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FINITE ELEMENT METHOD SCHEMES OF HIGHER ACCURACY FOR SOLVING NON-STATIONARY FOURTH-ORDER EQUATIONS

High-order Sobolev-type equations are mathematical models used in many applied problems. As is known, in many cases, it is difficult to obtain analytical solutions to high-order equations; therefore, they are mainly solved by numerical methods. At present, the method of straight lines is often used to solve non-stationary problems of mathematical physics; in this method, discretization is first realized only in spatial variables, and the resulting system of ordinary differential equations of high dimension is solved by finite difference methods or finite elements of higher accuracy. In this study, for a system of ordinary differential equations of the fourth order, new multi-parameter difference schemes of higher accuracy based on the finite element method are constructed. The presence of parameters in the scheme makes it possible to regularize the schemes in order to optimize the implementation algorithm and the accuracy of the scheme. The stability and convergence of the constructed difference schemes are also proved, and accuracy estimates are obtained on their basis. An algorithm for the implementation of the constructed difference schemes is presented. The results obtained can be further applied in the numerical solution to initial-boundary value problems for the equations of dynamics of a compressible stratified rotating fluid, magnetic gas dynamics, ion-acoustic waves in a magnetized plasma, spin waves in magnets, cold plasma in an external magnetic field, etc.

Key words: finite element method, difference schemes, stability, convergence, accuracy.

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Стационер емес төртінші тәртібті теңдеулерді шешу үшін жоғары дәлдікті шекті элементтік әдіс схемалары

Жоғары ретті Соболев типті теңдеулер көптеген қолданбалы есептердің математикалық модельдері болып табылады. Белгілі болғандай, көптеген жағдайларда жоғары ретті теңдеулердің аналитикалық шешімдерін алу қиын, сондықтан олар негізінен сандық әдістермен шешіледі. Соңғы кезде математикалық физиканың стационарлық емес есептерін шешу үшін сызықтар әдісі жиі қолданылады, онда дискретизация алдымен тек кеңістіктік айнымалылар бойынша жүзеге асырылады, ал алынған үлкен өлшемді қарапайым дифференциалдық теңдеулер жүйесі ақырлы айырмдық немесе жоғары дәлдіктегі ақырлы элементтер әдістерімен шешіледі. Бұл жұмыста төртінші ретті қарапайым дифференциалдық теңдеулер жүйесі үшін ақырлы элементтер әдісіне негізделген жоғары дәлдіктегі жаңа көппараметрлі айырмдық схемалары құрылып және зерттелген. Схемада параметрлердің болуы схемалардың дәлдігін жоғарғы ретке келтіруге және жүзеге асыру алгоритмін оңтайландыруға мүмкіндік береді. Сондай-ақ, құрылған айырмдық схемаларының тұрақтылығы мен жинақтылығы дәлелденді және олардың негізінде дәлдік бағалары алынды. Құрылған айырмадық схемаларын жүзеге

асыру алгоритмі берілді. Алынған нәтижелерді ары қарай сығылатын стратификацияланған айналмалы сұйықтық динамикасының, магниттік газ динамикасының, магниттелген плазмадағы иондық-дыбыстық толқындардың, магнетиктердегі спиндік толқындардың, сыртқы магнит өрісіндегі суық плазманың және т. б. теңдеулері үшін бастапқы-шекара есептерді сандық шешуде қолдануға болады.
Түйін сөздер: ақырлы элементтер әдісі, айырымдық схемалар, тұрақтылық, жинақтылық, дәлдік.

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Схемы метода конечных элементов повышенной точности для решения нестационарных уравнений четвертого порядка

Уравнения Соболевского типа высокого порядка являются математическими моделями многих прикладных задач. Как известно, во многих случаях получить аналитические решения уравнений высокого порядка затруднительно, поэтому, они в основном решаются численными методами. В последнее время для решения нестационарных задач математической физики часто применяют метод прямых, в котором дискретизация сначала проводится только по пространственным переменным, а полученная система обыкновенных дифференциальных уравнений высокой размерности решается методами конечных разностей или конечных элементов повышенной точности. В данной работе для системы обыкновенных дифференциальных уравнений четвертого порядка построены и исследованы новые многопараметрические разностные схемы повышенной точности на основе метода конечных элементов. Наличие параметров в схеме позволяет произвести регуляризацию схем с целью оптимизации алгоритма реализации и точности схемы. Также доказаны устойчивость и сходимость построенных разностных схем и на их основе получены оценки точности. Приведен алгоритм реализации построенных разностных схем. Полученные результаты могут найти дальнейшее применение при численном решений начально-краевых задач для уравнений динамики сжимаемой стратифицированной вращающейся жидкости, магнитной газовой динамики, ионно-звуковых волн в замагниченной плазме, спиновых волн в магнетиках, холодной плазмы во внешнем магнитном поле и т.п.
Ключевые слова: метод конечных элементов, разностные схемы, устойчивость, сходимость, точность.

1 Introduction

Recently, in the numerical solution of non-stationary partial differential equations, semi-discrete methods have been more often used, where differential operators with respect to spatial variables are approximated by the finite difference method or the finite element method, and the time variable is kept in differential form. The result is a system of ordinary differential equations of large dimensions [1–6]. In [1], for the abstract Cauchy problem for a system of second order ordinary differential equations, a two-parameter difference scheme of the fourth-order accuracy was constructed on the basis of the finite element method. The increase in accuracy is achieved due to a special choice of test functions involved in the replacement of the differential equation by some integral identity. The convergence of the scheme of the fourth-order accuracy to the solution of the original problem and its derivative is proved. These results are generalized in [2], [3], where three-parameter two-layer difference schemes of the fourth-order accuracy are constructed by a similar method for a system of second-order ordinary differential equations. On the basis of these results, initial-boundary value problems for Sobolev-type equations were studied in [4–6]. In particular, in [4], the problem of the internal wave propagation in a weakly stratified fluid was studied, and in [5], the convergence of the finite element method scheme for the equation of internal waves was considered. In [6], the convergence of the scheme of the finite element method for the equation of gravitational-gyroscopic waves in a stratified fluid was studied. Similar studies were conducted in [7–12] for various non-stationary initial-boundary value problems. In particular, in [7], a second-order accuracy estimate was obtained by the finite difference method for a fourth-order nonlinear Sobolev-type equation. In [8], [9], based on the Runge-Kutta methods, numerical methods were constructed and studied for the general fourth-order partial differential equation, where the spatial variables were approximated by the finite difference method. In [10], based on the tau-method, and in [11], [12], using the finite difference method, solutions to fourth-order ordinary differential equations were studied. Fourth-order accuracy estimates were obtained, and methods for obtaining a higher order of accuracy were indicated.

In this paper, based on the studies given in [1], the authors construct new parametric difference schemes for a system of fourth order ordinary differential equations of the form

$$D\overset{\overline{IV}}{u} + B\ddot{u} + Au = f, \quad t_0 < t \le T, \tag{1}$$

$$\overset{\overline{m}}{u}(t_0) = u_{0,m}, \quad m = \overline{0,3}, \tag{2}$$

where D, B and A are linear constants independent of t, operators from $H \to H$, $D^* = D > 0$, $B^* = B \ge 0$, $A^* = A > 0$; $\forall t \ge 0$, $u = u(t), f = f(t) \in H$- is the Hilbert space, $\overset{\overline{IV}}{u} = d^4u/dt^4, \ddot{u} = d^2u/dt^2$.

Let us give some examples of partial differential equations; spatial approximation of these equations leads to the solution of the abstract Cauchy problem (1), (2).

1. Dynamic equations for a compressible stratified rotating fluid are [13]

$$\frac{1}{c^2}\frac{\partial^4 u}{\partial t^4} = \frac{\partial^2}{\partial t^2}\left\{\Delta_3 u - \left(\beta^2 + \frac{\alpha^2}{c^2}\right)u\right\} + \omega_0^2\Delta_2 u + \alpha^2\frac{\partial^2 u}{\partial x_3^2} - \alpha^2\beta^2 u, \tag{3}$$

where $u = (x,t)$ is the flow velocity, $x = (x_1, x_2, x_3)$, $\Delta_2 = \partial^2u/\partial x_1^2 + \partial^2u/\partial x_2^2$, $\Delta_3 = \Delta_2 + \partial^2u/\partial x_3^2$, c is the speed of sound, ω_0^2 is the Väisälä-Brent frequency, α, β are some constants.

2. Equation of magnetic gas dynamics is [14]

$$\frac{\partial^4 u}{\partial t^4} - (a^2+b^2)\frac{\partial^2}{\partial t^2}\Delta_3 u + a^2b^2\frac{\partial^2}{\partial x_1^2}\Delta_3 u = f(x,t), \tag{4}$$

where a, b are some constants.

3. The equation of ion-acoustic waves in a "magnetized" plasma is [15]

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial t^2} + \omega_{B_i}^2\right)\left(\Delta_3 u - \frac{1}{r_D^2}u\right) + \omega_{p_i}^2\frac{\partial^2}{\partial t^2}\Delta_3 u + \omega_{p_i}^2\omega_{B_i}^2\frac{\partial^2 u}{\partial x_3^2} = f(x,t), \tag{5}$$

where $r_D^2 = T_e^2/(4\pi e^2 n_0)$ is Debye radius, $\omega_{B_i} = eB_0/(Mc)$ is the ion gyro-frequency, $\omega_{p_i}^2 = 4\pi e^2 n_0/M$ is the Langmuir ion frequency, M is mass, c is the speed of light in vacuum, B_0 is the external constant magnetic field, n_0 is the unperturbed particle density, e is the absolute value of electron charge, T_e is the electron temperature.

2 Constructing a scheme

Consider the abstract Cauchy problem of finding a solution to equation (1) that satisfies initial conditions (2). The generalized solution of equation (1) is defined as continuous function $u(t) \in C^3[0,T]$ satisfying the following integral identity for arbitrary function $\vartheta(t) \in C^2(t_н, t_к)$

$$\int_{t_н}^{t_к} (D\ddot{u}\ddot{\vartheta} - B\dot{u}\dot{\vartheta} + Au\vartheta)dt + \Big[D\dddot{u}\vartheta - D\ddot{u}\dot{\vartheta} + B\dot{u}\vartheta\Big]\Big|_{t_н}^{t_к} = \int_{t_н}^{t_к} (f, \vartheta)\, dt, \tag{6}$$

where $0 \le t_н \le t_к \le T$, $\dot{u} = du/dt$, $\dddot{u} = d^3u/dt^3$.

Let us introduce in $[0,T]$ uniform grid $\overline{\omega}_\tau = \{t_n = n\tau,\ n = 0, 1, ...;\ \tau > 0\}$. On each of the intervals (t_n, t_{n+1}), an approximate solution to problems (1), (2) is sought in the form of quintic polynomials

$$y(t) = \varphi_{00}^n(t)y^n + \varphi_{01}^n(t)y^{n+1} + \varphi_{10}^n(t)\dot{y}^n + \varphi_{11}^n(t)\dot{y}^{n+1} + \varphi_{20}^n(t)\ddot{y}^n + \varphi_{21}^n(t)\ddot{y}^{n+1}, \tag{7}$$

where $y^n = y(t_n)$, $y^{n+1} = y(t_{n+1})$, $\dot{y}^n = dy(t_n)/dt$, $\dot{y}^{n+1} = dy(t_{n+1})/dt$, $\ddot{y}^n = d^2y(t_n)/dt^2$, $\ddot{y}^{n+1} = d^2y(t_{n+1})/dt^2$. Next, we need to find the basis functions $\varphi_{k0}^n(t)$, $\varphi_{k1}^n(t)$, $k = 0, 1, 2$. We assume that

$$y(t) = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3 + a_4\xi^4 + a_5\xi^5, \tag{8}$$

where $\xi = (t - t_n)/(t_{n+1} - t_n) = (t - t_n)/\tau$. Coefficients $a_0, a_1, a_2, a_3, a_4, a_5$ are determined by the following conditions:

$$\begin{aligned}
&y^n(\xi) = y(0) = a_0,\ \dot{y}^n(\xi) = \left(\frac{dy}{d\xi}\right)_{\xi=0} = a_1,\\
&y^{n+1}(\xi) = y(1) = a_0 + a_1 + a_2 + a_3 + a_4 + a_5,\\
&\dot{y}^{n+1}(\xi) = \left(\frac{dy}{d\xi}\right)_{\xi=1} = a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5,\\
&\ddot{y}^n(\xi) = \left(\frac{d^2y}{d\xi^2}\right)_{\xi=0} = 2a_2,\ \ddot{y}^{n+1}(\xi) = \left(\frac{d^2y}{d\xi^2}\right)_{\xi=1} = 2a_2 + 6a_3 + 12a_4 + 20a_5.
\end{aligned} \tag{9}$$

Solving systems (9), we obtain the following expressions for the coefficients

$$a_0 = y^n,\ a_1 = \dot{y}^n, a_2 = 0.5\ddot{y}^n,\ a_3 = 10(y^{n+1} - y^n) - 2(2\dot{y}^{n+1} - 3\dot{y}^n) - 0.5(\ddot{y}^{n+1} - 3\ddot{y}^n),$$

$$a_4 = -15(y^{n+1} - y^n) + (7\dot{y}^{n+1} + 8\dot{y}^n) - 0.5(2\ddot{y}^{n+1} - 3\ddot{y}^n),$$

$$a_5 = 6(y^{n+1} - y^n) - 3(2\dot{y}^{n+1} + \dot{y}^n) + 0.5(\ddot{y}^{n+1} - \ddot{y}^n).$$

Therefore, expression (8) takes the following form

$$\begin{aligned}
&y(t) = (-6\xi^5 + 15\xi^4 + 6\xi^5 - 10\xi^3 + 1)y^n + (6\xi^5 - 15\xi^4 + 10\xi^3)y^{n+1}\\
&+\tau(-3\xi^5 + 8\xi^4 - 6\xi^3 + \xi)\dot{y}^n + \tau(-3\xi^5 + 7\xi^4 - 4\xi^3)\dot{y}^{n+1}\\
&+\tau^2(-\xi^5/2 + 3\xi^4/2 - 3\xi^3/2 + \xi^2/2)\ddot{y}^n + \tau(\xi^5/2 - \xi^4 + \xi^3/2)\ddot{y}^{n+1}.
\end{aligned} \tag{10}$$

Comparing (7) and (10), we find expressions for $\varphi^n_{k0}(t),\ \varphi^n_{k1}(t),\ k=0,1,2$:

$$\begin{aligned}
&\varphi^n_{00}(t) = -6\xi^5+15\xi^4+6\xi^5-10\xi^3+1,\ \varphi^n_{01}(t)=6\xi^5-15\xi^4+10\xi^3,\\
&\varphi^n_{10}(t) = \tau(-3\xi^5+8\xi^4-6\xi^3+\xi),\ \varphi^n_{11}(t)=\tau(-3\xi^5+7\xi^4-4\xi^3),\\
&\varphi^n_{20}(t) = \tau^2(-\xi^5/2+3\xi^4/2-3\xi^3/2+\xi^2/2),\ \varphi^n_{21}(t)=\tau(\xi^5/2-\xi^4+\xi^3/2),\\
&\xi=(t-t_n)/\tau.
\end{aligned} \tag{11}$$

Now, in (6) instead of $u(t)$ we substitute $y(t)$ and take $t_{\text{н}}=t_n,\ t_{\text{к}}=t_{n+1}$. Then with (11) from (6), we obtain

$$\begin{aligned}
&\left[D\int\limits_{t_n}^{t_{n+1}}\ddot\varphi_{01}\ddot\vartheta dt - B\int\limits_{t_n}^{t_{n+1}}\dot\varphi_{01}\dot\vartheta dt + A\int\limits_{t_n}^{t_{n+1}}\varphi_{01}\vartheta dt + \left(D\dddot\varphi_{01}\vartheta - D\ddot\varphi_{01}\dot\vartheta + B\dot\varphi_{01}\vartheta\right)\Big|_{t_n}^{t_{n+1}}\right]\hat y\\
&+\left[D\int\limits_{t_n}^{t_{n+1}}\ddot\varphi_{11}\ddot\vartheta dt - B\int\limits_{t_n}^{t_{n+1}}\dot\varphi_{11}\dot\vartheta dt + A\int\limits_{t_n}^{t_{n+1}}\varphi_{11}\vartheta dt + \left(D\dddot\varphi_{11}\vartheta - D\ddot\varphi_{11}\dot\vartheta + B\dot\varphi_{11}\vartheta\right)\Big|_{t_n}^{t_{n+1}}\right]\hat{\dot y}\\
&+\left[D\int\limits_{t_n}^{t_{n+1}}\ddot\varphi_{21}\ddot\vartheta dt - B\int\limits_{t_n}^{t_{n+1}}\dot\varphi_{21}\dot\vartheta dt + A\int\limits_{t_n}^{t_{n+1}}\varphi_{21}\vartheta dt + \left(D\dddot\varphi_{21}\vartheta - D\ddot\varphi_{21}\dot\vartheta + B\dot\varphi_{21}\vartheta\right)\Big|_{t_n}^{t_{n+1}}\right]\hat{\ddot y}\\
&+\left[D\int\limits_{t_n}^{t_{n+1}}\ddot\varphi_{00}\ddot\vartheta dt - B\int\limits_{t_n}^{t_{n+1}}\dot\varphi_{00}\dot\vartheta dt + A\int\limits_{t_n}^{t_{n+1}}\varphi_{00}\vartheta dt + \left(D\dddot\varphi_{00}\vartheta - D\ddot\varphi_{00}\dot\vartheta + B\dot\varphi_{00}\vartheta\right)\Big|_{t_n}^{t_{n+1}}\right]y\\
&+\left[D\int\limits_{t_n}^{t_{n+1}}\ddot\varphi_{10}\ddot\vartheta dt - B\int\limits_{t_n}^{t_{n+1}}\dot\varphi_{10}\dot\vartheta dt + A\int\limits_{t_n}^{t_{n+1}}\varphi_{10}\vartheta dt + \left(D\dddot\varphi_{10}\vartheta - D\ddot\varphi_{10}\dot\vartheta + B\dot\varphi_{10}\vartheta\right)\Big|_{t_n}^{t_{n+1}}\right]\dot y\\
&+\left[D\int\limits_{t_n}^{t_{n+1}}\ddot\varphi_{20}\ddot\vartheta dt - B\int\limits_{t_n}^{t_{n+1}}\dot\varphi_{20}\dot\vartheta dt + A\int\limits_{t_n}^{t_{n+1}}\varphi_{20}\vartheta dt + \left(D\dddot\varphi_{20}\vartheta - D\ddot\varphi_{20}\dot\vartheta + B\dot\varphi_{20}\vartheta\right)\Big|_{t_n}^{t_{n+1}}\right]\ddot y\\
&= \int\limits_{t_n}^{t_{n+1}} f\vartheta dt.
\end{aligned} \tag{12}$$

Here $y=y^n$, $\hat y = y^{n+1}$, $\dot y=\dot y^n$, $\hat{\dot y} = \dot y^{n+1}$, $\ddot y = \ddot y^n$, $\hat{\ddot y} = \ddot y^{n+1}$.

Further, choosing ϑ, we obtain difference approximation (1). Since equation (12) contains three unknowns $\hat y$, $\hat{\dot y}$ and $\hat{\ddot y}$, three functions $\vartheta(t)$ should be chosen. $\vartheta_1(t)$, $\vartheta_2(t)$ and $\vartheta_3(t)$ are taken as the following linear combinations of interpolation functions $\varphi_{km}(t)$ with parameters σ_1, σ_2 and σ_3:

$$\begin{aligned}
&\vartheta_1(t) = \sigma_1\varphi_{00} + (1-\sigma_1)\varphi_{01},\\
&\vartheta_2(t) = \sigma_2\varphi_{10} + (1-\sigma_2)\varphi_{11},\\
&\vartheta_3(t) = \sigma_3\varphi_{20} + (1-\sigma_3)\varphi_{21},
\end{aligned} \tag{13}$$

and parameters σ_1, σ_2, σ_3 are chosen from the condition of the maximum order of approximation of the resulting difference equations. Substituting successively (13) into (12) (at $f \equiv 0$), we obtain the following system

$$M\hat{Y} + GY = 0, \tag{14}$$

where

$$\hat{Y} = \begin{pmatrix} \hat{y} \\ \hat{\dot{y}} \\ \hat{\ddot{y}} \end{pmatrix},\ Y = \begin{pmatrix} y \\ \dot{y} \\ \ddot{y} \end{pmatrix},\ M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix},\ G = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix},$$

$$m_{11} = \frac{\tau}{4}A + (1-2\sigma_1)\left(\frac{540}{7\tau^2}D - \frac{10}{7\tau}B + \frac{131}{924}\tau A\right),$$

$$m_{12} = -\frac{6}{\tau^2}D + \frac{1}{2}B - \frac{1}{20}\tau^2 A + (1-2\sigma_1)\left(-\frac{270}{7\tau^2}D + \frac{5}{7}B - \frac{4}{231}\tau^2 A\right),$$

$$m_{13} = \frac{21}{7\tau}D + \frac{1}{240}\tau^2 A + (1-2\sigma_1)\left(\frac{45}{7\tau}D - \frac{1}{84}\tau B + \frac{5}{5544}\tau^2 A\right),$$

$$m_{21} = -\frac{60}{7\tau^2}D + \frac{3}{14}B - \frac{4}{231}\tau^2 A + (1-2\sigma_2)\left(\frac{1}{20}\tau^2 A\right),$$

$$m_{22} = \frac{30}{7\tau}D - \frac{3}{28}\tau B + \frac{5}{1848}\tau^3 A + (1-2\sigma_2)\left(\frac{6}{5\tau}D - \frac{17}{140}\tau B + \frac{31}{2520}\tau^2 A\right),$$

$$m_{23} = -\frac{5}{7}D + \frac{1}{168}\tau^2 B - \frac{17}{110880}\tau^4 A + (1-2\sigma_2)\left(-\frac{6}{10}D + \frac{3}{280}\tau^2 B - \frac{11}{10080}\tau^4 A\right),$$

$$m_{31} = \frac{1}{240}\tau^3 A + (1-2\sigma_3)\left(\frac{3}{7\tau}D - \frac{1}{84}\tau B + \frac{5}{5544}\tau^3 A\right),$$

$$m_{32} = -\frac{1}{10}D + \frac{3}{280}\tau^2 B - \frac{11}{10080}\tau^4 A + (1-2\sigma_3)\left(-\frac{3}{14}D + \frac{1}{168}\tau^2 B - \frac{17}{110880}\tau^4 A\right),$$

$$m_{33} = \frac{1}{20}\tau\, D - \frac{1}{840}\tau^3 B + (1-2\sigma_3)\left(\frac{1}{28}\tau\, D - \frac{1}{2520}\tau^3 B\right),$$

$$g_{11} = \frac{1}{4}\tau\, A + (1-2\sigma_1)\left(-\frac{540}{7\tau^3}D + \frac{10}{7\tau}B - \frac{131}{924}\tau\, A\right),$$

$$g_{12} = \frac{6}{\tau^2}D - \frac{1}{2}B + \frac{1}{20}\tau^2 A + (1-2\sigma_3)\left(-\frac{270}{7\tau^2}D + \frac{5}{7}\tau\, B - \frac{4}{231}\tau^2 A\right),$$

$$g_{13} = \frac{3}{\tau}D + \frac{1}{240}\tau^3 A + (1-2\sigma_1)\left(-\frac{45}{7\tau}D + \frac{1}{84}\tau\, B - \frac{5}{5544}\tau^3 A\right),$$

$$g_{21} = \frac{60}{7\tau^2}D - \frac{3}{14}B + \frac{4}{231}\tau^2 A - (1-2\sigma_2)\left(\frac{1}{20}\tau^2 A\right),$$

$$g_{22} = \frac{30}{7\tau}D - \frac{3}{28}\tau\, B + \frac{5}{1848}\tau^3 A + (1-2\sigma_2)\left(-\frac{6}{5\,\tau}D + \frac{17}{140}\tau\, B - \frac{31}{2520}\tau^3 A\right),$$

$$g_{23} = \frac{5}{7}D - \frac{1}{168}\tau^2 B + \frac{17}{110880}\tau^4 A + (1-2\sigma_2)\left(-\frac{3}{5}D + \frac{3}{280}\tau^2 B - \frac{11}{10080}\tau^4 A\right),$$
$$g_{31} = \frac{1}{240}\tau^3 A + (1-2\sigma_3)\left(-\frac{3}{7\tau}D + \frac{1}{84}\tau B - \frac{5}{5544}\tau^3 A\right),$$
$$g_{32} = \frac{1}{10}D - \frac{3}{280}\tau^2 B + \frac{11}{10080}\tau^4 A + (1-2\sigma_3)\left(-\frac{3}{14}D + \frac{1}{168}\tau^2 B - \frac{17}{110880}\tau^4 A\right),$$
$$g_{33} = \frac{1}{20}\tau D - \frac{1}{840}\tau^3 B + (1-2\sigma_3)\left(-\frac{1}{28}\tau D + \frac{1}{2520}\tau^3 B\right).$$

Having made some transformations with system (14), we obtain the following difference equations approximating (1):

$$\begin{gathered}
\left(D - \frac{\tau^2}{12}B\right)\frac{\hat{\dot{y}} - \dot{y}}{\tau} - \frac{\tau^2}{12}A\frac{\hat{y} + y}{2} - D\frac{\hat{\ddot{y}} + \ddot{y}}{2} \\
-(1-2\sigma_1)\left\{\left(-\frac{90}{7}D + \frac{5\tau^2}{21}B - \frac{4\tau^3}{693}A\right)\frac{\hat{\dot{y}} + \dot{y}}{2} + \left(\frac{45\tau^2}{42}D - \frac{\tau^3}{504}B\right)\frac{\hat{\ddot{y}} - \ddot{y}}{\tau}\right\} = 0, \\
\left(D - \frac{\tau^2}{40}B\right)\frac{\hat{y} - y}{\tau} - \left(D - \frac{\tau^2}{40}B\right)\frac{\hat{\dot{y}} + \dot{y}}{2} + \frac{\tau^2}{12}D\frac{\hat{\ddot{y}} - \ddot{y}}{\tau} \\
-(1-2\sigma_2)\left\{\left(\frac{7\tau^3}{50}D - \frac{17\tau^3}{1200}B\right)\frac{\hat{\dot{y}} - \dot{y}}{\tau} - \left(\frac{7\tau^3}{50}D + \frac{\tau^3}{800}B\right)\frac{\hat{\ddot{y}} + \ddot{y}}{2}\right\}, \\
\left(D - \frac{3\tau^2}{28}B\right)\frac{\hat{\dot{y}} - \dot{y}}{\tau} - \left(D - \frac{\tau^2}{42}B\right)\frac{\hat{\ddot{y}} + \ddot{y}}{2} - \frac{\tau^2}{12}A\frac{\hat{y} + y}{2} \\
-10(1-2\sigma_3)\left(\frac{3}{7\tau}D - \frac{\tau}{84}B + \frac{5\tau^3}{5544}A\right)\frac{\hat{y} - y}{\tau} \\
-(1-2\sigma_3)\left\{\left(-\frac{30}{7\tau}D + \frac{5\tau}{42}B + \frac{17\tau^3}{5544}A\right)\frac{\hat{\dot{y}} + \dot{y}}{2} - \left(\frac{10\tau}{28}D - \frac{\tau^3}{252}B\right)\frac{\hat{\ddot{y}} - \ddot{y}}{\tau}\right\} = 0.
\end{gathered} \tag{15}$$

On sufficiently smooth solutions $u(t)$ of equation (1), the approximation error of the resulting scheme (15) is

$$\psi_1 = (1-2\sigma_1)\left(\frac{30}{70}D\overline{u}^{(1)} - \frac{45}{84}\tau^2 D\overline{u}^{(3)} + \frac{5}{4}\tau^2 B\overline{u}^{(1)}\right)$$
$$+\frac{\tau^4}{12}\left(\frac{1}{24}B\overline{u}^{(4)} + \frac{1}{8}A\overline{u}^{(3)} + \frac{1}{32}D\overline{u}^{(4)}\right) + O(\tau^6),$$
$$\psi_2 = (1-2\sigma_2)\left(\frac{7}{50}\tau^3 D\overline{u}^{(2)} - \frac{7}{50}\tau^2 D\overline{u}^{(3)}\right) + \frac{\tau^4}{24}\left(-\frac{1}{40}B\overline{u}^{(3)} - \frac{1}{12}D\overline{u}^{(5)}\right) + O(\tau^6),$$
$$\psi_3 = (1-2\sigma_3)\left(-\frac{30}{7\tau}D\overline{u}^{(1)} + \frac{5}{42}\tau B\overline{u}^{(1)} - \frac{50}{56}\tau\, D\overline{u}^{(3)} + \frac{19}{1008}\tau^3 B\overline{u}^{(3)}\right)$$
$$+\frac{\tau^4}{36}\left(\frac{1}{7}B\overline{u}^{(4)} - A\overline{u}^{(2)}\right) + O(\tau^6),$$

where $\overline{u}^k = d^k u(t_n + \tau/2)/dt^k$. If we choose $\sigma_1 = \sigma_2 = \sigma_3 = 1/2$, then scheme (15) will have the fourth-order of approximation and the form of the scheme is greatly simplified:

$$\begin{aligned}
&\left(D - \frac{\tau^2}{12}B\right)\frac{\overset{\wedge}{\dot y} - \dot y}{\tau} - \frac{\tau^2}{12}A\frac{\hat{y} + y}{2} - D\frac{\overset{\wedge}{\ddot y} + \ddot y}{2} = 0,\\
&\left(D - \frac{\tau^2}{40}B\right)\frac{\hat{y} - y}{\tau} - \left(D - \frac{\tau^2}{40}B\right)\frac{\overset{\wedge}{\dot y} + \dot y}{2} + \frac{\tau^2}{12}D\frac{\overset{\wedge}{\ddot y} - \ddot y}{\tau} = 0,\\
&\left(D - \frac{3\tau^2}{28}B\right)\frac{\overset{\wedge}{\dot y} - \dot y}{\tau} - \left(D - \frac{\tau^2}{42}B\right)\frac{\overset{\wedge}{\ddot y} + \ddot y}{2} - \frac{\tau^2}{12}A\frac{\hat{y} + y}{2} = 0.
\end{aligned} \tag{16}$$

Difference scheme (16) is a two-layer vector scheme. Each grid node $\overline{\omega}_\tau$ is triple - it defines three values: $y^n,\ \dot y^n,\ \ddot y^n$.

Let $\sigma_1 = \sigma_2 = \sigma_3 = 1/2$ in (13). Then we get

$$\vartheta_1 = 1/2,\ \vartheta_2 = \vartheta_2^{(5)} = \tau(3\xi^5 + 15\xi^4/2 - 5\xi^3 + \xi/2) = \tau\xi(1-\xi)(\xi - 1/2)(3\xi^2 - 3\xi - 1),$$
$$\vartheta_3 = \vartheta_3^{(4)} = \tau^2\xi^2(\xi-1)^2/4,\ \xi = (t - t_n)/\tau.$$

Function $\vartheta_2^{(5)}$ is an odd function with respect to point $t = t_n + \tau/2$, i.e. the middle point of interval (t_n, t_{n+1}). Then, leaving function $\vartheta_1 = 1/2$ unchanged, ϑ_2 and ϑ_3 are chosen as a linear function, an odd one with respect to $t = t_n + \tau/2$: $\vartheta_2 = \vartheta_2^{(5)} = \tau(\xi - 1/2)(3\xi^2 - 3\xi - 1)$, $\vartheta_3 = \tau^2/4$. As a result, we get the following scheme

$$\begin{aligned}
&\left(D - \frac{\tau^2}{12}B\right)\frac{\overset{\wedge}{\dot y} - \dot y}{\tau} - \frac{\tau^2}{12}A\frac{\hat{y} + y}{2} - D\frac{\overset{\wedge}{\ddot y} + \ddot y}{2} = 0,\\
&\left(D - \frac{\tau^2}{60}B\right)\frac{\hat{y} - y}{\tau} - \left(D - \frac{\tau^2}{60}B\right)\frac{\overset{\wedge}{\dot y} + \dot y}{2} + \frac{\tau^2}{12}D\frac{\overset{\wedge}{\ddot y} - \ddot y}{\tau} = 0,\\
&\left(D - \frac{\tau^2}{10}B\right)\frac{\overset{\wedge}{\dot y} - \dot y}{\tau} - \left(D - \frac{\tau^2}{60}B\right)\frac{\overset{\wedge}{\ddot y} + \ddot y}{2} - \frac{\tau^2}{12}A\frac{\hat{y} + y}{2} = 0.
\end{aligned} \tag{17}$$

Approximation errors coincide with scheme (16) $\psi_1 = O(\tau^4)$, $\psi_2 = O(\tau^4)$, $\psi_3 = O(\tau^4)$.

3 Construction of a parametric family of schemes

As seen from the form of schemes (16) and (17), by choosing function ϑ_2 we can obtain the following parametric family of schemes:

$$\begin{aligned}
&D_\eta \dot y_t - \eta\tau^2 A y^{(0.5)} - D\ddot y^{(0.5)} = 0,\\
&D_\gamma y_t - D_\gamma \dot y^{(0.5)} + \eta\tau^2 D\ddot y_t = 0,\\
&D_\alpha \dot y_t - D_\beta \ddot y^{(0.5)} - \eta\tau^2 A y^{(0.5)} = 0,
\end{aligned} \tag{18}$$

where $D_m = D - m\tau^2 B$, $\ m = \alpha,\ \beta,\ \gamma,\ \eta,\ \ y_t = (\hat{y} - y)/\tau,\ \ \dot y_t = (\overset{\wedge}{\dot y} - \dot y)/\tau,\ \ddot y_t = (\overset{\wedge}{\ddot y} - \ddot y)/\tau$, $y^{(0.5)} = (\hat{y} + y)/2,\ \dot y^{(0.5)} = (\overset{\wedge}{\dot y} + \dot y)/2,\ \ddot y^{(0.5)} = (\overset{\wedge}{\ddot y} + \ddot y)/2.$

Here, the approximation errors are

$$\psi_1 = \frac{\tau^2}{12}(D\overline{u}^{(4)} + 12\eta B\overline{u}^{(2)} + 12\eta A\overline{u}) + O(\tau^4),$$

$$\psi_2 = -\tau^2[(1/12 - \eta)D\overline{u}^{(3)}] + O(\tau^4),$$

$$\psi_3 = \frac{\tau^2}{12}[D\overline{u}^{(4)} + 12(\alpha - \beta)B\overline{u}^{(2)} + 12\eta A\overline{u})] + O(\tau^4),$$

where $\overline{u}^{(k)} = d^k\overline{u}(t_n + \tau/2)/dt^k$.

For the fourth order of approximation, it is sufficient that the parameters of scheme (18) be related by the following relations

$$\alpha - \beta = 1/12, \ \eta = 1/12, \tag{19}$$

γ is an arbitrary constant. Therefore, parameters (19) are the conditions of the fourth-order approximation of scheme (18). We choose function ϑ_2 such that the corresponding equation coincides with the second equation of scheme (18). This function is sought in the following form

$$\vartheta_2^{(\gamma,\eta)} = s_1\vartheta_2^{(1)} + s_2\vartheta_2^{(5)}, \tag{20}$$

with parameters s_1, s_2, to be determined, where $\vartheta_2^{(1)} = \tau(\xi - 1/2)$, $\vartheta_2^{(5)} = \tau(3\xi^5 + 15\xi^4/2 - 5\xi^3 + \xi/2)$.

Substituting (20) into (12), we obtain the difference equation

$$\left(s_1\left(D - \frac{1}{60}\tau^2 B\right) - \frac{1}{7}s_2\left(D - \frac{1}{40}\tau^2 B\right)\right)\frac{\hat{y} - y}{\tau}$$

$$-\left(s_1\left(D - \frac{1}{60}\tau^2 B\right) - \frac{1}{7}s_2\left(D - \frac{1}{40}\tau^2 B\right)\right)\frac{\overset{\wedge}{\dot{y}} + \dot{y}}{2} + \left(\frac{1}{12}s_1 - \frac{1}{84}s_2\right)\tau^2 D\frac{\overset{\wedge}{\ddot{y}} - \ddot{y}}{\tau} = 0.$$

To match it with the second equation in (18), it is necessary that s_1, s_2 satisfy the following relations:

$$s_1 - \frac{s_2}{7} = 1, \ -\frac{s_1}{60} + \frac{s_2}{280} = -\gamma, \ \frac{s_1}{12} - \frac{s_2}{84} = \eta.$$

From this system we find

$$s_1 = 3 - 120\gamma, \ s_2 = 14 - 840\gamma, \ \eta = 1/12.$$

Now we choose function ϑ_3 such that the corresponding equation coincides with the third equation of scheme (18). This function is sought in the following form

$$\vartheta_3^{(\alpha,\beta,\eta)} = s_3\vartheta_3^{(2)} + s_4\vartheta_3^{(4)}, \tag{21}$$

with parameters s_3, s_4, to be determined, where $\vartheta_3^{(2)} = \tau^2\xi(\xi - 1)/2$, $\vartheta_3^{(4)} = \tau^2\xi^2(\xi - 1)^2/4$.

Substituting (21) into (12), we obtain the difference equation

$$\left(s_3\left(D-\frac{1}{10}\tau^2B\right)-\frac{1}{10}s_4\left(D-\frac{3}{28}\tau^2B\right)\right)\frac{\overset{\wedge}{\dot y}-\dot y}{\tau}-\left(s_3\left(D-\frac{1}{60}\tau^2B\right)\right.$$

$$\left.-\frac{1}{10}s_4\left(D-\frac{1}{42}\tau^2B\right)\right)\frac{\overset{\wedge}{\ddot y}+\ddot y}{2}-\left(\frac{1}{12}s_3-\frac{1}{120}s_4\right)\tau^2A\frac{\hat{y}+y}{2}=0.$$

To match it with the third equation in (18), it is necessary that s_3, s_4 satisfy the following relations:

$$s_3-\frac{1}{10}s_4=1,\ \ -\frac{1}{10}s_3+\frac{3}{280}s_4=\alpha,\ \ -\frac{1}{60}s_3+\frac{1}{420}s_4=\beta,\ \ \frac{1}{12}s_3-\frac{1}{120}s_4=\eta.$$

From this system we find

$$s_3=140\alpha+15,\ \ s_4=1400\alpha+140,\ \ \eta=1/12,\ \ \alpha-\beta=1/12.$$

Now, using as ϑ_3 constructed function $\vartheta_3^{(\alpha,\beta,\eta)}$, we can built a scheme with parameters $\alpha,\ \beta,\ \gamma,\ \eta$ for the case $f(t)\neq 0$:

$$\begin{aligned}
&D_\eta\dot y_t-\eta\tau^2Ay^{(0.5)}-D\ddot y^{(0.5)}=\varphi_1,\\
&D_\gamma y_t-D_\gamma\dot y^{(0.5)}+\eta\tau^2D\ddot y_t=\varphi_2,\\
&D_\alpha\dot y_t-D_\beta\ddot y^{(0.5)}-\eta\tau^2Ay^{(0.5)}=\varphi_3,
\end{aligned}\tag{22}$$

where

$$\varphi_1=-\frac{\tau}{6}\int\limits_{t_n}^{t_{n+1}}f(t)dt=-\frac{\tau^2}{6}\int\limits_0^1 f(t_n+\tau\xi)d\xi,$$

$$\varphi_2=-\frac{7\tau}{60}\int\limits_{t_n}^{t_{n+1}}f(t)\vartheta_2^{(\gamma,\eta)}(t)dt=-\frac{7\tau^2}{60}\int\limits_0^1 f(t_n+\tau\xi)[s_1\vartheta_2^{(1)}(\xi)+s_2\vartheta_2^{(5)}(\xi)]d\xi,$$

$$\varphi_3=-\frac{10}{\tau}\int\limits_{t_n}^{t_{n+1}}f(t)\vartheta_3^{(\alpha,\beta,\eta)}(t)dt=-10\int\limits_0^1 f(t_n+\tau\xi)[s_3\vartheta_3^{(2)}+s_4\vartheta_3^{(4)}]d\xi.$$

The first initial condition is approximated exactly, and the remaining initial conditions are approximated as in [16], by the fourth order, using the Taylor series and the initial equation:

$$\begin{aligned}
&y^0=u_{0,0},\ \dot y^0=u_{0,1}+\frac{\tau}{2}\left(E-\frac{\tau^2}{12}D^{-1}B\right)u_{0,2}+\frac{\tau^2}{6}u_{0,3}+\frac{\tau^3}{24}D^{-1}[f(0)-Au_{0,0}],\\
&\ddot y^0=u_{0,2}+\tau u_{0,3}+\frac{\tau^2}{2}D^{-1}[f(0)-Bu_{0,2}-Au_{0,0}]+\frac{\tau^3}{4}D^{-1}[\dot f(0)-B\dot u_{0,2}-A\dot u_{0,0}].
\end{aligned}\tag{23}$$

It is easy to check that scheme (22), (23) has the fourth order of approximation error on smooth solutions if conditions (19) are satisfied.

4 Stability of the scheme

Let us introduce into consideration a real finite-dimensional Hilbert space $H^3 = H \oplus H \oplus H$ (direct sum of spaces H) with scalar product

$$(U, V)_S = (SU, V) = \sum_{\alpha=1}^{3} (SU_\alpha, V_\alpha)$$

and norm

$$\|U\|_S^2 = (U, U)_S = \sum_{\alpha=1}^{3} \|u_\alpha\|_S^2,\ U, V \in H,\ U = (u_1, u_2, u_3),\ V = (\vartheta_1, \vartheta_2, \vartheta_3).$$

Scheme (22) after simple transformations takes the following form

$$\begin{aligned} &D_\beta D_\eta \dot{y}_t - \eta\tau^2 D_\beta A y^{(0.5)} - D_\beta D \ddot{y}^{(0.5)} = D_\beta \varphi_1, \\ &AD_\gamma y_t - AD_\gamma \dot{y}^{(0.5)} + \eta\tau^2 AD\ddot{y}_t = A\varphi_2, \\ &AD_\alpha \dot{y}_t - AD_\beta \ddot{y}^{(0.5)} - \eta\tau^2 A^2 y^{(0.5)} = A\varphi_3. \end{aligned} \tag{24}$$

We introduce $Y^n = (\ddot{y}^n,\ \dot{y}^n,\ y^n) \in H^3$. Then, scheme (23) can be represented as:

$$\overline{\mathrm{B}} Y_t + \mathrm{U}\frac{\hat{Y} + Y}{2} = \Phi,$$

where

$$\overline{\mathrm{B}} = \begin{pmatrix} 0 & D_\beta D_\eta & 0 \\ \eta\tau^2 AD & 0 & AD_\gamma \\ 0 & \eta\tau^2 AD_\alpha & 0 \end{pmatrix},\ U = \begin{pmatrix} -D_\beta D & 0 & -\eta\tau^2 D_\beta A \\ 0 & -AD_\gamma & 0 \\ -\eta\tau^2 AD_\beta & 0 & -\eta^2\tau^2 A^2 \end{pmatrix}.$$

Operators $\overline{\mathrm{B}},\ \mathrm{U}$ act from H^3 into H^3, $\Phi = (D_\beta\varphi_1,\ A\varphi_2,\ \eta\tau^2 A\varphi_3) \in H^3$.

Canonical form of scheme notation (24)

$$BY_t + UY = \Phi,$$

corresponds to operator $B = \overline{B} + 0.5\tau U$ [17].

Based on Theorem 6 from [17, p. 193], we obtain the following theorem.

Theorem 1. *Let*

$$D_m > 0,\ \alpha,\ \beta,\ \gamma,\ \eta, \tag{25}$$

$$\mathrm{U} = \mathrm{U}^* \geq -c_* E,\ c_* = const > 0, \tag{26}$$

U *is a constant operator and the following conditions are met*

$$B \geq \varepsilon E + 0.5\tau U',\ U' = \mathrm{U} + c_1 E,\ \varepsilon = const > 0, \tag{27}$$

where $c_1 = 2c_*$. Then for the solution of scheme (22), (23) the a priori estimate holds

$$\|Y(t_{n+1})\|_{U'}^2 \le e^{\theta_{n+1}} \left(\|Y(0)\|_{U'}^2 + \frac{1+t_{n+1}}{2\varepsilon} \sum_{k=0}^{n} \tau \|\Phi_k\|^2 \right), \; \theta_{n+1} = 2c_* \frac{t_{n+1}+1}{\varepsilon}. \tag{28}$$

Condition (25) imposes restrictions on the grid step τ:

$$D > m\tau^2 B, \; m = max(\alpha, \beta, \gamma, \eta). \tag{29}$$

Condition (26) is satisfied since operator U is sign-undefined in structure, and (27) will be satisfied if the stability condition (29) is met.

5 Convergence of the scheme

Let us introduce the error of the scheme $z^n = y^n - u(t_n)$, $\dot{z}^n = \dot{y}^n - \dot{u}(t_n)$, $\ddot{z}^n = \ddot{y}^n - \ddot{u}(t_n)$, where $u(t)$ is the solution of problem (1), (2). Substituting $y^n = u(t_n) + z^n$, $\dot{y}^n = \dot{u}(t_n) + \dot{z}^n$, $\ddot{y}^n = \ddot{u}(t_n) + \ddot{z}^n$ into (22), we obtain the problem for the error of the scheme

$$D_\eta \dot{z}_t - \eta\tau^2 A z^{(0.5)} - D\ddot{z}^{(0.5)} = \psi_1,$$
$$D_\gamma z_t - D_\gamma \dot{z}^{(0.5)} + \eta\tau^2 D\ddot{z}_t = \psi_2,$$
$$D_\alpha \dot{z}_t - D_\beta \ddot{z}^{(0.5)} - \eta\tau^2 A z^{(0.5)} = \psi_3,$$

with appropriate initial conditions. Therefore, based on Theorem 1, we obtain the convergence of the scheme with the fourth order, i.e., the following theorem holds.

Theorem 2. *Let the conditions of Theorem 1 and* (19) *be satisfied. Then, on the basis of estimate* (28), *we obtain that scheme* (22), (23) *converges to the solution of problem* (1), (2) *with the fourth order so, its solution satisfies the accuracy estimates:*

$$\|y^n - u(t_n)\|_{U'} \le M\tau^4, \; \|\dot{y}^n - \dot{u}(t_n)\|_{U'} \le M\tau^4, \; \|\ddot{y}^n - \ddot{u}(t_n)\|_{U'} \le M\tau^4.$$

6 Algorithm for implementing the scheme

Let us consider one of the possible algorithms for implementing scheme (26). We rewrite it in the following form

$$\begin{cases} m_{11}\hat{y} + m_{12}\hat{\dot{y}} + m_{13}\hat{\ddot{y}} = \Phi_1, \\ m_{21}\hat{y} + m_{22}\hat{\dot{y}} + m_{23}\hat{\ddot{y}} = \Phi_2, \\ m_{31}\hat{y} + m_{32}\hat{\dot{y}} + m_{33}\hat{\ddot{y}} = \Phi_3. \end{cases} \tag{30}$$

Here

$$m_{11} = -\eta\frac{\tau^3}{2}A, \; m_{12} = D_\eta, \; m_{13} = -\frac{\tau}{2}D, \; m_{21} = D_\gamma, \; m_{22} = -\frac{\tau}{2}D_\gamma, \; m_{23} = \eta\tau^2 D,$$
$$m_{31} = -\eta\frac{\tau^3}{2}A, \; m_{32} = D_\alpha, \; m_{33} = -\frac{\tau}{2}D_\beta, \Phi_1 = \tau\varphi_1 + \eta\frac{\tau^3}{2}Ay + D_\eta\dot{y} + \frac{\tau}{2}D\ddot{y},$$
$$\Phi_2 = \tau\varphi_2 + D_\gamma y + \frac{\tau}{2}D_\gamma\dot{y} + \eta\tau^2 D\ddot{y}, \; \Phi_3 = \tau\varphi_3 + \eta\frac{\tau^3}{2}y + D_\alpha\ddot{y} + \frac{\tau}{2}D_\beta\ddot{y}.$$

Assuming the mutual commutability of operators D, B and A, we exclude $\hat{\dot{y}}$ from the system of equations (30). As a result, we obtain the following system of equations

$$\begin{cases} g_{11}\hat{y} + g_{12}\hat{\dot{y}} = \widetilde{\Phi}_1, \\ g_{21}\hat{y} + g_{22}\hat{\dot{y}} = \widetilde{\Phi}_2, \end{cases} \tag{31}$$

where $g_{11} = m_{23}m_{11} - m_{13}m_{21}$, $g_{12} = m_{23}m_{12} - m_{13}m_{22}$, $g_{21} = m_{33}m_{11} - m_{13}m_{31}$, $g_{22} = m_{33}m_{12} - m_{13}m_{32}$, $\widetilde{\Phi}_1 = m_{23}\Phi_1 - m_{13}\Phi_2$, $\widetilde{\Phi}_2 = m_{33}\Phi_1 - m_{13}\Phi_3$.

Further, excluding $\hat{\dot{y}}$ from (31) we obtain

$$C\hat{y} = F, \tag{32}$$

where $C = g_{22}g_{11} - g_{12}g_{21}$, $F = g_{21}\widetilde{\Phi}_1 - g_{12}\widetilde{\Phi}_2$.

Finding $\hat{y}$ from (32), we determine $\hat{\dot{y}}$ from one of equations (31), for example, from the first equation

$$C_1\hat{\dot{y}} = F_1,$$

where $C_1 = g_{22}g_{12}$, $F_1 = g_{22}\widetilde{\Phi}_1 - g_{22}g_{11}\hat{y}$. Further, the value of $\hat{\ddot{y}}$ is found from system (30), for example, from the first equation $C_2\hat{\ddot{y}} = F_2$, where $C_2 = m_{13}$, $F_2 = \Phi_1 - m_{11}\hat{y} - m_{12}\hat{\dot{y}}$.

7 Conclusions

New multi-parameter difference schemes of the fourth-order accuracy are constructed and investigated for systems of the fourth order ordinary differential equations. The constructed schemes can be interpreted as schemes of the finite element method, since the approximate solution is determined from the condition of orthogonality of function $\vartheta_k(t)$, which differs from the interpolation functions $\varphi_{kl}(t)$. Then we can say that these schemes are based on the Galerkin-Petrov method [16]. On the other hand, scheme (22) is obtained from the integral identity (3), replacing differential equation (1), when interpolated by quintic polynomials the solutions at each grid step ω_τ, and it can be considered as schemes of the integro-interpolation method [17].

The family of schemes (22) has certain disadvantages and advantages. We note the following:

1. The scheme is conditionally stable and for its implementation, it is necessary to spend approximately three times more arithmetic operations than for conventional schemes of the finite difference method. However, this scheme allows choosing large time steps to achieve a certain accuracy.

2. The advantages of the schemes include the following: a) higher order of accuracy; b) in addition to the solution itself, we obtain its first and second derivatives (with the same accuracy order); for example, for problems of fluctuations in a continuous medium, in addition to displacements, we simultaneously determine velocity and acceleration; c) using the interpolation representation (7), if necessary, it is possible to obtain a solution and its first and second derivatives at any time point $t \in (t_n,\, t_{n+1})$; d) since the schemes are two-layer, it is possible to use variable step τ without loss of accuracy.

Thus, we can state the advantages of scheme (22) in solving various initial and initial-boundary value problems for partial differential equations of the type (3)-(5).

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