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SPECTRUM OF THE CESÁRO-HARDY OPERATOR IN LORENTZ $L_{p,q}(0,1)$ SPACES

The aim of this paper is to investigate the spectrum of the Cesáro-Hardy operator in Lorentz $L_{p,q}$ spaces over (0,1). In this paper we extended Leibowitz's results for L_p space to Lorentz spaces. Note that L_p space is a special case of Lorentz spaces when indexes p and q coincide. Interestingly, we obtained the same results as for L_p space. The point spectrum is obtained by solving an Euler differential equation of first order. We used the operator P_ξ to find the resolvent set of the Cesáro-Hardy operator. This operator was defined in Boyd's work in [1]. Boundedness of the operator P_ξ on L_p was proved in the same paper. But its boundedness on $L_{p,q}$ was proved in this paper by using $L_{p,q}$ norm of the dilation operator. Here, we also used the Boyd's theorem, which describes boundedness of operators on rearrangement invariant spaces. We verified conditions of Boyd's theorem. It allows us to obtain a bounded inverse of the operator $\lambda I - C$ for some complex numbers λ.

Keywords: Cesáro-Hardy operator, spectrum, point spectrum, Lorentz $L_{p,q}$ spaces, rearrangement invariant spaces.

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Лоренц $L_{p,q}(0,1)$ кеңістіктерінде анықталған Чезаро-Харди операторының спектрі

Бұл мақаланың мақсаты Чезаро-Харди операторының $(0,1)$ аралығында анықталған Лоренц $L_{p,q}$ кеңістіктеріндегі спекртін зерттеу болып табылады. Бұл мақалада біз Лейбоуицтің L_p кеңістігі үшін алған нәтижелерін Лоренц кеңістіктеріне кеңейттік. Лебег L_p кеңістігі p мен q индекстері тең болған кездегі Лоренц кеңістіктерінің дербес жағдайы екенін айта кеткен жөн. Біздің қол жеткізген нәтижеміз L_p кеңістігі үшін алынған нәтижемен бірдей болды. Нүктелік спектр Эйлердің бірінші ретті дифференциалдық теңдеуін шешу арқылы табылды. Чезаро-Харди операторының резольвент жиынын анықтау үшін P_ξ операторын қолдандық. Бұл оператор Бойдтың [1] жұмысында анықталаған. ξ комплекс санының кейбір мәндерінде P_ξ операторының L_p кеңістігінде шенелгендігін де дәл сол мақаладан көруге болады. Алайда бұл оператордың $L_{p,q}$ кеңістіктерінде шенелгендігі осы мақалада дәлелденді. Ол үшін біз E_s кеңейту операторының $L_{p,q}$ кеңістіктерін өз-өзіне бейнелеген кездегі нормасын қолдандық. Сонымен қатар, Бойдтың операторлардың инвариантты қайта реттеу кеңістіктеріндегі шенелгендігін сипаттайтын теоремасын қолдандық. Біз Бойд теоремасының шарттарын тексердік, бұл бізге λ комплекс санының кейбір мәндері үшін $\lambda I - C$ операторының шенелген кері операторын табуға мүмкіндік береді.

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Спектр оператора Чезаро-Харди в пространствах Лоренца $L_{p,q}(0,1)$

Целью данной статьи является исследование спектра оператора Чезаро-Харди в пространствах Лоренца $L_{p,q}$ на интервале $(0,1)$. В этой статье мы расширили результаты Лейбовица для пространства L_p на пространства Лоренца. Отметим, что пространство L_p является частным случаем пространств Лоренца, когда индексы p и q совпадают. Что интересно, мы получили те же результаты, что и для пространства L_p. Точечный спектр получается путем решения дифференциального уравнения Эйлера первого порядка. Для нахождения резольвентного множества оператора Чезаро-Харди мы использовали оператор P_ξ. Этот оператор был определен в работе Бойда в [1]. В этой же работе Бойда доказана ограниченность оператора P_ξ на L_p для некоторых значении комплексного числа ξ. Но его ограниченность на $L_{p,q}$ была доказана в этой статье с помощью $L_{p,q}$ нормы оператора растяжения E_s. Здесь мы также использовали теорему Бойда, описывающую ограниченность операторов в инвариантных перестановочных пространствах. Мы проверили условия теоремы Бойда. Он позволяет нам получить ограниченный обратный оператор $(\lambda I - C)^{-1}$ для некоторых значении комплексного числа λ.

Ключевые слова: оператор Чезаро-Харди, спектр, точечный спектр, пространства Лоренца $L_{p,q}$, перестановочные инвариантные пространства.

1 Introduction

Let $1 < p < \infty$, $1 \leq q \leq \infty$. Define Cesáro-Hardy operator C on the space $L_{p,q}(0;1)$ by

$$Cf(t) := \frac{1}{t}\int_0^t f(s)\,ds, \quad t \in (0,1) \tag{1}$$

Boundedness of C acting on $L_{p,q}(0,\infty)$ can be shown directly by Hardy's inequality [2, Lemma II 3.9]. Let us show its bondedness on $L_{p,q}(0,1)$. Indeed, if $f \in L_{p,q}(0,\infty)$ and $\tilde{f}(t) = f(t)$ $(0 < t < 1)$, $\tilde{f}(t) = 0$ $(1 \leq t < \infty)$, then

$$Cf\|_{L_{p,q}(0,1)} = \|C\tilde{f}\|_{L_{p,q}(0,\infty)} \leq \frac{p}{p-1}\|\tilde{f}\|_{L_{p,q}(0,\infty)} = \|f\|_{L_{p,q}(0,1)}.$$

It follows from the general theory of bounded linear operators on Banach spaces that the spectrum of C is non-empty set. The aim of this paper is to determine the spectrum and the point spectrum of the Cesáro-Hardy operator in $L_{p,q}(0,1)$ spaces. The spectrum and the point spectrum of the Cesáro-Hardy operator in $L_2(0,\infty)$ and $L_2(0,1)$ was shown by Brown, Halmos and Shields in [3]. Boyd extended their results to $L_p(0,\infty)$ $(1 < p \leq \infty)$ [1]. Later, Leibowitz investigated the spectrum and the point spectrum of the Cesáro-Hardy operator in Lebesgue L_p $(1 < p \leq \infty)$ spaces over $(0,1)$ [4]. The spectrum and the point spectrum of the Cesáro-Hardy operator in $C(0,\infty)$, $C^\infty(0,\infty)$ and $L_{p,loc}(0,\infty)$ were studied in [5]. About the spectrum and the point spectrum of the Cesáro-Hardy operator in the ultradifferentiable function spaces can be found from [6]. Boyd described the spectrum of the Cesáro-Hardy operator in rearrangement invariant spaces in [1], but its proof was not found. Therefore, we proved the spectrum of the Cesáro-Hardy operator in Lorentz $L_{p,q}(0,1)$ spaces ourselves and decided to publish our results.

2 Preliminaries

In this section we present some notions and notations from the general theory of linear bounded operators on Banach spaces.

Definition 1 *Let T be a bounded linear operator mapping a Banach space X into itself. The spectrum of T (will be denoted by $\sigma(T)$) is the set of all $\lambda \in \mathbb{C}$ for which the operator $\lambda I - T$ does not have an inverse that is a bounded linear operator.*

Definition 2 *[7, Definition 1.13] Let $T : X \to X$ be a bounded linear operator. Define*

$$\sigma_{pt}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \quad \textit{is not injective} \quad \Leftrightarrow \quad Ker\,(\lambda I - C) \neq \{0\}\};$$

$$\sigma_c(T) = \left\{\lambda \in \mathbb{C} : \lambda I - T \quad \textit{is injective with} \quad \overline{Im\,(\lambda I - T)} = X,\ \textit{but} \quad Im\,(\lambda I - T) \neq X\right\};$$

$$\sigma_r(T) = \left\{\lambda \in \mathbb{C} : \lambda I - T \quad \textit{is injective, but} \quad \overline{Im\,(\lambda I - T)} \neq X\right\}.$$

$\sigma_{pt}(T)$, $\sigma_c(T)$ and $\sigma_r(T)$ are called respectively the point spectrum, the continuous spectrum and the residual spectrum of T in X. Clearly $\sigma_{pt}(T)$, $\sigma_c(T)$ and $\sigma_r(T)$ are disjoint and

$$\sigma(T) = \sigma_{pt}(T) \cup \sigma_c(T) \cup \sigma_r(T).$$

Definition 3 *The resolvent set $\rho(T)$ of T in X is defined to be $\mathbb{C} \backslash \sigma(T)$ or in other words, the resolvent set of T is the set of all $\lambda \in \mathbb{C}$ for which the operator $\lambda I - T$ have an inverse that is a bounded linear operator.*

Definition 4 *[8, Definition 2.1] The function $\mu_f : [0, \infty] \to [0, \infty]$ of function f defined by*

$$\mu_f(\lambda) = \mu\,\{x \in \mathbb{R} : |f(t)| > \lambda\}, \quad (\lambda \geq 0)$$

is called the distribution function of f.

Definition 5 *[8, Definition 2.2] The decreasing rearrangement of f is the function $f^* : [0, \infty) \to [0, \infty]$ defined by*

$$f^* = \inf\,\{\lambda \geq 0 : \mu_f(\lambda) \leq t\}, \quad (t \geq 0).$$

Definition 6 *Let $0 < p \leq \infty$, $0 < q \leq \infty$. Then the Lorentz $L_{p,q}(0,1)$ space is the set of all Lebesgue measurable functions f such that $\|f\|_{L_{p,q}(0,1)} < \infty$, where*

$$\|f\|_{L_{p,q}} = \begin{cases} \left(\int\limits_0^1 \left(t^{\frac{1}{p}} f^*(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} & \textit{if } 1 < p < \infty,\ 1 \leq q < \infty, \\ \sup\limits_{t>0} t^{\frac{1}{p}} f^*(t) & \textit{if } 0 < p \leq \infty,\ q = \infty. \end{cases}$$

The Lorentz $L_{p,q}$ space is the generalization of the Lebesgue L_p space. If we take $p = q$, $L_{p,q}$ coincides with L_p and

$$\|f\|_{L_{p,p}} = \|f\|_{L_p}, \quad (f \in L_p).$$

The space $L_{\infty,q}$, for finite q, is trivial, since it contains only the zero-function [8].

The linear map P_ξ is defined by

$$P_\xi f(t) = \int_0^1 s^{-\xi} f(st)\,ds. \tag{2}$$

The operator P_ξ is bounded on L_p $(1 \leq p \leq \infty)$ if and only if $Re\,(\xi) < \frac{p-1}{p}$ [1, Lemma 1].

Definition 7 *[2, Definition II 4.1] Let (Ω, Σ, μ) be a totally σ-finite measure space and let $\mathcal{M}^{+}(\Omega)$ denote the class of non-negative measurable functions on Ω.*

Let $\rho : \mathcal{M}^{+}(\Omega) \to [0, \infty]$ satisfy the following five conditions for all f, g, $f_n \in \mathcal{M}^{+}(\Omega)$, all measurable sets $E \in \Sigma$ with $\mu(E) < \infty$ and all constants $a \geq 0$

1. $\rho(f) = 0 \Leftrightarrow f = 0$ *a.e.*,
   $\rho(f + g) \leq \rho(f) + \rho(g)$, $\rho(af) = a\rho(f)$
2. $0 \leq g \leq f$ *a.e.* $\Rightarrow \rho(g) \leq \rho(f)$
3. $0 \leq f_n \uparrow f$ *a.e* $\Rightarrow \rho(f_n) \uparrow \rho(f)$
4. $\mu(E) < \infty \Rightarrow \rho(\chi_E) < \infty$
5. $\mu(E) < \infty \Rightarrow \int_E f d\mu < C_E \rho(f)$

for some constant C_E, $0 < C_E < \infty$, depending on E and ρ, but independent of f.

Then ρ is said to be rearrangement invariant if $\rho(f) = \rho(g)$ for every equimeasurable functions f and g in $\mathcal{M}^{+}(\Omega)$. In this case, the Banach function space $Y = Y(\rho)$ generated by ρ is said to be a rearrangement invariant space.

Throughout this paper we study the case when $\Omega = (0, 1)$.

The Lorentz $L_{p,q}(0, 1)$ space is an example of rearrangement invariant spaces [2].

Let Y be a rearrangement invariant space. The dilation operator $E_s : Y \to Y$ for $0 < s < \infty$ is defined by

$$(E_s f)(t) = f(st). \tag{3}$$

Let $h(s; Y)$ denote the norm of (3) as a mapping from Y into itself [9].

The operator norm of E_s from $L_{p,q}(0, 1)$ to $L_{p,q}(0, 1)$ is $h(s; L_{p,q}(0, 1)) = s^{-\frac{1}{p}}$. The $h(s; L_{p,\infty}(0, 1))$ norm for $q = \infty$ is also equal to $s^{-\frac{1}{p}}$. [2, Theorem IV 4.3].

3 Spectrum of the Cesáro-Hardy operator in $L_{p,q}(0, 1)$

In this section, we find the spectrum of the Cesáro-Hardy operator in Lorentz $L_{p,q}(0, 1)$ space. The following theorem is the main result of this section.

Theorem 1 *Let $1 < p < \infty$ and $1 \leq q < \infty$ and let $C : L_{p,q}(0, 1) \to L_{p,q}(0, 1)$ be the Cesáro-Hardy operator defined by (1). Then the spectrum of the operator C in $L_{p,q}(0, 1)$ is the set*

$$\sigma(C) = \left\{ \lambda \in \mathbb{C} : Re\left(\frac{1}{\lambda}\right) \geq \frac{p-1}{p} \right\}$$

or equivalently

$$\sigma(C) = \left\{ \lambda \in \mathbb{C} : \left| \lambda - \frac{p'}{2} \right| \leq \frac{p'}{2} \right\}.$$

And the point spectrum is the set

$$\sigma_{pt}(C) = \left\{ \lambda \in \mathbb{C} : Re\left(\frac{1}{\lambda}\right) > \frac{p-1}{p} \right\}$$

or

$$\sigma_{pt}(C) = \left\{ \lambda \in \mathbb{C} : \left| \lambda - \frac{p'}{2} \right| < \frac{p'}{2} \right\},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. Every rearrangement invariant space Y on $(0,1)$ satisfies $L_\infty \subseteq Y \subseteq L_1$ (see [2, Corollary II 6.7]). If $f \in L_{p,q}(0,1)$, then $f \in L_1(0,1)$.

Let $\lambda = 0$. Since the function Cf is continuous on $(0,1)$, the range of the operator C does not coincide with the whole space $L_{p,q}(0,1)$. Hence, $\lambda = 0$ belongs to the spectrum of C.

If $\lambda \neq 0$, in this case we consider the equation $Cf = \lambda f$. By fundamental theorem of calculus, we observe that f is differentiable. Hence, we obtain

$$\lambda f'(t) + (\lambda - 1) f(t) = 0. \tag{4}$$

The solution of (4) will be $f(t) = t^{\frac{1-\lambda}{\lambda}}$. Our next aim is to find out in which cases of λ the function $f(t) = t^{\frac{1-\lambda}{\lambda}}$ belongs to $L_{p,q}(0,1)$.

Let $Re\left(\frac{1-\lambda}{\lambda}\right) \leq 0$, then the function $t^{qRe\left(\frac{1-\lambda}{\lambda}\right)}$ is non-increasing. Its decreasing rearrangement will be $\left(t^{qRe\left(\frac{1-\lambda}{\lambda}\right)}\right)^* = t^{qRe\left(\frac{1-\lambda}{\lambda}\right)}$ [8, Theorem 2.4].

$$\begin{aligned} \|f\|^q_{L_{p,q}(0;1)} &= \int_0^1 \left(t^{\frac{1}{p}} \left(t^{\frac{1-\lambda}{\lambda}} \right)^* \right)^q \frac{dt}{t} = \int_0^1 t^{\frac{q}{p}-1} \left(t^{qRe\left(\frac{1-\lambda}{\lambda}\right)} \right)^* dt \\ &= \int_0^1 t^{\frac{q}{p}-1+qRe\frac{1}{\lambda}-q}\, dt. \end{aligned}$$

Therefore, $\|f\|_{L_{p,q}(0,1)} < \infty$ whenever λ satisfies $\left\{\lambda \in \mathbb{C} : Re\left(\frac{1}{\lambda}\right) > \frac{p-1}{1}\right\} \cap \left\{\lambda \in \mathbb{C} : Re\left(\frac{1-\lambda}{\lambda}\right) \leq 0\right\}$. Or we can write this as $\left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{p'}{2}\right| < \frac{p'}{2}\right\} \cap \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{2}\right| \geq \frac{1}{2}\right\}$. Since $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, it follows that $1 < p' < \infty$, which means that this set is not empty.

Next, we consider the case when $Re\left(\frac{1-\lambda}{\lambda}\right) > 0$, then the function $t^{qRe\left(\frac{1-\lambda}{\lambda}\right)}$ is increasing. We have to find its decreasing rearrangement. Distribution function of $t^{qRe\left(\frac{1-\lambda}{\lambda}\right)}$ can be calculated by

$$\mu(u) = 1 - u^{1/qRe\left(\frac{1-\lambda}{\lambda}\right)}, \quad u \in (0,1)$$

and decreasing rearrangement becomes

$$\left(t^{qRe\left(\frac{1-\lambda}{\lambda}\right)}\right)^* = (1-t)^{qRe\left(\frac{1-\lambda}{\lambda}\right)}, \quad t \in (0,1).$$

Hence,

$$\begin{aligned} \|f\|^q_{Lp,q(0,1)} &= \int_0^1 t^{\frac{q}{p}-1} \left(t^{qRe\left(\frac{1-\lambda}{\lambda}\right)} \right)^* dt = \int_0^1 t^{\frac{q}{p}-1} (1-t)^{qRe\left(\frac{1-\lambda}{\lambda}\right)} dt \\ &= \int_0^{\frac{1}{2}} t^{\frac{q}{p}-1} (1-t)^{qRe\left(\frac{1-\lambda}{\lambda}\right)} dt + \int_{\frac{1}{2}}^1 t^{\frac{q}{p}-1} (1-t)^{qRe\left(\frac{1-\lambda}{\lambda}\right)} dt < \infty. \end{aligned}$$

The function $f(t) = t^{\frac{1-\lambda}{\lambda}} \in L_{p,q}(0,1)$ if and only if $Re\left(\frac{1}{\lambda}\right) > \frac{p-1}{p}$. Thus, the point spectrum of the operator C in $L_{p,q}(0;1)$ is the set

$$\sigma_{pt}(C) = \left\{\lambda \in \mathbb{C} : Re\left(\frac{1}{\lambda}\right) > \frac{p-1}{p}\right\} = \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{p'}{2}\right| < \frac{p'}{2}\right\}.$$

By [1, Lemma 1], the operator P_ξ defined by (2) is bounded on $L_p(0,1)$ for $1 < p < \infty$ if and only if $Re(\xi) < \frac{p-1}{p}$. Since $sh(s, L_{p,q}(0,1)) \to 0$ as $s \to 0+$ and $h(s, L_{p,q}(0,1)) \to 0$ as $s \to \infty$, it follows from [10, Corollary 1] that P_ξ is bounded on $L_{p,q}(0,1)$ if and only if $Re(\xi) < \frac{p-1}{p}$.

By Fubini's theorem, we obtain

$$\xi P_\xi C f = \xi C P_\xi f = (C - P_\xi) f. \tag{5}$$

We can see by (5) that the operator $\xi I + \xi^2 P_\xi$ for $\xi := \frac{1}{\lambda}$ is inverse of $\lambda I - C$ on $L_{p,q}(0,1)$. Thus, the resolvent set of the operator C in $L_{p,q}(0,1)$ becomes

$$\rho(C) = \left\{\lambda \in \mathbb{C} : Re\left(\frac{1}{\lambda}\right) < \frac{p-1}{p}\right\}.$$

As we know from general theory of bounded linear operators on Banach spaces that spectrum of bounded operators is closed, we observe that the spectrum of the Cesáro-Hardy operator in $L_{p,q}(0,1)$ is the set

$$\sigma(C) = \left\{\lambda \in \mathbb{C} : Re\left(\frac{1}{\lambda}\right) \geq \frac{p-1}{p}\right\}.$$

The result of the theorem can be seen from Figure 1.

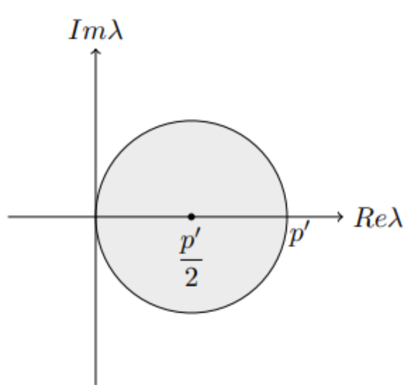

Figure 1: Spectrum of the Cesáro-Hardy operator in $L_{p,q}(0,1)$

□

The following theorem describes the spectrum and the point spectrum of the Cesáro-Hardy operator in $L_{p,\infty}(0,1)$.

Theorem 2 *Let $1 < p < \infty$ and let $C : L_{p,\infty}(0,1) \to L_{p,\infty}(0,1)$ be the Cesáro-Hardy operator. Then the spectrum of C in $L_{p,\infty}(0,1)$ is the set*

$$\sigma(C) = \left\{ \lambda \in \mathbb{C} : Re\left(\frac{1}{\lambda}\right) \geq \frac{p-1}{p} \right\}$$

and its point spectrum is

$$\sigma_{pt}(C) = \left\{ \lambda \in \mathbb{C} : Re\left(\frac{1}{\lambda}\right) \geq \frac{p-1}{p} \right\} \backslash \{0\}.$$

Proof. First, let $\lambda = 0$. The equation $Cf = 0$ has only trivial solution, which implies that $KerC = \{0\}$.

However, since the range of C contains only differentiable functions on $(0,1)$, it follows that C is not surjective. Hence, $\lambda = 0 \in \sigma(C) \backslash \sigma_{\text{pt}}(C)$.

Next, we consider the case when $\lambda \neq 0$. The function $f(t) = t^{\frac{1-\lambda}{\lambda}}$ that satisfies the equation $(\lambda I - C) f = 0$ belongs to $L_{p,\infty}(0,1)$ if and only if $Re\left(\frac{1}{\lambda}\right) \geq \frac{p-1}{p}$. Indeed, if $Re\left(\frac{1-\lambda}{\lambda}\right) \leq 0$, then the function $f(t) = t^{Re\left(\frac{1-\lambda}{\lambda}\right)}$ is non-increasing, hence $\left(t^{Re\left(\frac{1-\lambda}{\lambda}\right)}\right)^* = t^{Re\left(\frac{1-\lambda}{\lambda}\right)}$ [8, Theorem 2.4]. Therefore,

$$\|f\|_{L_{p,\infty}(0;1)} = \sup_{0<t<1} t^{\frac{1}{p}} \left(t^{\frac{1-\lambda}{\lambda}}\right)^* = \sup_{0<t<1} t^{\frac{1}{p}+Re\left(\frac{1}{\lambda}\right)-1}.$$

Let $Re\left(\frac{1-\lambda}{\lambda}\right) > 0$, then the function $t^{Re\left(\frac{1-\lambda}{\lambda}\right)}$ is increasing and its decreasing rearrangement will be $\left(t^{Re\left(\frac{1-\lambda}{\lambda}\right)}\right)^* = (1-t)^{Re\left(\frac{1}{\lambda}\right)-1}$.

$$\|f\|_{L_{p,\infty}(0;1)} = \sup_{0<t<1} t^{\frac{1}{p}} (1-t)^{Re\left(\frac{1}{\lambda}\right)-1} < \infty.$$

It follows that the point spectrum of C in $L_{p,\infty}(0,1)$ is the set

$$\sigma_{pt}(C) = \left\{ \lambda \in \mathbb{C} : Re\left(\frac{1}{\lambda}\right) \geq \frac{p-1}{p} \right\} \backslash \{0\}.$$

And again by [1, Lemma 1], the operator P_ξ is bounded on $L_p(0,1)$ for $1 < p < \infty$ if and only if $Re(\xi) < \frac{p-1}{p}$. Since $sh(s; L_{p,\infty}(0,1)) \to 0$ as $s \to 0+$ and $h(s; L_{p,\infty}(0,1)) \to 0$ as $s \to \infty$, it follows from [10, Corollary 1] that P_ξ is bounded on $L_{p,\infty}(0,1)$ if and only if $Re(\xi) < \frac{p-1}{p}$.

If $\xi := \frac{1}{\lambda}$, it is easy to see by (5) that the operator $\xi I + \xi^2 P_\xi$ is inverse of $\lambda I - C$ on $L_{p,\infty}(0,1)$. This implies that the resolvent set of the operator C in $L_{p,\infty}(0,1)$ will be

$$\rho(C) = \left\{ \lambda \in \mathbb{C} : Re\left(\frac{1}{\lambda}\right) < \frac{p-1}{p} \right\}.$$

Thus, the spectrum of the Cesáro-Hardy operator in $L_{p,\infty}(0,1)$ is the set

$$\sigma(C) = \left\{ \lambda \in \mathbb{C} : Re\left(\frac{1}{\lambda}\right) \geq \frac{p-1}{p} \right\}.$$

□

Remark 1 *The case $p = q = \infty$ was investigated in [4, Theorem, p. 29].*

4 Conclusion

In this paper, we investigated the spectrum of the Cesáro-Hardy operator in Lorentz $L_{p,q}$ spaces over a finite interval. We obtained that the spectrum of the operator C in the space $L_{p,q}(0,1)$ for $1 < p < \infty$ and $1 \leq q < \infty$ is the closed disk with center $\left(\frac{p'}{2}, 0\right)$ and radius $\frac{p'}{2}$. And the point spectrum of the Cesáro-Hardy operator in $L_{p,q}(0,1)$ is interior of this disk. The spectrum of the Cesáro-Hardy operator in the space $L_{p,\infty}(0,1)$ $(1 < p < \infty)$ was the same as for finite q. But the point spectrum of the operator C in $L_{p,\infty}(0,1)$ was the set $\sigma_{\text{pt}}(C) = \left\{\lambda \in \mathbb{C} : Re\left(\frac{1}{\lambda}\right) \geq \frac{p-1}{p}\right\} \setminus \{0\}$. We also determined the resolvent set of the Cesáro-Hardy operator in the spaces $L_{p,q}(0,1)$ and $L_{p,\infty}(0,1)$. Here, we used the operator P_ξ and proved its bondedness on $L_{p,q}(0,1)$. It helped us to show bounded inverse of the operator $\lambda I - C$ for some complex numbers λ.

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