is used.

Let ω denote the mesh on D, where $\omega = \omega_N \times \omega_\tau$

$$\omega_N = \left\{x_i = ih,\ i = 1, 2, \ldots, N-1;\ h_i = x_i - x_{i-1}\right\},$$

$$\omega_\tau = \left\{t_j = j\tau,\ j = 1, 2, \ldots, M;\ \tau = \frac{T}{M}\right\}, \omega_N^+ = \omega_N \cup \{x = 0, l\},\ \overline{\omega}_\tau = \omega_\tau \cup \{t = 0\}.$$

Let the mesh function v be defined on ω_N. The notations are as the following

$$v_x = \frac{v_{i+1} - v_i}{h_{i+1}},\ v_{\overline{x}} = \frac{v_i - v_{i-1}}{h_i},\ v_{\hat{x}} = \frac{v_{i+1} - v_i}{\hbar_i},\ v_{\overline{x}\hat{x}} = \frac{v_x - v_{\overline{x}}}{\hbar_i}.$$

Let the function $g(t)$ be defined on mesh ω_τ. Then the formulas are the following:

$$g_t = \frac{g_{j+1} - g_i}{\tau},\ g_{t} = \frac{g_j - g_{j-1}}{\tau},\ g_{\bar{t}t} = \frac{g_{j+1} - 2g_j + g_{j-1}}{\tau^2} \quad \text{(Samarskii, 2001)}.$$

Bakhvalov mesh points (Boglaev 1984; Boglaev, 2006) are as the following

$$x_i = \begin{cases} -\alpha^{-1}\varepsilon \ln\Big(1 - \Big(1-\varepsilon\Big)\dfrac{4i}{N}\Big), \quad I = 0,1,\dots,\ \dfrac{N}{4},\ x_i \in [0,\sigma_1], \ \text{if} \ \sigma_1 < \dfrac{l}{4}; \\ -\alpha^{-1}\varepsilon \ln\Big(1 - \Big(1-e^{-\dfrac{\alpha l}{4\varepsilon}}\Big)\dfrac{4i}{N}\Big), \quad i = 0,1,\dots,\ \dfrac{N}{4},\ x_i \in [0,\sigma_1], \ \text{if} \ \sigma_1 = \dfrac{l}{4}; \\ \sigma_1 + \Big(i - \Big(\dfrac{N}{4}\Big)\Big)h^{(1)}, \quad i = \dfrac{N}{4}+1,\dots,\dfrac{3N}{4}, \quad x_i \in [\sigma_1,\sigma_2],\ h^{(1)} = \dfrac{2(\sigma_2-\sigma_1)}{N}; \\ \sigma_2 - \alpha^{-1}\varepsilon \ln\Big(1 - \Big(1-\varepsilon\Big)\Big(\dfrac{4\Big(i-\dfrac{3N}{4}\Big)}{N}\Big)\Big), \quad i = \dfrac{3N}{4}+1,\dots,N, \quad x_i \in [\sigma_2, l]; \\ \sigma_2 - \alpha^{-1}\varepsilon \ln\Big(1 - \Big(1-e^{-\dfrac{\alpha l}{4\varepsilon}}\Big)\Big(\dfrac{4\Big(i-\dfrac{3N}{4}\Big)}{N}\Big)\Big), \quad i = \dfrac{3N}{4}+1,\dots,N, \quad x_i \in [\sigma_2, l], \quad \sigma_2 = \dfrac{3l}{4}, \end{cases}$$

where

$$\sigma_1 = \min\left\{\frac{l}{4}, -\alpha^{-1}\varepsilon\ln\varepsilon\right\}, \sigma_2 = l - \sigma_1.$$

The approach of generating difference method is through the integral identity:

$$\tau^{-1}\int\limits_{t_{j-1}}^{t_{j+1}}\left[\hbar_i^{-1}\int\limits_{x_{i-1}}^{x_{i+1}} L_1\left[\frac{\partial^2 u}{\partial t^2}\right]\varphi_i(x)dx + \hbar_i^{-1}\int\limits_{x_{j-1}}^{x_{j+1}} L_2[u]\varphi_i(x)dx\right]\chi_j(t)dt =$$

$$= \tau^{-1}\int\limits_{t_{j-1}}^{t_{j+1}}\left[\hbar_i^{-1}\left[\int\limits_{x_{j-1}}^{x_{j+1}} f(x,t)\varphi_i(x)dx\right]\right]\chi_j(t)dt. \tag{9}$$

where the basis functions

$$\varphi_i(x) = \begin{cases} \varphi_i^{(1)}(x) \equiv \dfrac{(x - x_{i-1})}{h_i}, & x \in [x_{i-1}, x_i], \\ \varphi_i^{(2)}(x) \equiv \dfrac{(x_{i+1} - x)}{h_{i+1}}, & x \in [x_i, x_{i+1}], \\ 0, & x \notin (x_{i-1}, x_{i+1}), \end{cases}$$

and

$$\chi_j(t) = \begin{cases} \chi_j^{(1)}(t) \equiv \dfrac{(t - t_{i-1})}{\tau}, & t \in [t_{j-1}, t_j], \\ \chi_j^{(2)}(t) \equiv \dfrac{(t_{i+1} - t)}{\tau}, & t \in [t_j, t_{j+1}], \\ 0, & t \notin (t_{j-1}, t_{j+1}). \end{cases}$$

Applying interpolating quadrature rules in [1], we find:

$$\ell u_i^j = \ell_1(u^j_{\bar{t}t,i}) + \ell_2(u_i^j) + R_j^i = f_i^j,$$

where

$$\ell_1(u^j_{\bar{t}t,i}) = -\varepsilon u^j_{\bar{t}t\overline{x}\hat{x},i} + a_i u^j_{\bar{t}t}, \quad \ell_2(u_i^j) = -\varepsilon u_{\overline{x}\hat{x},i} + b_i^j u_i^j.$$

Here the remainder terms are denoted by

$$R_j^i = \varepsilon (R^{(0)})_x + R^{(1)},$$

where

$$R^{(0)} = R_1^{(0)}, \quad R^{(1)} = R^* + R_1^{(1)} - R_2,$$

$$R^* = \tau^{-1} \int\limits_{t_{j-1}}^{t_{j+1}} R_i^* \chi_j(t) dt, \quad R_i^* = R_1^*(t) + R_2^*(t) - R_3^*(t),$$

and

$$R_1^{(0)} = \tau^{-1} \int\limits_{t_{j-1}}^{t_{j+1}} d\eta \left[\frac{\partial^2 u(x_i,\eta)}{\partial t^2} \right]_x \left[\int\limits_{\eta}^{t_{j+1}} T_1(t-\eta)\chi_j(t)dt - T_1(t_j-\eta) \right],$$

$$R_1^{(1)} = \tau^{-1} \int\limits_{t_{j-1}}^{t_{j+1}} d\eta b(x_i,\eta) \left[\frac{\partial^2 u(x_i,\eta)}{\partial t^2} \right]_x \left[\int\limits_{\eta}^{t_{j+1}} T_1(t-\eta)\chi_j(t)dt - T_1(t_j-\eta) \right],$$

$$R_2 = \tau^{-1} \int\limits_{t_{j-1}}^{t_{j+1}} d\eta \left[\frac{\partial^2 f(x_i,\eta)}{\partial t^2} \right]_x \left[\int\limits_{\eta}^{t_{j+1}} T_1(t-\eta)\chi_j(t)dt - T_1(t_j-\eta) \right],$$

and in R_i^* we have

$$R_{1,i}^*(t) = \hbar_i^{-1} \int\limits_{x_{i-1}}^{x_{j+1}} [a(x) - a(x_i)] \frac{\partial u}{\partial t} \varphi_i(x) dx$$

$$R_{2,i}^*(t) = \left[h^{-1} \int\limits_{x_{i-1}}^{x_{i+1}} L_2[u] \varphi_i(x) dx - (-\varepsilon u_{\overline{x}\hat{x},i} + b(x_i,t)u(x_i,t)) \right],$$

$$R_{3,i}^*(t) = \hbar_i^{-1} \int\limits_{x_{j-1}}^{x_{j+1}} [f(x,t) - f(x_i,t)] \varphi_i(x) dx.$$

Then, it follows that

$$\ell u \equiv -\varepsilon u^j_{\bar{t}t\overline{x}\hat{x},i} + a_i u^j_{\bar{t}t} - \varepsilon u_{\overline{x}\hat{x},i} + b_i^j u_i^j + R = f_i^j, \quad (x,t) \in \omega. \tag{10}$$

For the initial condition (3), the following relation is written:

$$\hbar_i^{-1}\tau^{-1}\int_{t_0}^{t_1}\int_{x_{i-1}}^{x_{i+1}}(Lu-f)\varphi_i(x)\chi_0^{(2)}(t)dxdt=0,\quad x_i\in\omega.$$

From here, we get

$$-\varepsilon u^0_{t\overline{x}\hat{x}}+a_iu^0_t+r=\phi,\quad \phi=-\varepsilon\psi_{\overline{x}\hat{x}}+a_i\psi_i+\frac{\tau}{2}\varepsilon\varphi_{\overline{x}\hat{x}}-\frac{\tau}{2}b_i^0\varphi_i+\frac{\tau}{2}f_i^0 \tag{11}$$

with the remainder term

$$r=-\varepsilon(r^{(0)})_x+r^{(1)},$$

that

$$r^{(0)}=\int_{t_0}^{t_1}d\eta\left(\frac{\partial^2 f(x_i,\eta)}{\partial t^2}\right)_x\left[\int_{\eta}^{t_1}T_1(t-\eta)\chi_0^{(2)}(t)dt-T_1(t_j-\eta)\right],$$

$$r^{(1)}=\int_{t_0}^{t_1}R_i^*(t)\chi_0^{(2)}(t)dt$$

$$+\int_{t_0}^{t_1}d\eta b(x_i,\eta)\frac{\partial^2 u(x_i,\eta)}{\partial t^2}\left[\int_{\eta}^{t_1}T_1(t-\eta)\chi_0^{(2)}(t)dt-T_1(t_j-\eta)\right]$$

$$-\int_{t_0}^{t_1}d\eta\frac{\partial^2 f(x_i,\eta)}{\partial t^2}\left[\int_{\eta}^{t_1}T_1(t-\eta)\chi_0^{(2)}(t)dt-T_1(t_j-\eta)\right], \tag{12}$$

where the basis function $\chi_0(t)$ is given by

$$\chi_0(t)=\begin{cases}\chi_0^{(2)}(t)\equiv\dfrac{t_1-t}{\tau}, & t\in(t_0,t_1),\\ 0, & t\notin(t_0,t_1).\end{cases}$$

For the periodical boundary conditions, we use the integral identity as the form

$$\hbar_i^{-1}\tau^{-1}\int_{t_{j-1}}^{t_{j+1}}\int_{x_0}^{x_1}(Lu-f)\varphi_0(x)\chi_j(t)dxdt=0,\quad t\in\omega_\tau.$$

From here, we analogously find:

$$-\varepsilon u^j_{\overline{t}t\overline{x}\hat{x},N}+a_0u^j_{\overline{t}t,N}-\varepsilon u^j_{\overline{x}\hat{x},N}+b_0^ju_i^j+r^*=f_0^j,\quad t\in\omega_\tau. \tag{13}$$

Here, the remainder term is shown that

$$r^* = -\varepsilon r^{*(0)} + r^{*(1)}$$

where

$$r^{*(0)} = \tau^{-1} \int_{t_{j-1}}^{t_{j+1}} d\eta \left(\frac{\partial^2 f(x_i, \eta)}{\partial t^2} \right)_{x,0}$$

$$\left[\int_{\eta}^{t_1} T_1(t-\eta)\chi_j(t)dt - T_1(t_j - \eta) \right],$$

and

$$r^{*(1)} = \tau^{-1} \int_{t_{j-1}}^{t_{j+1}} r_i^*(t)\chi_j(t)dt + {r_1}^{*(1)} - {r_2}^*.$$

Here, we can write the ${r_1}^{*(1)}$ and ${r_2}^*$ terms in the last the remainder term as follows

$${r_1}^{*(1)} = \tau^{-1}\hbar_i^{-1} \int_{t_{j-1}}^{t_{j+1}} d\eta b(x_0, \eta) \frac{\partial^2 u(0, \eta)}{\partial t^2}$$

$$\left[\int_{\eta}^{t_1} T_1(t-\eta)\chi_j(t)dt - T_1(t_j - \eta) \right],$$

$${r_2}^* = \tau^{-1}\hbar_i^{-1} \int_{t_{j-1}}^{t_{j+1}} d\eta \frac{\partial^2}{\partial t^2} f_0(\eta)$$

$$\left[\int_{\eta}^{t_1} T_1(t-\eta)\chi_j(t)dt - T_1(t_j - \eta) \right],$$

$$\tau^{-1} \int_{t_{j-1}}^{t_j} r_i^*(t)\chi_j(t)dt = \tau^{-1} \int_{t_{j-1}}^{t_{j+1}} [r_{1,i}^*(t) + r_{2,i}^*(t) - r_{3,i}^*(t)]\chi_j(t)dt,$$

where

$$r_{1,i}^*(t) = \int_{x_0}^{x_1} [a(x) - a_0] \frac{\partial^2 u}{\partial t^2} {\varphi_0}^{(2)}(x)dx,$$

$$r^*_{2,i}(t) = \int\limits_{x_0}^{x_1} L_2 u\varphi_0^{(2)}(x)dx - l_2^{(1)}[u],$$

$$r^*_{3,i}(t) = \int\limits_{x_0}^{x_1} [f(x,t) - f_0]{\varphi_0}^{(2)}(x)dx$$

and the basis function $\varphi_0(x)$ is defined by

$$\varphi_0(x) = \begin{cases} \varphi_0^{(2)}(x) \equiv \dfrac{(x_0 - x)}{h_1}, & x_0 < x < x_1, \\ \varphi_N^{(1)}(x) \equiv \dfrac{(x - x_{N-1})}{h_N}, & x_{N-1} < x < x_N, \\ 0, & x \notin (x_0, x_1) \cup (x_{N-1}, x_N). \end{cases}$$

Based on the relations (10), (11) and (13), we propose the following difference scheme for the approximating (1)–(4):

$$\ell y \equiv -\varepsilon y^j_{\bar{t}t\bar{x}\hat{x},i} + a_i y^j_{\bar{t}t} - \varepsilon y_{\bar{x}\hat{x},i} + b^j_i y^j_i = f^j_i, \quad (x,t) \in \omega \tag{14}$$

$$y(x,0) = \varphi(x), \quad x \in \omega_N, \tag{15}$$

$$\ell^{(0)} y \equiv -\varepsilon y^0_{\bar{t}\bar{x}\hat{x}} + a_i y^0_t = \phi \tag{16}$$

$$y(0,t) = y(l,t), \quad y(h_1,t) = y(l+h_N,t), \quad t \in \omega_\tau, \tag{17}$$

$$\ell^{(1)} y \equiv -\varepsilon y^j_{\bar{t}t\bar{x}\hat{x},N} + a_0 y^j_{\bar{t}t,N} - \varepsilon y^j_{\bar{x}\hat{x},N} + b^j_0 y^j_i = f^j_0, \quad t \in \omega_\tau, \tag{18}$$

where

$$\phi = -\varepsilon\psi_{\bar{x}\hat{x}} + a_i\psi_i + \frac{\tau}{2}\varepsilon\varphi_{\bar{x}\hat{x}} - \frac{\tau}{2}b^0_i\varphi_i + \frac{\tau}{2}f^0_i, \quad i = 1,2,\ldots,N; \quad j = 1,2,\ldots,M.$$

4 Error Analysis

Let u be the solution of (1)–(5) and y be the solution of (14)–(18). The error function $z = y - u$ is a solution to the following discrete problem:

$$\ell_1(z^j_{\bar{t},i}) + \ell_2(z^j_i) = R^i_j. \tag{19}$$

$$z(0,t_j) = 0, \quad 0 \le i \le N, \tag{20}$$

$$z(0,t)=z(l,t),\quad z(h_1,t)=z(l+h_N,t),\quad t\in\omega_\tau, \tag{21}$$

$$\ell^{(0)}z\equiv-\varepsilon z^0_{\bar{t}t\bar{x}\hat{x}}+a_iyz^0_t=r \tag{22}$$

$$\ell^{(1)}z\equiv-\varepsilon z^j_{\bar{t}t\bar{x}\hat{x},N}+a_0z^j_{\bar{t}\bar{t},N}-\varepsilon z^j_{\bar{x}\hat{x},N}+b^j_0yz^j_i=r^*,\quad t\in\omega_\tau, \tag{23}$$

Lemma 2 *Under the conditions:* $C_0\tau<1\left(C_0=\max\left(\frac{b^*+1}{2\alpha},\gamma_*^{-1}\right)\right)$, $1-\frac{\tau^2}{2}\geq\gamma_*>0$, *the following estimates are satisfied for the solution of the problem* (19)–(23)

$$\|z_t\|+\varepsilon\|z_{t\hat{x}}\|\leq C(N^{-2}+\tau^2). \tag{24}$$

See Duru, 2004 for proof.

5 Numerical Results

Let's write the problem (14)–(18) explicitly:

$$A_1y^{j+1}_N-C_1y^{j+1}_1+B_1y^{j+1}_2=-F_1,\quad i=1,\ldots,N-1$$

$$A_iy^{j+1}_{i+1}-C_iy^{j+1}_i+B_iy^{j+1}_{i-1}=-F_i,\quad i=1,\ldots,N-1;\quad j=2,\ldots,M-1 \tag{25}$$

$$A_Ny^{j+1}_{N-1}-C_Ny^{j+1}_N+B_Ny^{j+1}_1=-F_N,$$

$$A^*_1y^1_{i-1}-C^*_1y^1_i+B^*_1y^1_{i+1}=-F^*_i,\quad i=1,\ldots,N-1,$$

where

$$A_i=-\varepsilon\tau^{-2}\hbar_i^{-1}h_i^{-1},$$

$$B_i=-\varepsilon\tau^{-2}\hbar_i^{-1}h_{i+1}^{-1},$$

$$C_i=-\varepsilon\hbar_i^{-1}\tau^{-2}\left(h_{i+1}^{-1}+h_i^{-1}\right)-a_i\tau^{-2},$$

$$F_i=-f^j_i+\left(b^j_i-2a_i\tau^{-2}\right)y^j_i+a_i\tau^{-2}y^{j-1}_i+\varepsilon\left(2\tau^{-2}-1\right)y^j_{\bar{x}\hat{x},i}-\varepsilon\tau^{-2}y^{j-1}_{\bar{x}\hat{x},i}$$

$$A^*_1=-\varepsilon\tau^{-1}\hbar_i^{-1}h_i^{-1},$$

$$B^*_1=-\varepsilon\tau^{-1}\hbar_i^{-1}h_{i+1}^{-1},$$

$$C^*_1=-\varepsilon\hbar_i^{-1}\tau^{-1}\left(h_{i+1}^{-1}+h_i^{-1}\right)-a_i\tau^{-1},$$

$$F^*_1=\varepsilon\psi_{\bar{x}\hat{x}}-a_i\psi_i-\varepsilon\left(\frac{\tau}{2}-\tau^{-1}\right)\varphi_{\bar{x}\hat{x}}+\left(\frac{\tau}{2}-a_i\tau^{-1}\right)b^0_i\varphi_i-\frac{\tau}{2}f^0_i,$$

$$A_N = -\varepsilon\tau^{-2}\hbar_0^{-1}h_0^{-1},$$
$$B_N = -\varepsilon\tau^{-2}\hbar_0^{-1}h_1^{-1},$$
$$C_N = -\varepsilon\hbar_0^{-1}\tau^{-2}\left(h_1^{-1}+h_0^{-1}\right) - a_N\tau^{-2},$$
$$F_N = -f_0^j + (b_0^j - 2a_0\tau^{-2})y_N^j + a_i\tau^{-2}y_N^{j-1} + \varepsilon(2\tau^{-2}-1)y_{\overline{x}\hat{x},N}^j - \varepsilon\tau^{-2}y_{\overline{x}\hat{x},N}^{j-1}.$$

The linear equation system (25) will be solved by the elimination algorithm given below [16]. For these coefficients, the elimination algorithm is defined as follows

$$\alpha_2 = \frac{B_1}{C_1},\quad \gamma_2 = \frac{A_1}{C_1},\quad \beta_2 = \frac{F_1}{C_1},\quad C_i - \alpha_i A_i \neq 0,$$

$$\alpha_{i+1} = \frac{B_i}{C_i - \alpha_i A_i},\quad \beta_{i+1} = \frac{F_i + A_i\beta_i}{C_i - \alpha_i A_i},\quad \gamma_{i+1} = \frac{A_i\gamma_i}{C_i - \alpha_i A_i}\quad i = 1,\ldots,N-1,$$
$$p_{N-1} = \beta_N,\quad q_{N-1} = \beta_N + \gamma_N,$$
$$p_i = \alpha_{i+1}p_{i+1} + \beta_{i+1},\quad q_i = \alpha_{i+1}q_{i+1} + \gamma_{i+1},\quad i = N-1, dots, 1,$$
$$y_N = \frac{\beta_{N+1} + \alpha_{N+1}p_1}{1 - \alpha_{N+1}q_1 - \gamma_{N+1}},$$
$$y_i = p_i + y_N q_i,\ i = N-1,\ldots,0.$$

To test the order of uniform convergence of the samples, we define the absolute errors and the convergence rate as follows

$$e^N = \max_{0\le i<N}\left|y_i^N - y_{2i}^{2N}\right|$$

and

$$p = \frac{\ln\left(e^N/e^{2N}\right)}{\ln 2}.$$

Example 1. Consider the following periodic problem on $D = (0,1)\times(0,1]$

$$-\varepsilon\frac{\partial^4 u}{\partial t^2\partial x^2} + (1+e^{\sin(2\pi x)})\frac{\partial^2 u}{\partial t^2} - \varepsilon\frac{\partial^2 u}{\partial x^2} + (t+\sin(2\pi x))u = e^{-t}\sin(t)(e^{\sin(2\pi x)} + \sin(2\pi x) + t + 1)$$
$$u(x,0) = \sin(2\pi x),$$
$$\frac{\partial u}{\partial t}(x,0) = -\sin(2\pi x),$$
$$u(0,t) = u(1,t),$$
$$\frac{\partial u}{\partial x}(x,0) - \frac{\partial u}{\partial x}(x,1) = 0.$$

The obtained results are given in Table 1.

Table 1. Maximum point-wise errors and the order of convergence rate for $N = 8$ and $M = 10$.

ε	r_0	p_1	r_1	p_2	r_2	$\overline{p}$
2^{-3}	0.09348526	1.7747	0.02732148	1.9396	0.00712245	1.8571
2^{-4}	0.09675640	1.7142	0.02948816	1.8696	0.00806897	1.7919
2^{-5}	0.09331897	1.7085	0.02855243	1.8846	0.00773220	1.7965
2^{-6}	0.09402811	1.7444	0.0280626	1.8611	0.00772448	1.8027

Example 2. Consider the problem on $D = (0,1) \times (0,1]$

$$-\varepsilon\frac{\partial^4 u}{\partial t^2 \partial x^2} + (1 + e^{\cos(2\pi x)})\frac{\partial^2 u}{\partial t^2} - \varepsilon\frac{\partial^2 u}{\partial x^2} + (t + cos(2\pi x)t)u =$$

$$= e^{-t}\sin(t)(e^{\cos(2\pi x)} + \cos(2\pi x) + t + 1)$$

$$u(x,0) = -sin(\pi x),$$

$$\frac{\partial u}{\partial t}(x,0) = -sin(\pi x),$$

$$u(0,t) = u(1,t),$$

$$\frac{\partial u}{\partial x}(x,0) - \frac{\partial u}{\partial x}(x,1) = 0.$$

The computed results are summarized in Table 2.

Table 2. Maximum point-wise errors and the order of convergence rate for N=8 and M=10.

ε	r_0	p_1	r_1	p_2	r_2	$\overline{p}$
2^{-3}	0.10025268	1.8574	0.02766639	1.9126	0.00734822	1.8850
2^{-4}	0.09669872	1.8467	0.02688412	1.9344	0.00703359	1.8905
2^{-5}	0.09337445	1.8413	0.02605667	1.9343	0.00681726	1.8878
2^{-6}	0.10048995	1.8736	0.02742225	1.9239	0.00722681	1.8987

6 Discussion and Conclusion

We suggested a new difference scheme to solve singularly perturbed equations. By using the energy inequalities, the stability of the solution to the continuous problem was shown. Difference schemes were constructed using interpolation quadrature rules with integral terms and weight functions as linear basis functions. The remainder term with integral form relieves the conditions on the solution, such as continuity. The linear basis functions are chosen in such a way that the method error of the terms with the highest order derivative is zero. Error analysis is performed in discrete norm and the convergence rate is $O(N^{-2} + \tau^2)$.

Different numerical methods can be used for the problem we are considering. Exponential fitted difference schemes and similarly established difference schemes on a piecewise regular network can be used. Each of these has advantages and disadvantages. The method we used in this study is advantageous in terms of memory savings and computational cost in the computer.

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