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ON AN INVERSE PROBLEM WITH AN INTEGRAL OVERDETERMINATION CONDITION FOR THE BURGERS EQUATION

In this paper we consider one inverse problem for the Burgers equation with an integral overdetermination and periodic boundary conditions in a domain that is trapezoid. Using an integral overdetermination, boundary and initial conditions, we reduce the inverse problem to the study of an already direct initial boundary value problem for the loaded Burgers equation. Next, we use a one-to-one transformation of independent variables to move from a trapezoid to a rectangular domain, where we study an auxiliary problem, for which the methods of Faedo-Galerkin, a priori estimates and functional analysis have been proved a theorem on its unique solvability in Sobolev classes. Note that the obtained a priori estimates are uniform with respect to the summation index of the approximate solution and do not depend on time. Further, on the basis of this theorem, due to the correspondence of spaces, theorems on the unique solvability of the original inverse problem are proved. Also, for the selected initial data, the paper presents graphs of the initial-boundary problem for the loaded Burgers equation and the desired function of the inverse problem, which together constitute the solution of the original inverse problem.
Key words: Burgers equation, inverse problem, a priori estimates, Galerkin method.

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Бургерс теңдеуіне қойылған және интегралдық қосымша шарты бар кері есеп туралы

Осы жұмыста біз трапеция болып табылатын облыстағы Бюргерс теңдеуі үшін қойылған және интегралдық қосымша шарты және периодикалық шекаралық шарттары бар кері есепті зерттейміз. Қосымша интегралдық шартын, бастапқы және шекаралық шарттарды пайдаланып, біз кері есепті жүктелген Бюргерс теңдеуі үшін қойылған бастапқы шекаралық есепті зерттеуге келтіреміз. Әрі қарай біз тәуелсіз айнымалылардың өзара керіленетін түрлендіруі көмегімен трапециядан тікбұрышты облысқа көшеміз. Содан кейін осы облыста көмекші есепті зерттеп, Фаедо-Галеркин әдісі, априорлы бағалаулар әдісі мен функционалдық талдау көмегімен осы есептің Соболев кластарындағы бірмәнді шешімділігі туралы теоремаларды дәлелдедік. Айта кетсек, алынған априорлы бағалаулар жуықтау шешімінің сумма алыну индексі бойынша бірқалыпты және уакытқа тауелсіз. Одан кейін осы теорема арқылы кеңістіктердің сәйкестігі негізінде кері есептің бірмәнді шешімділігі туралы теоремалар дәлелденеді. Бұған қоса біз арнайы алынған бастапқы шарттар үшін Бюргерс теңдеуіне қойылған бастапқы шекаралық есептің және кері есептің ізедлінді функциясының графиктерін келтіреміз.
Түйін сөздер: Бургерс теңдеуі, кері есеп, априорлы бағалаулар, Галеркин әдісі.

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Об одной обратной задаче с интегральным условием переопределения для уравнения Бюргерса

В данной работе нами исследуется одна обратная задача для уравнения Бюргерса с интегральным условием переопределения и периодическими граничными условиями в области, представленной трапецией. Используя дополнительное интегральное условие, граничные и начальные условия мы обратную задачу сводим к исследованию уже прямой начально граничной задачи для нагруженного уравнения Бюргерса. Далее с помощью взаимооднозначного преобразования независимых переменных мы переходим от трапеции к прямоугольной области. И уже в этой области мы исследуем вспомогательную задачу, для которой методами Фаедо-Галеркина, априорных оценок и функционального анализа была доказана теорема об её однозначной разрешимости в классах Соболева. Отметим, что полученные априорные оценки являются равномерными относительно индекса суммирования приближенного решения и не зависят от времени. Далее на основе данной теоремы, в силу соответствия пространств доказываются теоремы об однозначной разрешимости исходной обратной задачи. Также мы для выбранных начальных данных в работе приводим графики начально граничной задачи для нагруженного уравнения Бюргерса и искомой функции обратной задачи, которые вместе составляют решение исходной обратной задачи.

Ключевые слова: уравнение Бюргерса, обратная задача, априорные оценки, метод Галеркина.

Introduction

The simplest equation combining both nonlinear propagation effects and diffusive effects is Burgers' equation

$$c_t + cc_x = \nu c_{xx}. \tag{1}$$

For the first time, this nonlinear parabolic partial differential equation was introduced by J.M. Burgers [16] in 1948. Since then, the study of the Burgers equation has a long history, which is noted in many papers. Here we present only a small part. In [7] it was shown that (1) is an exact equation for waves described by

$$\rho_t + q_x = 0, \quad q = Q(\rho) - \nu\rho_x,$$

in the case that $Q(\rho)$ is a quadratic function of ρ. Although the Burgers equation is unphysical [12], it is nevertheless relevant to various areas of applied mathematics, such as fluid mechanics, nonlinear acoustics, gas dynamics, and traffic flow [5, 10]. An important role in mathematical physics is played by various modifications of the Burgers equation, such as the generalized Burgers equation [6, 14, 21], Burgers-Fisher equation [34], Korteweg-de Vries-Burgers equation [32,39], Rosenau-Burgers [22,36] and others. The Korteweg-de Vries-Burgers equation is obtained when in the models describing propagation of undular bores in shallow water and in fluids containing gas bubbles a smoothing effect is added and produces a third phenomenon, dissipation (second-order term) [3, 17]. Since the Burgers equation, in a sense, is a one-dimensional simplified analogue of the Navier-Stokes equation, the Burgers equation is very often used as a test example for Navier-Stokes [20, 33].

As for the problems with periodic conditions for the Burgers equation and its modification, in the work [31] they were derived to simulate wave propagation in a prestressed thick elastic tube filled with a viscous liquid using the long-wave approximation and the perturbation method. Such problems also arise in hydrodynamics as models of long waves in a viscous liquid flowing down an inclined plane, and to describe drift waves in plasma [18].

The applied importance of inverse problems is so great (it arises in various fields of human activity: seismology, mineral exploration, biology, medicine, quality control of industrial products, etc.) that puts them in several urgent problems of modern mathematics. The variety of inverse problems in comparison with their direct counterparts is huge. And many inverse problems that were derived from classical and basic direct problems are still waiting for theoretical and numerical research. Here we will also focus only on some of them.

In the works [1, 2, 8, 9, 19, 23, 24] questions of the solvability of the inverse problems of determining the right-hand side and the unknown coefficient of the desired function with the integral redefenition condition are studied. Among the recent papers on inverse problems for the Burgers equation, we note only [11,30]. In the work [2] and [9] were considered the inverse problems heat conduction and the Burgers equation with periodic boundary conditions.

It is known that by the Hopf-Cole transformation [4, 13] the Burgers equation can be reduced to the heat equation [11,25,28]. Theorems on the existence, depending on the initial and boundary conditions, of a unique and non-unique solution to the inverse problem for the one-dimensional Burgers equation in a rectangular domain were proved in [11]. Similar results for the direct problem for the Burgers equation:

$$\begin{cases} w_t + ww_x - w_{xx} = 0,\ 0 < x < t,\ t > 0, \\ w|_{x=0} = 0,\ w|_{x=t} = 0. \end{cases} \tag{2}$$

but already in the angular domain were obtained in [25], and it was shown that boundary value problem (2) in the corresponding weight Lebesgue class, where the weight is determined by the nature of the degeneracy of the domain, along with the trivial solution, has a nontrivial solution. In [28] the results obtained in [25] were extended to the case of inhomogeneous boundary conditions.

On inverse problems for parabolic equations in degenerating domains, we can note the works [26, 27, 29].

As regards boundary value problems for the Burgers equation in domains that are not rectangles, we would like to mention the closest works [37, 38, 40]. In work [37] in the non-rectangular domain that can be transformed into rectangle, the correctness of the boundary value problem for the Burgers equation was established in Sobolev spaces. These studies were continued in work [38], where the boundary value problem is already considered in a degenerate triangular domain. In [40] new regularity results for the non-homogeneous Burgers equation in domains that can be transformed into rectangles were obtained.

In this paper, in contrast to work [40], we consider the inverse problem for the Burgers equation in the non-rectangular domain that can be transformed into rectangle. The solvability issues of the inverse problem for the Burgers equation and associated initial boundary value problem for the loaded Burgers equation are studied in Sobolev classes. The results obtained for the latter problem are used in proving theorems on the unique solvability of the inital inverse problem. A separate work will be devoted to the inverse problem in a degenerating domain.

The paper is organized as follows. Section 1 is this Introduction. Statements of the generalized initial boundary value problem for the loaded Burgers equation, the original inverse problem, and associated with it initial boundary value problem for the loaded Burgers equation are given in Section 2. The main results are also given here. Section 3 is devoted to the proof of the theorem on the unique solvability of the generalized initial boundary value problem for the loaded Burgers equation. Theorems on the solvability of the inverse problem and the associated initial-boundary value problem are proved in Section 4. Graphs of the solution to the initial boundary value problem for the loaded Burgers equation, the initial inverse problem, and the desired function are presented in Section 5. A brief conclusion completes the work.

1 Statements of the problems and main results

Recall that $L^p(0,t)$ and $H^m(0,t)$ are the usual spaces of Lebesgue and Sobolev [15], for $1 \le p \le \infty$, $t \in (t_0,T)$, $t_0 > 0$ and $m \in \mathbb{Z}$. We introduce the notation, $\forall t \in (t_0,T)$:

$$H^2_{per}(0,t) = \left\{ w \in H^2(0,t): \ w(0,t) = w(t,t), \ \partial_x w(0,t) = \partial_x w(t,t) \right\}.$$

The article is concerned with the following two questions: the first one is to study the existence and uniqueness of the solution to the inverse problem for Burgers equation in the domain $Q_{xt} = \{x,t \,|\, 0 < x < t, \ t_0 < t < T < \infty, \ t_0 > 0\}$ where $\Omega_t = \{0 < x < t, t_0 > 0\}$ is a cross section of the domain Q_{xt} for a fixed value of the variable $t \in (t_0,T)$: to find a couple of functions $\{u(x,t), \ \lambda(t)\}$ from the conditions

$$\partial_t u + u\partial_x u - \nu\partial_x^2 u = \lambda(t)f(x), \ \ (x,t) \in Q_{xt}, \tag{3}$$

$$\partial_x^j u(0,t) = \partial_x^j u(t,t), \ j = 0, \ 1; \ t \in (t_0,T), \tag{4}$$

$$u(x,t_0) = 0, \ x \in (0,t_0), \tag{5}$$

$$\int_0^t u(x,t)dx = E(t), \ t \in [t_0,T], \tag{6}$$

where $\nu = const > 0$ is a given constant and functions $f(x)$, $E(t)$ satisfy the conditions

$$\begin{cases} f(x) \in L^\infty(t_0,T; L^\infty(0,t)) \equiv L^\infty(Q_{xt}), \ \bar{f}(t) \equiv \int_0^t f(x)dx \neq 0, \ \forall t \in [t_0,T], \\ E(t) \in W^{1,\infty}(t_0,T). \end{cases} \tag{7}$$

Note that (7) implies: $\bar{f}(t) \in L^\infty(t_0,T)$ and there exists such ε that $|\bar{f}(t)| \ge \varepsilon > 0$, $\forall t \in [t_0,T]$.

Integrating the equation (3) with respect to x over the sections Ω_t we will have

$$E'(t) - u(t,t) = \lambda(t)\bar{f}(t), \ t \in (t_0,T),$$

and for the unknown function $\lambda(t)$ we have the following formula:

$$\lambda(t) = -h(t)u(t,t) + g(t), \ t \in (t_0,T), \ \text{where } h(t) = \frac{1}{\bar{f}(t)}, \ g(t) = \frac{E'(t)}{\bar{f}(t)}. \tag{8}$$

Thus, from (3)–(5) and (8) we obtain the following initial boundary value problem for the loaded Burgers equation

$$\partial_t u + u\partial_x u - \nu\partial_x^2 u + h(t)f(x)u(t,t) = g(t)f(x), \ \ (x,t) \in Q_{xt}, \tag{9}$$

$$\partial_x^j u(0,t) = \partial_x^j u(t,t), \ j = 0, \ 1; \ t \in (t_0,T), \tag{10}$$

$$u(x,t_0) = 0, \ x \in (0,t_0). \tag{11}$$

The following theorems are valid:

Theorem 1 *Let conditions* (7) *be met. Then initial boundary value problem* (9)–(11) *has a unique solution*

$$u \in H^{2,1}_{per}(Q_{xt}) \equiv \left\{L^2(t_0,T;H^2_{per}(0,t)) \cap H^1(t_0,T;L^2(0,t))\right\}.$$

Theorem 2 (Main result) *Let conditions* (7) *be satisfied. Then the inverse problem* (3)–(6) *has a unique solution*

$$u \in H^{2,1}_{per}(Q_{xt}) \equiv \left\{L^2(t_0,T;H^2_{per}(0,t)) \cap H^1(t_0,T;L^2(0,t))\right\}, \ \lambda(t) \in L^\infty(t_0,T),$$

where $u(x,t)$ is a solution to the initial boundary value problem (9)–(11)*, $\lambda(t)$ is determined by formula* (8)*.*

The second question concerns the following generalized initial boundary value problem for the loaded Burgers equation in a rectangular domain $Q_{yt} = \{y,t|\ y \in (0,1),\ t \in (t_0,T),\ t_0 > 0,\ T < \infty\}$

$$\partial_t w + a_1(t) w \partial_y w - a_2(t)\partial_y^2 w + a_3(y,t)\partial_y w + a_4(y,t) w(1,t) = q(y,t), \tag{12}$$

$$\partial_y^j w(0,t) = \partial_y^j w(1,t),\ j = 0,\ 1;\ t \in (t_0,T), \tag{13}$$

$$w(y,t_0) = 0,\ \ 0 < y < 1, \tag{14}$$

where $q(y,t) \in L^2(Q_{yt})$. We also assume that there are positive constants $\varepsilon_i,\ i = \overline{1,6}$, that the given functions $a_1(t) \in C^1([t_0,T]),\ a_2(t),\ a_3(y,t),\ \partial_y a_3(y,t), a_4(y,t) \in C(\bar{Q}_{yt})$ satisfy the conditions

$$\begin{gathered} a_1'(t) \le 0,\ \varepsilon_1 \le a_1(t) \le \varepsilon_2,\ \varepsilon_3 \le a_2(t) \le \varepsilon_4,\ \forall t \in [t_0,T], \\ |a_3(y,t)| \le \varepsilon_5,\ |\partial_y a_3(y,t)| \le \varepsilon_5,\ |a_4(y,t)| \le \varepsilon_6,\ \ \forall y \in (0,1),\ \forall t \in [t_0,T]. \end{gathered} \tag{15}$$

Theorem 3 *Let $q \in L^2(Q_{yt})$ and conditions* (15) *be satisfied. Then initial boundary value problem* (12)–(14) *has a unique solution*

$$w \in H^{2,1}_{per}(Q_{yt}) \equiv L^2(t_0,T;H^2_{per}(0,1)) \cap H^1(t_0,T;L^2(0,1)).$$

2 Proof of Theorem 3

First, we prove a theorem on the unique solvability of a generalized initial boundary value problem for the loaded Burgers equation in a rectangular domain. The results obtained for the latter problem are used in proving theorems on the solvability of the initial inverse problem.

2.1 Approximate problem

Let us multiply the equations (12) scalarly in $L^2(0,1)$ by the function $v \in H^2_{per}(0,1)$. As a result, taking into account the initial (14) and the boundary conditions (13) we will have a weak formulation of the problem (12)–(14):

$$\int_0^1 \partial_t w v dy + a_1(t)\int_0^1 w\partial_y w v dy + a_2(t)\int_0^1 \partial_y w \partial_y v dy \int_0^1 a_3(y,t)\partial_y w v dy +$$

$$+w(1,t)\int_0^1 a_4(y,t)vdy=\int\limits_0^1 qvdy,\ \ \forall\, v\in H^2_{per}(0,1), \tag{16}$$

$$w(y,t_0)=0,\ \ y\in(0,1). \tag{17}$$

To apply the Faedo-Galerkin method, we need to solve the following spectral problem:

$$-Y''(y)=\lambda^2 Y(y),\ y\in(0,1), \tag{18}$$

$$Y(0)=Y(1),\ \ Y'(0)=Y'(1). \tag{19}$$

The solution to the problem (18)-(19) is a system of orthogonal eigenfunctions $Y_k(y)=e^{ik(2\pi y)}$ with eigenvalues $\lambda_k^2=(2\pi k)^2,\ k\in\mathbb{Z},\ \mathbb{Z}=\{0,\pm1,\pm2,\dots\}$.

We introduce the following approximate solution

$$w_N(y,t)=\sum_{j=-N}^{N} c_{Nj}(t)Y_j(y),\ \ w_N(y,t_0)=\sum_{j=-N}^{N} c_{Nj}(t_0)Y_j(y), \tag{20}$$

which we will satisfy the problem (16)–(17):

$$\int\limits_0^1 \partial_t w_N Y_j dy + a_1(t)\int\limits_0^1 w_N\partial_y w_N Y_j dy + a_2(t)\int\limits_0^1 \partial_y w_N \partial_y Y_j dy + \int\limits_0^1 a_3(y,t)\partial_y w_N Y_j dy$$

$$+w_N(1,t)\int_0^1 a_4(y,t)Y_j dy=\int\limits_0^1 qY_j dy,\ \ j=\overline{-N,N}, \tag{21}$$

$$w_N(y,t_0)=0,\ \ y\in(0,1), \tag{22}$$

for all $t\in[t_0,T]$.

2.2 Solution of the approximate problem (21)–(22)

Lemma 1 *The problem* (21)–(22) *has a unique solution* $C_N=\{c_{Nj}(t)\}_{j=-N}^{N}$.

Proof. As we have mentioned earlier, the system of functions $\{Y_k(y)\}_{k\in\mathbb{Z}}$ forms an orthogonal basis in $L^2(0,1)$. Let W_N be Gram matrix, and $(\cdot,\cdot)$ the scalar product $L^2(0,1)$, then for any finite N there is

$$\det\{W_N\}=\|(Y_k(y),Y_j(y))\|_{k,j=-N}^{N}\neq 0.$$

Next, if for all $j=\overline{-N,N}$ we introduce the notation

$$G_N(t)=\{q_j(t)\},\ j=\overline{-N,N},\ \ P_N(t)=\{p_j(t)\},\ j=\overline{-N,N},$$

$$C_N(t)=\{c_{Nj}(t)\},\ j=\overline{-N,N}\ \ \ A_N=(\partial_y Y_k(y),\partial_y Y_j(y))_{k,j=-N}^{N}$$

where

$$q_j(t)=\int\limits_0^1 qY_j(y)dy,\ \ p_j(t)=-a_1(t)\int\limits_0^1 w_N\partial_y w_N Y_j(y)dy$$

$$-\int\limits_0^1 a_3(y,t)\partial_y w_N(y,t)Y_j(y)dy-w_N(1,t)\int_0^1 a_4(y,t)Y_j dy,\ j=\overline{-N,N},$$

then the problem (21)–(22) will be equivalent to the following Cauchy problem for a finite system of the following nonlinear ordinary differential equations

$$C_N'(t)=W_N^{-1}\left[-a_2(t)A_NC_N(t)+P_N(t)+G_N(t)\right],\ \ C_N(t_0)=0. \tag{23}$$

Since $q(y,t)\in L^2(Q_{yt})$, then $q_j(t)$ is a square-integrable function, and the function $p_j(t)$ well defined. In this regard, the Cauchy problem (23) is uniquely solvable on some interval $[t_0,T_0]$, where $T_0\le T$. However, because of a priori estimates from Lemmas 2–4 in Section 2.3, we get that the solution $C_N(t)$ can be continued to a finite time T.

Thus, for each fixed finite N we find the functions $C_N(t)=\{c_{Nj}(t),\ j=\overline{-N,N}\}$ as a solution to the Cauchy problem (23), and with them the unique approximate solution $w_N(y,t)$ of problem (16)–(17). Lemma 1 is fully proved.

2.3 A priori estimates for the approximate solution (20) of problem (21)–(22)

To further prove Theorem 3, we need to prove a number of lemmas on a priori estimates.

Lemma 2 *There is a positive constant K_1, independent of N, that for all $t\in(t_0,T]$ there is an estimate*

$$\|w_N(y,t)\|^2_{L^2(0,1)}+\varepsilon_3\int\limits_{t_0}^t \|\partial_y w_N(y,\tau)\|^2_{L^2(0,1)}d\tau\le K_1. \tag{24}$$

Proof. Multiplying (21) by $c_{Nj}(t)$, and summing the result by j from $-N$ to N, we get

$$\frac{1}{2}\frac{d}{dt}\int\limits_0^1|w_N(y,t)|^2dy+a_2(t)\int\limits_0^1|\partial_y w_N(y,t)|^2dy=-\int\limits_0^1 a_3(y,t)\partial_y w_N(y,t)w_N(y,t)dy$$

$$-w_N(1,t)\int_0^1 a_4(y,t)w_N(y,t)dy+\int\limits_0^1 q(y,t)w_N(y,t)dy. \tag{25}$$

Now, integrating (25) by t from t_0 to t and using ε-Cauchy inequalities

$$-\int\limits_0^1 a_3(y,t)\partial_y w_N(y,t)w_N(y,t)dy\le\frac{\varepsilon_3}{4}\|\partial_y w_N(y,t)\|^2_{L^2(0,1)}+\frac{\varepsilon_5^2}{\varepsilon_3}\|w_N(y,t)\|^2_{L^2(0,1)},$$

$$\int\limits_0^1 q(y,t)w_N(y,t)dy \le \frac{1}{2}\|q(y,t)\|^2_{L^2(0,1)} + \frac{1}{2}\|w_N(y,t)\|^2_{L^2(0,1)},$$

$$-w_N(1,t)\int_0^1 a_4(y,t)w_N(y,t)dy \le \varepsilon_6|w_N(1,t)|\|w_N(y,t)\|_{L^2(0,1)}$$

$$\le \varepsilon_6\|w_N(y,t)\|_{L^\infty(0,1)}\|w_N(y,t)\|_{L^2(0,1)} \le \varepsilon_6\Bigg(\|w_N(y,t)\|_{L^2(0,1)}$$

$$+\|\partial_y w_N(y,t)\|_{L^2(0,1)}\Bigg)\|w_N(y,t)\|_{L^2(0,1)}$$

$$\le \left(\varepsilon_6 + \frac{\varepsilon_6^2}{\varepsilon_3}\right)\|w_N(y,t)\|^2_{L^2(0,1)} + \frac{\varepsilon_3}{4}\|\partial_y w_N(y,t)\|^2_{L^2(0,1)},$$

we will have

$$\|w_N(y,t)\|^2_{L^2(0,1)} + \varepsilon_3\int\limits_{t_0}^t \|\partial_y w_N(y,\tau)\|^2_{L^2(0,1)}d\tau$$

$$\le A_0\int\limits_{t_0}^t \|w_N(y,\tau)\|^2_{L^2(0,1)}d\tau + \int\limits_{t_0}^T \|q(y,\tau)\|^2_{L^2(0,1)}d\tau, \tag{26}$$

where $A_0 = \left(\frac{2\varepsilon_5^2+2\varepsilon_6^2}{\varepsilon_3} + 2\varepsilon_6 + 1\right)$. By using estimate (26) we establish the required estimate of the Lemma 2 where K_1 depends only on A_0 and q, and does not depend on N.

Lemma 3 *For a positive constant K_2, independent of N, for all $t \in (t_0, T]$ there is an inequality:*

$$\|\partial_y w_N(y,t)\|^2_{L^2(0,1)} + \varepsilon_3\int\limits_{t_0}^t \|\partial_y^2 w_N(y,\tau)\|^2_{L^2(0,1)}d\tau \le K_2. \tag{27}$$

Proof. Considering equality

$$\sum_{j=-N}^{N} c_{Nj}\lambda_j^2 Y_j(y) = -\sum_{j=-N}^{N} c_{Nj}\partial_y^2 Y_j(y) = -\partial_y^2 w_N(y,t),$$

which follows from (18), (20) and multiplying the equality (21) by $c_{Nj}\lambda_j^2$, then summing the result by j from $-N$ to N, we get

$$\frac{1}{2}\frac{d}{dt}\|\partial_y w_N(y,t)\|^2_{L^2(0,1)} + a_2(t)\|\partial_y^2 w_N(y,t)\|^2_{L^2(0,1)} =$$

$$= a_1(t)\left(w_N(y,t)\partial_y w_N(y,t), \partial_y^2 w_N(y,t)\right) + \left(a_3(y,t)\partial_y w_N(y,t), \partial_y^2 w_N(y,t)\right)$$

$$+w_N(1,t)\left(a_4(y,t), \partial_y^2 w_N(y,t)\right) - \left(q(y,t), \partial_y^2 w_N(y,t)\right)$$

$$\leq \varepsilon_2 \left| \Big(w_N(y,t)\partial_y w_N(y,t), \partial_y^2 w_N(y,t) \Big) \right| + \varepsilon_5 \left| \big(\partial_y w_N(y,t), \partial_y^2 w_N(y,t) \big) \right|$$

$$+\varepsilon_6 |w_N(1,t)| \|\partial_y^2 w_N(y,t)\|_{L^2(0,1)} + \left| \big(q(y,t), \partial_y^2 w_N(y,t) \big) \right|. \tag{28}$$

First, we will consider estimates of nonlinear terms in (28). We have

$$\left| \Big(w_N(y,t)\partial_y w_N(y,t), \partial_y^2 w_N(y,t) \Big) \right|$$

$$\leq \|w_N(y,t)\|_{L^4(0,1)} \|\partial_y^2 w_N(y,t)\|_{L^2(0,1)} \|\partial_y w_N(y,t)\|_{L^4(0,1)}$$

$$\leq \|w_N(y,t)\|_{L^4(0,1)} \|\partial_y w_N(y,t)\|_{H^1(0,1)} \|\partial_y w_N(y,t)\|_{L^4(0,1)}. \tag{29}$$

Further, taking into account the interpolation inequality from ([35], Theorems 5.8–5.9, p.140–141), $\forall\, \partial_y w_N(y,t) \in H^1(0,1)$

$$\varepsilon_2 \|\partial_y w_N(y,t)\|_{L^4(0,1)} \leq C \|\partial_y w_N(y,t)\|_{H^1(0,1)}^{1/2} \|\partial_y w_N(y,t)\|_{L^2(0,1)}^{1/2},$$

from (29) we will get

$$\varepsilon_2 \left| \Big(w_N(y,t)\partial_y w_N(y,t), \partial_y^2 w_N(y,t) \Big) \right|$$

$$\leq C \|w_N(y,t)\|_{L^4(0,1)} \|\partial_y w_N(y,t)\|_{H^1(0,1)}^{3/2} \|\partial_y w_N(y,t)\|_{L^2(0,1)}^{1/2}$$

$$\leq \frac{\varepsilon_3}{8} \|\partial_y^2 w_N(y,t)\|_{L^2(0,1)}^2 + \left[\frac{\varepsilon_3}{8} + A_1 \|w_N(y,t)\|_{L^4(0,1)}^4 \right] \|\partial_y w_N(y,t)\|_{L^2(0,1)}^2, \tag{30}$$

where $A_1 = \frac{54}{\varepsilon_3^3} C^4$ and the boundedness of $\|w_N(y,t)\|_{L^4(0,1)}$ follows from Lemma 2:

$$\|w_N(y,t)\|_{L^4(0,1)} \leq (T-t_0)^{1/4} \|w_N(y,t)\|_{L^\infty(0,1)} \leq C_0 \|w_N(y,t)\|_{H^1(0,1)} \leq K_1.$$

Here we used Young's inequality $(r^{-1} + s^{-1} = 1)$:

$$|UV| = \left| \big(\theta^{1/r} U\big) \left(\theta^{1/s} \frac{V}{\theta} \right) \right| \leq \frac{\theta}{r} |U|^r + \frac{\theta}{s\theta^s} |V|^s,$$

where $\theta = \frac{\varepsilon_3}{6}$, $r = \frac{4}{3}$, $s = 4$,

$$U = \|\partial_y w_N(y,t)\|_{H^1(0,1)}^{3/2}, \quad V = C \|w_N(y,t)\|_{L^4(0,1)} \|\partial_y w_N(y,t)\|_{L^2(0,1)}^{1/2}.$$

Next, for the last three terms of (28) we will have:

$$\varepsilon_5 \left| \Big(\partial_y w_N(y,t), \partial_y^2 w_N(y,t) \Big) \right| \leq \frac{\varepsilon_3}{8} \|\partial_y^2 w_N(y,t)\|_{L^2(0,1)}^2 + \frac{2\varepsilon_5^2}{\varepsilon_3} \|\partial_y w_N(y,t)\|_{L^2(0,1)}^2, \tag{31}$$

$$\varepsilon_6 |w_N(1,t)| \|\partial_y^2 w_N(y,t)\|_{L^2(0,1)} \leq \frac{2\varepsilon_6^2}{\varepsilon_3} |w_N(1,t)|^2 + \frac{\varepsilon_3}{8} \|\partial_y^2 w_N(y,t)\|_{L^2(0,1)}$$

$$\leq \left(\frac{K_1}{\widetilde{K}} \right)^2 \frac{2\varepsilon_6^2}{\varepsilon_3} + \frac{\varepsilon_3}{8} \|\partial_y^2 w_N(y,t)\|_{L^2(0,1)}, \tag{32}$$

$$\left|\left(q(y,t), \partial_y^2 w_N(y,t)\right)\right| \le \frac{\varepsilon_3}{8}\|\partial_y^2 w_N(y,t)\|^2_{L^2(0,1)} + \frac{2}{\varepsilon_3}\|q(y,t)\|^2_{L^2(0,1)}. \tag{33}$$

The estimate (32) follows from Lemma 2 and inequalities

$$\widetilde{K}|w_N(1,t)| \le \widetilde{K}\|w_N(y,t)\|_{L^2(t_0,T;L^\infty(0,1))} \le \|w_N(y,t)\|_{L^2(t_0,T;H^1(0,1))} \le K_1, \tag{34}$$

where K_1 is the constant from Lemma 2.

From (28), (30)–(33) we will have

$$\frac{d}{dt}\|\partial_y w_N(y,t)\|^2_{L^2(0,1)} + \varepsilon_3\|\partial_y^2 w_N(y,t)\|^2_{L^2(0,1)} \le A_2\|q(y,t)\|^2_{L^2(0,1)}$$

$$+\left[\frac{\varepsilon_3}{4} + 2A_1\|w_N(y,t)\|^4_{L^4(0,1)} + \frac{4\varepsilon_5^2}{\varepsilon_3}\right]\|\partial_y w_N(y,t)\|^2_{L^2(0,1)} + K_0, \tag{35}$$

where $A_2 = \frac{4}{\varepsilon_3}$, $K_0 = \left(\frac{K_1}{\widetilde{K}}\right)^2\frac{2\varepsilon_6^2}{\varepsilon_3}$, or, integrating (35) with respect to t from t_0 to t, we will get

$$\|\partial_y w_N(y,t)\|^2_{L^2(0,1)} + \varepsilon_3\int_{t_0}^{t}\|\partial_y^2 w_N(y,\tau)\|^2_{L^2(0,1)}d\tau \le A_2\|q(y,t)\|^2_{L^2(Q_{yt})}$$

$$+\int_{t_0}^{t} A_3(\tau)\|\partial_y w_N(y,\tau)\|^2_{L^2(0,1)}d\tau + K_0(T-t_0), \tag{36}$$

where $A_3(t) = \frac{\varepsilon_3}{4} + 2A_2\|w_N(y,t)\|^4_{L^4(0,1)} + \frac{4\varepsilon_5^2}{\varepsilon_3}$ and $A_3(t) \in L^\infty(t_0,T)$.

From the inequality (36), similarly as in the proof of Lemma 2, we obtain the desired estimate (26) where K_2 depends on A_2, $A_3(t)$, $q(y,t)$, K_0, and does not depend on N. Lemma 3 is fully proved.

Lemma 4 *For a positive constant K_3, independent of N, for all $t \in (t_0,T]$ there is an inequality:*

$$\|\partial_t w_N(y,t)\|^2_{L^2(Q_{yt})} \le K_3. \tag{37}$$

Proof. Let us satisfy the equation (12) to the approximate solution $w_N(y,t)$:

$$\partial_t w_N + a_1(t)w_N\partial_y w_N - a_2(t)\partial_y^2 w_N + a_3(y,t)\partial_y w_N + a_4(y,t)w_N(1,t) = q, \tag{38}$$

From the equation (38) we obtain

$$\|\partial_t w_N\|_{L^2(Q_{yt})} \le \varepsilon_2\|w_N\partial_y w_N\|_{L^2(Q_{yt})} + \varepsilon_4\|\partial_y^2 w_N\|_{L^2(Q_{yt})} + \varepsilon_5\|\partial_y w_N\|_{L^2(Q_{yt})}$$

$$+\varepsilon_6\frac{K_1}{\widetilde{K}}\|w_N(y,t)\|_{L^2(t_0,T;H^1(0,1))} + \|q\|_{L^2(Q_{yt})}, \tag{39}$$

Hence, given the Lemmas 2 and 3, we get

$$\|w_N\partial_y w_N\|_{L^2(Q_{yt})} = \left(\int_{t_0}^{T}\int_0^1 |w_N|^2|\partial_y w_N|^2 dydt\right)^{1/2} \le \left(\int_{Q_{yt}} |w_N|^4 dQ_{yt}\right)^{1/2}$$

$$\cdot\left(\int_{Q_{yt}} |\partial_y w_N|^4 dQ_{yt}\right)^{1/2} \le C\|w_N\|^2_{L^4(Q_{yt})}\|\partial_y w_N\|_{L^2(t_0,T;H^1(0,1))}\|\partial_y w_N\|_{L^2(t_0,T;L^2(0,1))}$$

$$\le \frac{C_1}{2}\left[\|\partial_y w_N\|^2_{L^2(t_0,T;H^1(0,1))} + \|\partial_y w_N\|^2_{L^2(t_0,T;L^2(0,1))}\right] \le C_1K_2. \tag{40}$$

The estimate (37) follows from (39), (40) and from the statements of Lemmas 2 and 3. Lemma 4 is fully proved.

2.4 Unique solvability of the problem (12)–(14)

Lemmas 2–4 show that sequences of Galerkin approximations

$$\{w_N(y,t),\ N = 0,1,2,...\} \text{ and } \{\partial_t w_N(y,t),\ N = 0,1,2,...\}$$

bounded in spaces

$$L^\infty(t_0,T;H^1(0,1)) \cap L^2(t_0,T;H^2_{per}(0,1)) \text{ and } L^2(t_0,T;L^2(0,1)),$$

respectively. Thus, we can extract weakly converging subsequences (for which we will keep the former index designations N):

$$w_N(y,t) \to w(y,t) \ \text{ weakly in } \ H^1(t_0,T;L^2(0,1)) \cap L^2(t_0,T;H^2_{per}(0,1)), \tag{41}$$

$$w_N(y,t) \to w(y,t) \ \text{ strongly in } \ L^2(t_0,T;L^2(0,1)) \ \text{ and a.e. in } \ Q_{yt}, \tag{42}$$

$$w_N(1,t) \to w(1,t) \ \text{ strongly in } \ L^2(t_0,T). \tag{43}$$

Lemma 5 *Let conditions* (15) *and* $q \in L^2(Q_{yt})$ *be satisfied. Then initial boundary value problem* (12)–(14) *has a weak solution in space* $H^{2,1}_{per}(Q_{yt})$.

Proof. Let us introduce the notation $v_j(y,t) = \varphi(t)Y_j(y)$, where $Y_j(y) \in H^2_{per}(0,1)$, $\varphi(t) \in C^\infty([t_0,T])$. Now, multiplying the integral identity (21) by the function $\varphi(t) \in C^\infty([t_0,T])$ and integrating the result with respect to t from t_0 to T, we will have

$$\int_{t_0}^{T}\int_0^1 \big[\partial_t w_N + a_1(t)w_N\partial_y w_N - a_2(t)\partial_y^2 w_N + a_3(y,t)\partial_y w_N$$

$$+w_N(1,t)a_4(y,t)\big]\, v_j\, dy\, dt = \int_{t_0}^{T}\int_0^1 q v_j\, dy\, dt,\ \ \forall\, \varphi(t) \in L^2(t_0,T),\ \forall\, j = \overline{-N,N}, \tag{44}$$

since the set of all linear combinations of $v_j(y,t)$ is dense in $L^2(t_0,T;H^2_{per}(0,1))$, then the integral identity (44) can be rewritten as

$$\int_{t_0}^{T}\int_0^1 \left[\partial_t w_N + a_1(t) w_N \partial_y w_N - a_2(t)\partial_y^2 w_N + a_3(y,t)\partial_y w_N\right.$$

$$\left.+w_N(1,t)a_4(y,t)\right] v\,dy\,dt = \int_{t_0}^{T}\int_0^1 qv\,dy\,dt,\ \ \forall\, v(y,t)\in L^2(t_0,T;H^2_{per}(0,1)). \tag{45}$$

In the integral identity (45) we pass to the limit at $N\to\infty$. In the expressions corresponding to the linear terms of the equation (12), the transition to the limits is carried out according to the ratios (41) and (43). As for the nonlinear term, here we have the following:

$$\int_{t_0}^{T}\int_0^1 a_1(t) w_N(y,t)\partial_y w_N(y,t) v(y,t)\,dy\,dt = \int_{t_0}^{T} a_1(t)\int_0^1 [w_N(y,t)$$

$$-w(y,t)]\partial_y w_N(y,t) v(y,t)\,dy\,dt + \int_{t_0}^{T} a_1(t)\int_0^1 w(y,t)\partial_y w_N(y,t) v(y,t)\,dy\,dt$$

$$\to \int_{t_0}^{T} a_1(t)\int_0^1 w(y,t)\partial_y w(y,t) v(y,t)\,dy\,dt, \tag{46}$$

since according to (41) and (42) there is a limit ratio

$$\int_{t_0}^{T} a_1(t)\int_0^1 [w_N(y,t)-w(y,t)]\partial_y w_N(y,t) v(y,t)\,dy\,dt \to 0.$$

So, passing to the limit at $N\to\infty$ in the integral identity (45) and taking into account the limit ratio (46), as well as the initial condition (22), we get

$$\int_{t_0}^{T}\int_0^1 \left[\partial_t w + a_1(t) w \partial_y w - a_2(t)\partial_y^2 w + a_3(y,t)\partial_y w\right.$$

$$\left.+w(1,t)a_4(y,t)\right] v\,dy\,dt = \int_{t_0}^{T}\int_0^1 qv\,dy\,dt,\ \ \forall\, v(y,t)\in L^2(t_0,T;H^2_{per}(0,1)), \tag{47}$$

$$\int_0^1 w(y,t_0)\psi(y)\,dy = 0,\ \ \forall\,\psi\in L^2(0,1). \tag{48}$$

Thus, from (47)–(48) we get that the weak limit function $w(y,t)$ satisfies the equation (12), boundary conditions (13) and initial condition (14). Lemma 5 is fully proved.

Lemma 6 *Under the conditions of Lemma 5 the solution $w \in H^{2,1}_{per}(Q_{yt})$ to initial boundary value problem* (12)–(14) *is unique.*

Proof. Let the initial boundary value problem (12)–(14) has two different solutions $w^{(1)}(y,t)$ and $w^{(2)}(y,t)$. Then their difference $w(y,t) = w^{(1)}(y,t) - w^{(2)}(y,t)$ will satisfy the following problem:

$$\partial_t w + a_1(t) w \partial_y w^{(1)} + a_1(t) w^{(2)} \partial_y w - a_2(t) \partial_y^2 w$$

$$+a_3(y,t)\partial_y w + a_4(y,t) w(1,t) = 0, \tag{49}$$

$$\partial_y^j w(0,t) = \partial_y^j w(1,t), \;\; j = 0,\, 1;\, t \in (t_0, T), \tag{50}$$

$$w(y,t_0) = 0, \;\; 0 < y < 1. \tag{51}$$

According to Lemmas 2 and 3 we have

$$w^{(i)}(y,t) \in L^\infty(t_0,T; H^1(0,1)) \cap L^2(t_0,T; H^2_{per}(0,1)), \;\; i = 1,2. \tag{52}$$

Multiplying the equation (49) by the function $w(y,t)$ scalarly in $L^2(0,1)$ and taking into account (50)–(52), we get

$$\frac{1}{2}\frac{d}{dt}\|w(y,t)\|^2_{L^2(0,1)} + a_2(t)\,\|\partial_y w(y,t)\|^2_{L^2(0,1)} = -\int_0^1 a_3(y,t) w \partial_y w dy$$

$$-a_1(t)\int_0^1 \left[w^2 \partial_y w^{(1)} + w^{(2)} w \partial_y w\right]\, dy - w(1,t)\int_0^1 a_4(y,t) w dy. \tag{53}$$

Let us estimate the right-hand part of (53). According to (52) and by Lemma 2 we obtain:

$$-\int_0^1 a_3(y,t) w \partial_y w dy \le \frac{\varepsilon_3}{8}\|\partial_y w(y,t)\|^2_{L^2(0,1)} + \frac{2\varepsilon_5^2}{\varepsilon_3}\|w(y,t)\|^2_{L^2(0,1)}, \tag{54}$$

$$-a_1(t)\int_0^1 \left[w^2 \partial_y w^{(1)} + w^{(2)} w \partial_y w\right]\, dy = -a_1(t)\int_0^1 \left[-2w^{(1)} w \partial_y w + w^{(2)} w \partial_y w\right]\, dy$$

$$\le \frac{2\varepsilon_2^2}{\varepsilon_3}\left[2\|w^{(1)}\|_{L^\infty(0,1)} + \|w^{(2)}\|_{L^\infty(0,1)}\right]^2 \|w(y,t)\|^2_{L^2(0,1)} + \frac{\varepsilon_3}{8}\|\partial_y w(y,t)\|^2_{L^2(0,1)}$$

$$\le A_4\|w(y,t)\|^2_{L^2(0,1)} + \frac{\varepsilon_3}{8}\|\partial_y w(y,t)\|^2_{L^2(0,1)}, \tag{55}$$

where $A_4 = \frac{2\varepsilon_2^2}{\varepsilon_3}\left[2\|w^{(1)}\|_{L^\infty(0,1)} + \|w^{(2)}\|_{L^\infty(0,1)}\right]^2$.

$$-w(1,t)\int_0^1 a_4(y,t) w dy \le \left(\varepsilon_6 + \frac{\varepsilon_6^2}{\varepsilon_3}\right)\|w(y,t)\|^2_{L^2(0,1)} + \frac{\varepsilon_3}{4}\|\partial_y w(y,t)\|^2_{L^2(0,1)} \tag{56}$$

Based on relations (53)–(56) we get

$$\frac{d}{dt}\|w(y,t)\|^2_{L^2(0,1)} + \varepsilon_3\,\|\partial_y w(y,t)\|^2_{L^2(0,1)} \le A_5\|w(y,t)\|^2_{L^2(0,1)} \;\; \forall\, t \in [t_0, T],$$

where $A_5 = \left(\frac{4\varepsilon_5^2+2\varepsilon_6^2}{\varepsilon_3} + 2A_4 + 2\varepsilon_6 \right).$

Hence, applying the Gronwall inequality, we get:

$$\|w(y,t)\|^2_{L^2(0,1)} \equiv 0, \ \forall\, t \in [t_0, T].$$

This means that $w^{(1)}(y,t) \equiv w^{(2)}(y,t)$ in $L^2(Q_{yt})$, i.e. the solution to initial boundary value problem (12)–(14) can only be one. Lemma 6 is fully proved.

Thus, from the statements of Lemmas 5 and 6 the validity of Theorem 3 follows. Theorem 3 is fully proved. Theorem 3 will also be used in the following sections when proving Theorems 1 and 2.

3 Proof of Theorems 1 and 2

Using reversible transformation of independent variables

$$y = y(x,t) = \frac{x}{t}, \ \ t = t; \ \ x = x(y,t) = yt, \ \ t = t; \tag{57}$$

we go from (x,t) to (y,t). In this case, the domain Q_{xt} will be transformed into a rectangular domain $Q_{yt} = \{y,t: \ 0 < y < 1, \ t_0 < t < T < \infty, \, t_0 > 0\}$.

The problem (9)-(11) takes the following form:

$$\partial_t w + \frac{1}{t} w \partial_y w - \frac{\nu}{t^2}\partial_y^2 w - \frac{y}{t}\partial_y w + h(t)\tilde{f}(y,t)w(1,t) = \tilde{q}(y,t), \tag{58}$$

$$w(0,t) = w(t,t), \ \partial_y w(0,t) = \partial_y w(t,t), \ \ t \in (t_0, T), \tag{59}$$

with initial condition

$$w(y,t_0) = 0, \ \ y \in (0,1), \tag{60}$$

where $w(y,t) = u(x(y,t),t)$, $\tilde{f}(y,t) = f(x(y,t))$, $\tilde{q}(y,t) = \tilde{f}(y,t)g(t) = f(x(y,t))g(t)$. Note that according to condition (7) $\tilde{f}(y,t) \in L^\infty(t_0,T; L^2(0,1))$.

Thus initial boundary value problem (58)–(60) is a special case of the first auxiliary problem (12)–(14), where

$$a_1(t) = \frac{1}{t}, \ a_2(t) = \frac{\nu}{t^2}, \ a_3(y,t) = -\frac{y}{t}, \ a_4(y,t) = h(t)\tilde{f}(y,t),$$

and the conditions (15) are provided. Therefore, as a consequence of the Theorem 3 we get

Theorem 4 *Let the conditions* (7) *and* $\tilde{q} \in L^2(Q_{yt})$ *be satisfied. Then initial boundary value problem* (58)–(60) *is uniquely solvable in space*

$$w(y,t) \in H^{2,1}_{per}(Q_{yt}).$$

Further, given the correspondence of spaces in domains Q_{xt} and Q_{yt}:

$$\tilde{q} \in L^2(Q_{yt}) \Longleftrightarrow f(x)g(t) \in L^2(Q_{xt}),$$

$$w \in H^{2,1}_{per}(Q_{yt}) = L^2(t_0, T; H^2_{per}(0,1)) \cap H^1(t_0, T; L^2(0,1)) \Longleftrightarrow u \in H^{2,1}_{per}(Q_{xt})$$

$$= L^2(t_0, T; H^2_{per}(0,t)) \cap H^1(t_0, T; L^2(0,t)),$$

we get the validity of Theorem 1.

Thus, we have proved the Theorem 1. Proof of the Theorem 2 follows from the proof of Theorem 1, where $u(x,t)$ is the solution to initial boundary value problem (9)–(11), $\lambda(t)$ is found by the formula (8). Thus, the coefficient inverse problem (3)–(6) is completely solved, the unknown coefficient $\lambda(t)$ is found.

4 Graphs of the solution and the desired function

Below we present graphs of approximate solutions of initial-boundary value problem (58)–(60), original inverse problem (3)–(6) and $\lambda(t)$, when $f(x) = 1.5 + \cos 2\pi x$, $E(t) = \cos 2\pi t$, $T = 3$, $t_0 = 1$. The solution of initial-boundary value problem (58)–(60) is found according to the formula (20), for $N = 0$ and $N = 1$, where $C_{Nj}(t)$ are solutions to the Cauchy problem (21)–(22) with appropriate coefficients, $Y_j(y)$ are solutions to the spectral problem (18)–(19). The solution to (3)–(6) is found as the solution to the initial boundary value problem for the loaded Burgers equation (9)–(11) from the solution of initial-boundary value problem (58)–(60) using (57), $\lambda(t)$ is determined by formula (8).

For problem (58)–(60) in Figure 1, the domain of change of variables (y,t) is the rectangle $Q_{yt} = \{y, t|\ 0 < y < 1,\ 1 < t < 3\}$, and the solution surface $w(y,t)$ is built over it. For problem (9)–(11) in Figure 2, the domain of change of variables (x,t) is the trapezoid $Q_{xt} = \{x, t \mid 0 < x < t,\ 1 < t < 3\}$, and the solution surface $u(x,t)$ is built over it. Figure 3 shows the graph of the desired function $\lambda(t)$. Thus, Figures 2-3 show graphs of the solution of the original inverse problem.

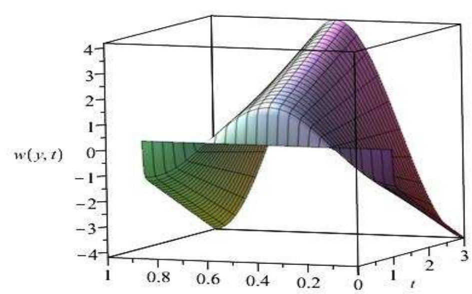

Figure 1: Graphs of solution to problem (58)–(60).

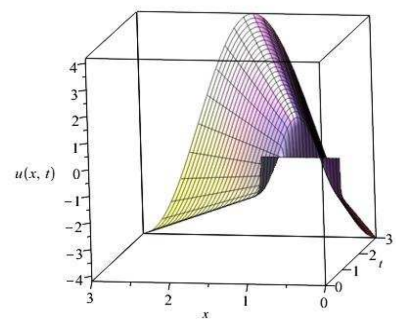

Figure 2: Graphs of solution to problem (9)–(11).

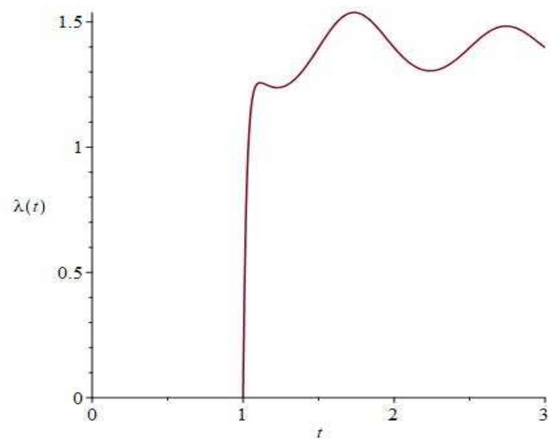

Figure 3: Graph of function $\lambda(t)$.

Conclusion

The paper establishes theorems on the solvability in Sobolev classes of the inverse problem for the Burgers equation with periodic boundary and an integral overdetermination and the associated initial boundary value problem for the loaded Burgers equation. Graphs of solution $w(y,t)$ to the initial boundary value problem for the loaded Burgers equation and of solution $\{u(x,t),\ \lambda(t)\}$ to the original inverse problem are presented.

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