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# A NUMERICAL SCHEME ON S-MESH FOR THE SINGULARLY PERTURBED INITIAL BOUNDARY VALUE SOBOLEV PROBLEMS WITH LARGE TIME DELAY

The purpose of this article is to provide a numerical method for time delay singularly perturbed Sobolev type equations. First, asymptotic estimates for the Sobolev problem solution with singular perturbation and delay parameters were obtained. This estimate showed that the solution depends on the initial data. It is constructed and examined to solve this problem using a finite difference technique on a specific piecewise uniform mesh (Shishkin mesh)whose solution converges pointwise independent of the singular perturbation parameter. A discrete norm was used to investigate the stability of difference schemes. It is showed that the completely discrete scheme converges with order $O\left(\tau^2 + N_l^{-2}\ln^2 N_l\right)$ in both space and time, independent of the perturbation parameter. Finally, with a test problem and numerical experiments, the theoretical accuracy and computational effectiveness of the proposed methods are further testified.
**Key words**: Delayed partial differential equation; Finite difference method; Shishkin mesh; Singular perturbation; Sobolev problem.

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**Кешiгуi үлкен болатын Соболевтiң сингулярлы ауытқыған бастапқы-шекаралық есебiнiң S торындағы сандық сұлбасы**

Бұл мақаланың мақсаты кешiгуi бар сингулярлы ауытқыған Соболев типтi теңдеулердiң сандық әдiсiн беру болып табылады. Бiрiншiден, сингулярлы ауытқыған мен кешiгу параметрлерi бар Соболев есебiн шешудiң асимптотикалық бағалаулары алынды. Бұл бағалау шешiмнiң бастапқы деректерге тәуелдi екенiн көрсеттi. Бұл есептi нақты кесiндiлiк-бiртектi торда (Шишкин торында) шектi-айырымдық әдiсiмен шешу құрастырылған және зерттелген, оның шешiмi сингулярлық күйзелiс параметрiне қарамастан нүктелiк мағынада жинақталады. Айырымдық сұлбалардың орнықтылығын зерттеу үшiн дискреттi норма қолданылды. Толық дискреттi сұлбаның кеңiстiкте де және уақыт бойынша да, сонымен қатар, сингулярлы ауытқу параметрiне қарамастан $O\left(\tau^2 + N_l^{-2}\ln^2 N_l\right)$ ретiмен жинақталатыны көрсетiлген. Соңында, сынақ есебiнiң және сандық тәжiрибелердiң көмегiмен ұсынылған әдiстердiң теориялық дәлдiгi мен есептеу тиiмдiлiгi қосымша расталды.

**Түйiн сөздер**: Кешiктiрiлген дербес дифференциалдық теңдеу, ақырлы айырымдық әдiсi, Шишкин торы, сингулярлық ауытқу; Соболев есебi.

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**Численная схема на S-сетке для сингулярно возмущенного начально-краевой задачи Соболева с большим запаздыванием**

Целью данной статьи является предоставление численного метода для сингулярно возмущенных уравнений соболевского типа с запаздыванием. Во-первых, получено асимптотические оценки решения для задачи Соболева с сингулярными параметрами возмущения и запаздывания. Эта оценка показала, что решение зависит от начальных данных. Построено и исследовано решение этой задачи методом конечных разностей на конкретной кусочно-равномерной сетке (на сетке Шишкина), решение которой сходится поточечно независимо от параметра сингулярного возмущения. Для исследования устойчивости разностных схем использовалась дискретная норма. Показано, что полностью дискретная схема сходится с порядком $O\left(\tau^2+N_l^{-2}\ln^2 N_l\right)$ как в пространстве, так и во времени, независимо от параметра возмущения. Наконец, с помощью тестовой задачи и численных экспериментов дополнительно подтверждается теоретическая точность и вычислительная эффективность предложенных методов.

**Ключевые слова**: Запаздывающее уравнение в частных производных, метод конечных разностей, сетка Шишкина, сингулярное возмущение, Проблема Соболева.

## 1 Introduction

In this study, the following linear singularly perturbed delay initial-boundary value Sobolev problem is taken into account in the domain $\bar{D}=\bar{\Omega}\times[0,T];\ \bar{\Omega}=[0,l]\,,\ \Omega=(0,l),\ D=\Omega\times(0,T]$:

$$Lu\equiv L_1\left[\frac{\partial^2 u(x,t)}{\partial t^2}\right]+L_2\left[u(x,t)\right]+c\left(t\right)u\left(x,t-r\right)=f\left(x,t\right),\qquad (x,t)\in D, \tag{1.1}$$

$$u\left(x,t\right)=\zeta\left(x,t\right),\quad (x,t)\in\bar{\Omega}\times[-r,0]\,, \tag{1.2}$$

$$\frac{\partial u}{\partial t}\left(x,0\right)=\psi\left(x\right),\ x\in[0,l]\,, \tag{1.3}$$

$$u(0,t)=u(l,t)=0,\ t\in(0,T]\,, \tag{1.4}$$

where

$$L_1\left[\frac{\partial^2 u(x,t)}{\partial t^2}\right]\equiv-\varepsilon\frac{\partial^4 u(x,t)}{\partial x^2\partial t^2}+a\left(x\right)\frac{\partial^2 u(x,t)}{\partial t^2},$$

$$L_2\left[u(x,t)\right]\equiv-\varepsilon\frac{\partial^2 u(x,t)}{\partial x^2}+b\left(x,t\right)u\left(x,t\right),$$

and $0<\varepsilon\ll 1$ is a perturbation parameter; $a$, $b$, $f$, $\zeta$and $\psi$ are sufficiently smooth functions, $r>0$ is delay parameter and $a(x)\geq\alpha>0$.

These types of problems arise in a variety of areas of mathematical physics and fluid mechanics. Sobolev type equations are also known as pseudoparabolic equations in some conditions. They are applied in the investigation of ion-acoustic waves in plasma, waves in transmission lines, and other physical models ( [7], [15], [16], [20]).

It has been discussed in ( [3], [11]- [14]) and ( [19]), how to solve equations of this type numerically in the usual conditions. The numerical explo Due to the boundary-layer behavior of the exact solutions, numerical simulations of singular perturbation scenarios has never been an easy process. Applied mathematicians are focusing more attention to the subject of singular perturbations, nonetheless ( [10]). Degenerate equations of the Sobolev type and equations of the Sobolev type with an alternating or non-invertible operator as the coefficient of the greatest derivative with respect to time were investigated from an abstract perspective in ( [10], [16]) . Many important consequences were obtained for systems of equations that were unresolved with respect to the highest derivative with respect to time ( [8], [14], [15]). In ( [13]), the local solubility of equations of the Sobolev type was taken into consideration.

In comparison to typical instantaneous singularly perturbed partial differential equations, singularly perturbed delay partial differential equations (DPDEs) offer more realistic models for phenomena in a variety of scientific domains that exhibit a time-lag or an aftereffect (PDEs). The derivatives of an unknown function are related to it by singularly perturbed PDEs when they are evaluated at the same instance. Singularly perturbed DPDEs, on the other hand, represent physical issues whose evolution depends not only on the system's current state but also on its historical development.

In recent decades, several authors have explored and developed singularly perturbed PDEs in great detail (see ( [4]) and the references therein). The theory and numerical solution of singularly perturbed DPDEs, however, are still in their development. The various approaches to the numerical solution of singularly perturbed delay initial-boundary value Sobolev problem can be found in ( [5], [6]). Additionally, the existence, uniqueness, and smoothness of exact solutions to the problems mentioned were investigated in ( [18], [22]).

The numerical solution of partial differential equations with two-time derivatives appearing as a small parameter in the highest order term is presented in this study. Since this type of problem's solutions quickly alter in the neighborhood of the endpoints, where they have an extremely steep gradient and their derivatives are unbounded. Standard discretization techniques are known to be unstable and to produce inaccurate results when used to solve singular perturbation problems for differential equations. Then, according to references ( [9]) and ( [9]), classical difference schemes do not converge to an exact solution.

The development of practical numerical methods for solving this type of equations, whose accuracy does not depend on the parameter $\varepsilon$, that is, methods that are uniformly convergent with respect to the parameter $\varepsilon$, is crucial for providing accurate results when the perturbation parameter is small ( [1], [2]). The method of approximation has the advantage that the base functions are selected such that, the error of the method on terms of higher order derivative is vanished. Because of this selection, there is no heavy conditions on the solution of the considered continuous problem under certain restrictions. By remainder term in integral form the continuity of the solution conditions are alleviates. Throughout this paper the constant $C$ will be the positive generic constant, independent of the singular perturbation parameter $\varepsilon$.

## 2 Estimation of Asymptotic Solution

In this section we show some useful asymptotic estimates of the exact solution of (1.1)-(1.4), which are essential for the rest of the article to analyze numerical solution.

**Lemma 1** *The solution $u(x,t)$ of the problem (1.1)-(1.4) satisfies the following inequality:*

$$\left\|\frac{\partial^{k+s}u}{\partial t^k \partial x^s}\right\| \leq C\left\{\varepsilon^{-s/2}\left[\|f\|^2_{L_2(D)} + \|\zeta(x,0)\|^2 + \varepsilon\,\|\zeta'(x,0)\|^2 + \|\psi\|^2 + \varepsilon\,\|\psi'\|^2 + \|u\,(x,t-r)\|^2\right] + s(s-1)\left[\|\zeta''(x,0)\|^2_{L_2(0,T)} + \|\psi''\|^2_{L_2(0,T)}\right]\right\}, \quad (2.1)$$

*where* $k, s = 0,1,2,\ 0 < t \le T$ *and* $\|.\| = \|.\|_{L_2(0,l)}$. *The constants* $C_i,\ i = 1,2,...$ *are independent of the small parameter as well as* $h$ *and* $\tau$ *in this part and the following sections.*

Refer ( [5]) for the idea of the proof.

**Lemma 2** *Under the assumptions* $a \in C^2[0,l]$, $b \in C_0^2\left(\bar{D}\right)$, $f \in C\left(\bar{D}\right)$ *and*

$$|a(0) - b(0,t)| \le C\varepsilon, \quad |a(l) - b(l,t)| \le C\varepsilon \tag{2.2}$$

*The following form represents the asymptotic expansion of the solution of problem (1.1)-(1.4) :*

$$u\left(x,t\right) = u_0\left(x,t\right) + \vartheta_0\left(\xi,t\right) + w_0\left(\eta,t\right) \tag{2.3}$$

$$+\sqrt{\varepsilon}\left[u_1\left(x,t\right) + \vartheta_1\left(\xi,t\right) + w_1\left(\eta,t\right)\right] + R^*\left(x,t\right),$$

*where the functions* $u_0\left(x,t\right)$, $u_1\left(x,t\right)$, $\vartheta_0\left(\xi,t\right)$, $w_0\left(\eta,t\right)$, $\vartheta_1\left(\xi,t\right)$, $w_1\left(\eta,t\right)$ *are as follows:*

$$\begin{cases} a(x)\frac{\partial^2 u_0}{\partial t^2} + b(x,t)u_0 + c(t)u_0\left(x,t-r\right) = f(x,t), \\ u_0(x,t-\tau) = \zeta(x,t),\ -r \le t \le 0; \frac{\partial u_0}{\partial t}(x,0) = \psi(x), \end{cases}$$

$$\begin{cases} a\left(x\right)\frac{\partial^2 u_1}{\partial t^2} + b\left(x,t\right)u_1 + c(t)u_1\left(x,t-r\right) = -\sqrt{\varepsilon}\left[\frac{\partial^4 u_0}{\partial t^2\partial x^2} + \frac{\partial^2 u_0}{\partial x^2}\right], \\ u_1\left(x,t\right) = 0, -r \le t \le 0;\ \frac{\partial u_1}{\partial t}\left(x,0\right) = 0, \end{cases}$$

$$\begin{cases} -\frac{\partial^2 \vartheta_0}{\partial t^2\partial \xi^2} + a\left(0\right)\frac{\partial^2 \vartheta_0}{\partial t^2} - \frac{\partial^2 \vartheta_0}{\partial \xi^2} + a(0)\vartheta_0 + c(t)\vartheta_0\left(x,t-r\right) = 0, \\ \vartheta_0(\xi,t) = 0,\ -r \le t \le 0;\ \frac{\partial \vartheta_0}{\partial t}(\xi,0) = 0, \\ \vartheta_0(0,t) = -u_0\left(0,t\right);\ \vartheta_0(\frac{l}{\sqrt{\varepsilon}},t) = 0, \end{cases}$$

$$\begin{cases} -\frac{\partial^4 w_0}{\partial t^2\partial \eta^2} + a(l)\frac{\partial^2 w_0}{\partial t^2} - \frac{\partial^2 w_0}{\partial \eta^2} + a(l)w_0 + c(t)w_0\left(x,t-r\right) = 0, \\ w_0(\eta,t) = 0, \quad -r \le t \le 0;\ \frac{\partial w_0}{\partial t}(\eta,0) = 0, \\ w_0(\frac{l}{\sqrt{\varepsilon}},t) = 0; w_0(0,t) = -u_0\left(l,t\right), \end{cases}$$

$$\begin{cases} -\frac{\partial^4 \vartheta_1}{\partial t^2\partial \xi^2} + a(0)\frac{\partial^2 \vartheta_1}{\partial t^2} - \frac{\partial^2 \vartheta_1}{\partial \xi^2} + a(0)\vartheta_1 + c(t)\vartheta_1\left(x,t-r\right) \\ = -\xi\frac{\partial b}{\partial x}(0,t)\vartheta_0 - \xi a'(0)\frac{\partial^2 \vartheta_0}{\partial t^2}, \\ \vartheta_1(\xi,t) = 0, \quad -r \le t \le 0;\ \frac{\partial \vartheta_1}{\partial t}(\xi,0) = 0, \\ \vartheta_1(0,t) = -u_1\left(0,t\right);\ \vartheta_1(\frac{l}{\sqrt{\varepsilon}},t) = 0, \end{cases}$$

$$\begin{cases} -\frac{\partial^4 w_1}{\partial t^2\partial \eta^2} + a(l)\frac{\partial^2 w_1}{\partial t^2} - \frac{\partial^2 w_1}{\partial \eta^2} + a(l)w_1 + c(t)w_1\left(x,t-r\right) \\ = -\eta\frac{\partial b}{\partial x}(l,t)\omega_0 - \eta a'(l)\frac{\partial^2 w_0}{\partial t^2}, \\ w_1(\eta,t) = 0, \quad -r \le t \le 0;\ \frac{\partial w_1}{\partial t}(\eta,0) = 0, \\ w_1(\frac{l}{\sqrt{\varepsilon}},t) = 0;\ w_1(0,t) = -u_1\left(l,t\right), \end{cases}$$

*where* $\xi = \frac{x}{\sqrt{\varepsilon}}, \eta = \frac{l-x}{\sqrt{\varepsilon}}$.
*It is possible to write the remaining term of the asymptotic expansion as follows:*

$$\varepsilon^s\left\|\frac{\partial^{k+s}R^*}{\partial t^k\partial x^s}\right\| \le C\varepsilon^{1-s/2} \quad k,\ s = 0,1,2.$$

One can find the similar proof given in ( [5]).

**Lemma 3** *Under the conditions (2.2), using*

$$\left|\frac{\partial^{k+s}\vartheta_0}{\partial t^k \partial x^s}\right| \le C\varepsilon^{-s/2}e^{-x\sqrt{a(0)/\varepsilon}}$$

*and*

$$\left|\frac{\partial^{k+s}w_0}{\partial t^k \partial x^s}\right| \le C\varepsilon^{-s/2}e^{-(l-x)\sqrt{a(l)/\varepsilon}}$$

*the following inequality can be expressed*

$$\left|\frac{\partial^{k+s}u}{\partial t^k \partial x^s}\right| \le C\left\{1+\varepsilon^{-s/2}\left[e^{-x\sqrt{a(0)/\varepsilon}}+e^{-(l-x)\sqrt{a(l)/\varepsilon}}\right]\right\}, \tag{2.4}$$

$$(x,t)\in \bar{D},\ k=0,\ 1,\ 2;\ s=0,\ 1,\ 2.$$

See ( [6]) for proof.

## 3 The Difference Scheme's Construction

Let a set of discretizing mesh nodes for the variable $D$ be given by

$$\omega=\omega_{N_l}\times\omega_{N_T},$$

with

$$\omega_{N_l}=\{0<x_1<x_2<...<x_{N_l}=l,\ h_i=x_i-x_{i-1}\},$$
$$\omega_{N_T}=\{t_j=j\tau,\ j=1,2,..,N_T-1;\tau=l/N_T\},$$

and

$$\omega_{N_l}^{+}=\omega_{N_l}\cup\{x_N=l\},\ \varpi_{N_l}=\omega_{N_l}\cup\{x_0=0,\ x_{N_l}=l\},$$
$$\varpi_{N_T}=\omega_{N_T}\cup\{t_0=0\},\ \varpi=\varpi_{N_l}\times\varpi_{N_T}.$$

At the conclusion of the section, the nonuniform mesh $\hat{\omega}_{N_T}$ specification will be given. For each mesh function $v_i=v(x_i)$ provided on $\varpi_{N_l}$, we define the following finite difference:.

$$v_{\bar{x}}=\frac{v_i-v_{i-1}}{h_i},\ v_x=\frac{v_{i+1}-v_i}{h_{i+1}}, v_{\hat{x}}=\frac{v_{i+1}-v_i}{\hbar_i}, v_{\bar{x}\hat{x},i}=\frac{v_{x,i}-v_{\bar{x},i}}{\hbar_i}$$

where $\hbar_i=\frac{1}{2}(h_i+h_{i+1})$.These are the inner products for the mesh functions $v_i$ and $w_i$ defined on $\varpi_{N_l}$ ( [15]):

$$(v,w)_0\equiv(v,w)_{0,\omega_{N_l}}=\sum_{i=1}^{N_l-1}hv_iw_i,$$

$$(v,w]\equiv(v,w)_{0,\omega_{N_l}^{+}}=\sum_{i=1}^{N_l}hv_iw_i,$$

$$[v,w)=\sum_{i=0}^{N_l-1}hv_iw_i.$$

Vanishing for $i=0$ and $i=N_l$ are stated with in addition to the norms for any mesh function $v_i$,

$$\|v\|_0^2 \equiv \|v\|_{0,w_{N_l}}^2 = (v,v)_0,$$

$$\|v\|_{\mathring{1}}^2 = (v_{\bar{x}}, v_{\bar{x}}] ,$$
$$\|v\|_1^2 = \|v\|_0^2 + \|v\|_{\mathring{1}}^2 ,$$
$$\|v\|_\infty = \|v\|_{\infty,w_{N_l}} = \max_{0\le i\le N_l} |v_i| .$$

We shall also use the notation,

$$g_{\bar{t},i}^j = \frac{g_i^j - g_i^{j-1}}{\tau},$$

$$g_{t,i}^j = \frac{g_i^{j+1} - g_i^j}{\tau},$$

$$g_{\bar{t}t,i}^j = \frac{g_i^{j+1} - 2g_i^j g_i^{j-1}}{\tau}.$$

where $g \equiv g_i^j \equiv g(x_i,t_j)$ defined on $\varpi_{N_l}$.

Using the basis function $\Phi_{ij}(x,t)$ and the interpolation quadrature formulae with the weight and remainder term in integral form ( [1] ) to subintervals $[t_{j-1},t_{j+1}]$ and $[x_{i-1},x_{i+1}]$ are created.

The integral identity is used as the method for generating difference

$$\hbar_i^{-1}\tau^{-1}\int_{t_{j-1}}^{t_{j+1}}\int_{x_{i-1}}^{x_{i+1}} [Lu - f(x,t)]\,\Phi_{ij}(x,t)dxdt = 0,\ 1\le i\le N_l-1,\ 1\le j\le N_T, \quad (3.1)$$

where the basis functions $\Phi_{ij}(x,t)$ have the form

$$\Phi_{ij}(x,t) = \varphi_i(x)\chi_j(t),$$

with the space's piecewise linear functions

$$\varphi_i(x) = \begin{cases} (x-x_{i-1})/h, & x_{i-1}\le x\le x_i; \\ (x_{i+1}-x)/h, & x_i\le x\le x_{i+1}; \\ 0, & \text{otherwise}, \end{cases}$$

and for the time

$$\chi_j(t) = 1 - \left|\frac{t-t_j}{\tau}\right| = \begin{cases} (t-t_{j-1})/\tau, & t_{j-1}<t<t_j; \\ (t_{j+1}-t)/\tau, & t_j<t<t_{j+1}; \\ 0, & \text{otherwise}. \end{cases}$$

Here, the $\varphi_i(x)$ and $\chi_j(t)$ functions, respectively, represent the solutions of the following problems:

$$\varepsilon\varphi_i''(x) = 0,\ x_{i-1}<x<x_i,\ \varphi_i(x_{i-1}) = 0,\ \ \varphi_i(x_i) = 1,$$

$$\varepsilon\varphi_i''(x) = 0,\ x_i < x < x_{i+1},\ \ \varphi_i(x_i) = 1,\ \ \varphi_i(x_{i+1}) = 0$$

and

$$\varepsilon\chi_j''(t) = 0,\ t_{j-1} < t < t_j,\ \chi_j(t_{j-1}) = 0,\ \ \chi_j(t_j) = 1,$$

$$\varepsilon\chi_j''(t) = 0,\ t_j < t < t_{j+1},\ \chi_j(t_j) = 1,\ \ \chi_j(t_{j+1}) = 0.$$

According to ( [1], [2]), we have the relation for the exact solution in (3.1),

$$\tau^{-1}\hbar_i^{-1}\int_{t_{j-1}}^{t_{j+1}}\int_{x_{i-1}}^{x_{i+1}} L_1 u\left(x,t\right)\varphi_i(x)\chi_j(t)dxdt = -\varepsilon u_{\bar{t}t\bar{x}\hat{x}} + a_i u_{\bar{t}t} + R_{a,i}^*,$$

where

$$\begin{aligned} R_{a,i}^* &= \hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}}\left[a\left(x\right)-a\left(x_i\right)\right]\frac{\partial^2 u(x,t)}{\partial t^2}\varphi_i\left(x\right)dx \\ &\quad + a\left(x_i\right)\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} d\xi\frac{\partial^4 u(\xi,t)}{\partial x^2\partial t^2}\int_{x_{i-1}}^{x_{i+1}}\left[T_1\left(x-\xi\right)\varphi_i\left(x\right)dx - T_1\left(x_i-\xi\right)\right].\end{aligned}$$

Furthermore, we have

$$\tau^{-1}\hbar_i^{-1}\int_{t_{j-1}}^{t_{j+1}}\int_{x_{i-1}}^{x_{i+1}} L_2 u\left(x,t\right)\varphi_i(x)\chi_j(t)dxdt = -\varepsilon u_{\bar{x}\hat{x},i}^j + b_i^j u_i^j + R_{b,i}^*,$$

and

$$\begin{aligned}\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} c\left(t\right)u\left(x,t-r\right)\varphi_i\left(x\right)dx &= \hbar_i^{-1}c\left(t\right)\int_{x_{i-1}}^{x_{i+1}} u\left(x,t-r\right)\varphi_i\left(x\right)dx \\ &= c\left(t\right)u\left(x_i,t-r\right) + R_{c,i}^*,\end{aligned}$$

where

$$R_{b,i}^* = \hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} b\left(x,t\right)u\left(x,t\right)\varphi_i\left(x\right)dx - b\left(x_i,t\right)u\left(x_i,t\right),$$

and

$$R_{c,i}^* = \hbar_i^{-1}c\left(t\right)\int_{x_{i-1}}^{x_{i+1}} dx\varphi_i\left(x\right)\int_{x_{i-1}}^{x_{i+1}}\frac{\partial^2 u(\xi,t-r)}{\partial x^2}\left[T_1\left(x-\xi\right)-T_1\left(x_i-\xi\right)\right]d\xi.$$

Finally, we have

$$\tau^{-1}\hbar_i^{-1}\int_{t_{j-1}}^{t_{j+1}}\int_{x_{i-1}}^{x_{i+1}} f\left(x,t\right)\varphi_i(x)\chi_j(t)dxdt = f_i^j + R_{f,i}^*,$$

where

$$R_{f,i}^* = \hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} dx\varphi_i\left(x\right)\int_{x_{i-1}}^{x_{i+1}}\frac{\partial^2 f(\xi,t)}{\partial x^2}\left[T_1\left(x-\xi\right)-T_1\left(x_i-\xi\right)\right]d\xi.$$

As a result, the relation for the solution to improve is given from (3.1)

$$-\varepsilon\left(\frac{\partial^2 u}{\partial t^2}\right)_{\bar{x}\hat{x}}+a_i\left(\frac{\partial^2 u(t)}{\partial t^2}\right)_i-\varepsilon u_{\bar{x}\hat{x},i}+b_i\left(t\right)u_i\left(t\right)+c\left(t\right)u_i\left(t-r\right)-f_i\left(t\right)+R_i^*=0, \quad (3.2)$$

where

$$R_i^*=R_{a,i}^*+R_{b,1}^*+R_{c,i}^*-R_{f,i}^*.$$

After that, we define the difference schemes with regard to the time variable. In this instance, we multiply both sides of the formula (3.2) by $\tau^{-1}\chi_j\left(t\right)$, then integrate in $[t_{j-1},t_{j+1}]$. From here, we get

$$\tau^{-1}\int_{t_{j-1}}^{t_{j+1}}\left[Lu_i-f\left(x_i,t\right)\right]\chi_j\left(t\right)dt=\tau^{-1}\int_{t_{j-1}}^{t_{j+1}}\left\{-\varepsilon\left(\frac{\partial^2 u}{\partial t^2}\right)_{\bar{x}\hat{x}}+a_i\left[\frac{\partial^2 u}{\partial t^2}\right]_i\right.$$
$$\left.-\varepsilon u_{\bar{x}\hat{x},i}+b_i u_i+c\left(t\right)u\left(x_i,t-r\right)-f\left(x_i,t\right)+R_i^*\chi_j\left(t\right)\right\}dt.$$

Furthermore, with $\tau=T/M$ and $rM/T=M_0$, we have $u\left(x_i,t_j-r\right)=u\left(x_i,t_{j-M_0}\right)$, and applying the same computing we get

$$\ell u-f(x_i,t_j):=-\varepsilon u_{\bar{t}t\bar{x}\hat{x}}+a_i u_{\bar{t}t}-\varepsilon u_{\bar{x}\hat{x}}+b_i^j u+c^j u_i^{j-M_0}-f_i^j+R=0, \quad (x,t)\in\omega_{N_l}\times\omega_{N_T}, \quad (3.3)$$

that the remainder term is

$$R=-\varepsilon\left(R^{(0)}\right)_{\bar{x}\hat{x}}+R^{(1)}+R_c,$$

where

$$R^{(0)} = \tau^{-1}\int_{t_{j-1}}^{t_{j+1}}d\eta\frac{\partial^2 u(x_i,\eta)}{\partial t^2}\left[\int_{t_{j-1}}^{t_{j+1}}T_1\left(t-\eta\right)\chi_j\left(t\right)dt-T_1\left(t_j-\eta\right)\right],$$
$$R^{(1)} = R_1^{(1)}+\tau^{-1}\int_{t_{j-1}}^{t_{j+1}}R_{a,i}^*\chi_j\left(t\right)dt,$$

and

$$R_1^{(1)}=R_{b,i}^j-R_{f,i}^j, \quad R_c=R_{c,i}^j.$$

Here $R_{b,i}^j$, $R_{c,i}^j$ and $R_{f,i}^j$ remainder terms are

$$R_{b,i}^j = \tau^{-1}\int_{t_{j-1}}^{t_{j+1}}\left\{\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}}d\xi b\left(\xi,t_j\right)\frac{\partial^2 u\left(\xi,t_j\right)}{\partial x^2}\right.$$
$$\left.\times\left[\int_{x_{i-1}}^{x_{i+1}}T_1\left(x-\xi\right)\varphi_i\left(x\right)dx-T_1\left(x_i-\xi\right)\right]\right\}\chi_j\left(t\right)dt$$
$$+\tau^{-1}\int_{t_{j-1}}^{t_{j+1}}d\eta b\left(x,\eta\right)\frac{\partial^2 u\left(\xi,t_j\right)}{\partial t^2}\left[\int_{t_{j-1}}^{t_{j+1}}T_1\left(t-\eta\right)\chi_j\left(t\right)dt-T_1\left(t_j-\eta\right)\right],$$

$$\begin{aligned} R_{c,i}^{j} &= \tau^{-1}\int_{t_{j-1}}^{t_{j+1}} d\eta c\left(\eta\right)\frac{\partial^2 u\left(x_i,\eta-r\right)}{\partial t^2}\left[\int_{t_{j-1}}^{t_{j+1}} T_1\left(t-\eta\right)\chi_j\left(t\right)dt - T_1\left(t_j-\eta\right)\right] \\ &+\tau^{-1}\int_{t_{j-1}}^{t_{j+1}} c\left(t\right)\left\{\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} d\xi\frac{\partial^2 u\left(\xi,t-r\right)}{\partial x^2}\right. \\ &\times\left.\left[\int_{x_{i-1}}^{x_{i+1}} T_1\left(x-\xi\right)\varphi_i\left(x\right)dx - T_1\left(x_i-\xi\right)\right]\right\}\chi_j\left(t\right)dt, \end{aligned}$$

$$\begin{aligned} R_{f,i}^{j} &= \tau^{-1}\int_{t_{j-1}}^{t_{j+1}} \left\{\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} d\xi\frac{\partial^2 f\left(\xi,t\right)}{\partial x^2}\right. \\ &\times\left.\left[\int_{x_{i-1}}^{x_{i+1}} T_1\left(x-\xi\right)\varphi_i\left(x\right)dx - T_1\left(x_i-\xi\right)\right]\right\}\chi_j\left(t\right)dt \\ &+\int_{t_{j-1}}^{t_{j+1}} d\eta\frac{\partial^2 f\left(x_i,\eta\right)}{\partial t^2}\left[\int_{t_{j-1}}^{t_{j+1}} T_1\left(t-\eta\right)\chi_j\left(t\right)dt - T_1\left(t_j-\eta\right)\right]. \end{aligned}$$

We take use of the following integral equality for the initial condition (1.3)

$$\hbar^{-1}\int_{t_0}^{t_1}\int_{x_{i-1}}^{x_{i+1}}\left[Lu-f\left(x,t\right)\right]\varphi_i\left(x\right)\chi_0\left(t\right)dxdt=0,$$

where the basis function $\chi_0\left(t\right)$ is given by

$$\chi_0\left(t\right)=1-\left|\frac{t-t_0}{\tau}\right|,\quad t_0\le t\le t_1.$$

In general, we get

$$\ell^{(0)}u\equiv-\varepsilon u_{t\bar{x}\hat{x}}^{0}+a_i u_t^0+c^0 u_i^{-M_0}+r=-\varepsilon\psi_{\bar{x}\hat{x}}+a_i\psi_i+\varepsilon\frac{\tau}{2}\zeta_{\bar{x}\hat{x}}^{0}-\frac{\tau}{2}b_i^0\zeta_i^0+\frac{\tau}{2}f_i^0, \tag{3.4}$$

where remainder term is

$$r=-\varepsilon\left(r^{(0)}\right)_{\bar{x}\hat{x}}+r^{(1)}+r_{c,i},$$

that here

$$\begin{aligned} r^{(0)} &= \int_{t_0}^{t_1} d\eta\frac{\partial^2 u\left(x_i,\eta\right)}{\partial t^2}\left[\int_{t_0}^{t_1} T_1\left(t-\eta\right)\chi_0\left(t\right)dt - T_1\left(t_0-\eta\right)\right], \\ r^{(1)} &= r_1^{(1)}+\int_{t_0}^{t_1} R_{a,i}^{*}\chi_0\left(t\right)dt, \\ r_1^{(1)} &= r_{b,i}^{0}-r_{f,i}^{0},\ r_{c,i}=r_{c,i}^{0}, \end{aligned}$$

and $r_{b,i}^{0}$, $r_{f,i}^{0}$ and $r_{c,i}^{0}$ the remainder terms are

$$\begin{aligned} r_{b,i}^{0} &= \int_{t_0}^{t_1} d\eta b\left(x_i,\eta\right)\frac{\partial^2 u\left(x_i,\eta\right)}{\partial t^2}\left[\int_{t_0}^{t_1} T_1\left(t-\eta\right)\chi_0\left(t\right)dt - T_1\left(t_1-\eta\right)\right] \\ &+\int_{t_0}^{t_1}\left\{\hbar^{-1}\int_{x_{i-1}}^{x_{i+1}} d\xi b\left(\xi,t\right)\frac{\partial^2 u\left(\xi,t\right)}{\partial x^2}\right. \\ &\times\left.\left[\int_{x_{i-1}}^{x_{i+1}} T_1\left(x-\xi\right)\varphi_i\left(x\right)dx - T_1\left(x_i-\xi\right)\right]\right\}\chi_0\left(t\right)dt, \end{aligned}$$

$$\begin{aligned} r_{f.i}^{0} &= \int_{t_0}^{t_1} d\eta \frac{\partial^2 f(x_i,\eta)}{\partial t^2}\left[\int_{t_0}^{t_1} T_1(t-\eta)\chi_j(t)\,dt - T_1(t_1-\eta)\right] \\ &+ \int_{t_0}^{t_1}\left\{\hbar^{-1}\int_{x_{i-1}}^{x_{i+1}} d\xi \frac{\partial^2 f(\xi,t)}{\partial x^2}\right. \\ &\times \left.\left[\int_{x_{i-1}}^{x_{i+1}} T_1(x-\xi)\varphi_i(x)\,dx - T_1(x_i-\xi)\right]\right\}\chi_0(t)\,dt, \end{aligned}$$

$$\begin{aligned} r_{c,i}^{0} &= \int_{t_0}^{t_1} d\eta c(\eta)\frac{\partial^2\left[u(x_i,\eta-r)\right]}{\partial t^2}\left[\int_{t_0}^{t_1} T_1(t-\eta)\chi_0(t)\,dt - T_1(t_1-\eta)\right] \\ &+ \int_{t_0}^{t_1} c(t)\left\{\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} d\xi \frac{\partial^2 u(\xi,t-r)}{\partial x^2}\right. \\ &\times \left.\left[\int_{x_{i-1}}^{x_{i+1}} T_1(x-\xi)\varphi_i(x)\,dx - T_1(x_i-\xi)\right]\right\}\chi_0(t)\,dt. \end{aligned}$$

Therefore based on (3.2), (3.3) and (3.4), we propose the following difference scheme for approximating (1.1)-(1.4):

$$\ell u \;:\; = -\varepsilon y_{\bar{t}t\hat{x}x} + a_i y_{\bar{t}t} - \varepsilon y_{\hat{x}x} + b_i^j y + c^j y_i^{j-M_0} = f_i^j, \tag{3.5}$$

$$y(x,0) = \zeta(x,t), \qquad x \in [-r,0], \tag{3.6}$$

$$\ell^{(0)} y \equiv -\varepsilon y^0_{t\bar{x}\hat{x}} + a_i y_t^0 + c^0 y_i^{-M_0} = \phi, \qquad x \in \omega_{N_l}, \tag{3.7}$$

$$y(0,t) = y(l,t) = 0, \qquad x \in \omega_{N_T}, \tag{3.8}$$

where

$$\phi = -\varepsilon\psi_{\bar{x}\hat{x}} + a_i\psi_i + \frac{\tau}{2}\varepsilon\zeta^0_{\bar{x}\hat{x}} - \frac{\tau}{2} b_i^0\zeta_i^0 + \frac{\tau}{2} f_i^0.$$

### 3.1 The Piecewise Uniform Mesh

We introduce a piecewise uniform mesh $\omega_n$ which will be implemented as follows. This is a piecewise uniform mesh that is enhanced in the boundary layer. The fitted special piecewise uniform mesh $\omega_n$ on the interval $[0,l]$ is created by dividing the interval into three subintervals $[0,\sigma_1]$, $[\sigma_1,\sigma_2]$ and $[\sigma_2,l]$, where

$$\sigma_1 = \min\left\{\frac{l}{4},\; -\alpha^{-1}\varepsilon\ln\varepsilon\right\},\; \sigma_2 = l - \sigma_1.$$

The mesh points are denoted with

$$x_i = \begin{cases} x_0 + (i-1)\frac{4\sigma_1}{N_l},\; i = 0,1,...,\frac{N_l}{4},\; x_i \in [0,\sigma_1],\; \sigma_1 < \frac{l}{4}, \\ \sigma_1 + (i-1-\frac{N_l}{4})\frac{2(\sigma_2-\sigma_1)}{N_l},\; i = \frac{N_l}{4}+1,...,\frac{3N_l}{4},\; x_i \in [\sigma_1,\sigma_2], \\ \sigma_2 - \left(i-1-\frac{3N_l}{4}\right)\frac{4\sigma_1}{N_l},\; i = \frac{3N_l}{4}+1,...,N,\; x_i \in [\sigma_2,l]. \end{cases}$$

## 4 Estimates of Error

Let $y_i$ and $u_i$ be solutions of (3.5)-(3.8) and (1.1)-(1.4) respectively. Let $z(x_i,t_i) = y(x_i,t_i) - u(x_i,t_i)$, $(x_i,t_i) \in \varpi_{N_l} \times \varpi_{N_T}$. We have

$$\begin{aligned} \ell z &= -\varepsilon z_{\bar{t}t\hat{x}x} + a_i z_{\bar{t}t} - \varepsilon z_{\hat{x}x} + b_i^j z + c^j z_i^{j-M_0} = R, \\ &= -\varepsilon \left(R^{(0)}\right)_{\bar{x}\hat{x}} + R^{(1)} + R_c,\ (x_i,t_i) \in \omega_{h\tau}, \qquad (4.1) \\ & \quad z(x,t) = 0,\ t \in [-r,0], \qquad (4.2) \\ \ell^{(0)} z &= -\varepsilon z^0_{t\hat{x}x} + a_i z^0_t + c^0 z_i^{-M_0} = r = -\varepsilon \left(r^{(0)}\right)_{\bar{x}\hat{x}} + r^{(1)} + r_{c,i}, \qquad (4.3) \\ & \quad z(0,t) = z(l,t) = 0. \qquad (4.4) \end{aligned}$$

For the convergence of the approximate solution, we prove the following theorem.

**Theorem 1** *Let $u_i$ and $y_i$ be solutions of (1.1)-(1.4) and (3.5)-(3.8) respectively. If $u(x,t) \in C_1^2(\bar{\Omega})$, then and under the following conditions*

$$C_0\tau < 1,\ C_0 = \max\left(\frac{b^* + c^* + 2}{2\alpha}, \gamma_*^{-1}\right),$$

*the error of the (4.1)-(4.4) satisfies,*

$$\begin{aligned} \left\|z_t^j\right\| + \sqrt{\varepsilon}\left\|z_{t\bar{x}}^j\right\| \le C &\left\{\varepsilon \left\|r^{(0)}\right\|^2 + \tau \left\|r^{(1)}\right\|^2 + \|r_c\|^2 \right. \\ &\left. + \tau \sum_{j'=1}^{j} \left[\varepsilon \left\|R^{(0)}(t_{j'})\right\|^2_{L_2(\omega_h^+)} + \left\|R^{(1)}(t_{j'})\right\|^2_{L_2(\omega_h)} + \left\|R_c^{j'}\right\|^2_{L_2(\omega_h)}\right]\right\},\ t \in \omega_\tau, \qquad (4.5) \end{aligned}$$

*where*

$$1 - \frac{\tau^2}{2} \ge \gamma_* > 0.$$

One can find the similar proof given in ( [5]).

**Theorem 2** *According to the assumptions made for the data in Lemma (1) for $u(x,t) \in C_1^2\left(\bar{\Omega}\right)$, the following estimate is valid:*

$$\|R\| \le C\left\{\tau^2 + h_i^2\left[1 + \varepsilon^{-1}\int_{x_{i-1}}^{x_{i+1}} \left(e^{-x\sqrt{\frac{\alpha}{\varepsilon}}} + e^{-(l-x)\sqrt{\frac{\alpha}{\varepsilon}}}\right) dx,\right]\right\}. \qquad (4.6)$$

For proof of the Lemma from

$$\begin{aligned} R_i^j &= -\varepsilon \left(R^{(0)}\right)_{\bar{x}\hat{x}} + R^{(1)} + R^j_{c,i}, \\ R^{(1)} &= R_1^{(1)} + \tau^{-1}\int_{t_{j-1}}^{t_{j+1}} R^*_{a,i}\chi_j(t)dt = R^j_{b,i} - R^j_{f,i} + \tau^{-1}\int_{t_{j-1}}^{t_{j+1}} R^*_{a,i}\chi_j(t)dt, \end{aligned}$$

we show that

$$\left\|R^j_{f,i}\right\| = O\left(\tau^2 + h_i^2\right)$$

and

$$\left\|R_{b,i}^{j}\right\| = C\left\{\tau^{2} + h_{i}^{2}\left[1 + \varepsilon^{-1}\int_{x_{i-1}}^{x_{i+1}}\left(e^{-x\sqrt{\frac{\alpha}{\varepsilon}}} + e^{-(l-x)\sqrt{\frac{\alpha}{\varepsilon}}}\right)dx\right]\right\}.$$

The similar procedure may be used to estimate the error for the terms $R^{(0)}$ and $R_{c,i}^{j}$. From

$$R_{f,i}^{j} = \tau^{-1}\int_{t_{j-1}}^{t_{j+1}} d\eta \frac{\partial^{2} f\left(x_{i},\eta\right)}{\partial t^{2}}\int_{t_{j-1}}^{t_{j+1}}\left[T_{1}\left(t-\eta\right)\chi_{j}(t)dt - T_{1}\left(t_{j}-\eta\right)\right]$$
$$+\tau^{-1}\int_{t_{j-1}}^{t_{j+1}}\left\{\hbar^{-1}\int_{x_{i-1}}^{x_{i+1}} d\xi \frac{\partial^{2} f\left(\xi,t\right)}{\partial x^{2}}\int_{x_{i-1}}^{x_{i+1}}\left[T_{1}\left(x-\xi\right)\varphi_{i}\left(x\right)dx - T_{1}\left(x_{i}-\xi\right)\right]\right\}\chi_{j}(t)dt,$$

the estimation that we get is as follows

$$\left\|R_{f,i}^{j}\right\| \leq \max_{t}\left|\frac{\partial^{2} f\left(x_{i},\eta\right)}{\partial t^{2}}\right|\int_{t_{j-1}}^{t_{j+1}}\left[T_{1}\left(t-\eta\right) - T_{1}\left(t_{j}-\eta\right)\right]d\eta$$
$$+\max_{x}\left|\frac{\partial^{2} f\left(\xi,t\right)}{\partial x^{2}}\right|\int_{x_{i-1}}^{x_{i+1}}\left[T_{1}\left(x-\xi\right) - T_{1}\left(x_{i}-\xi\right)\right]d\xi \leq C\left(\tau^{2} + h_{i}^{2}\right).$$

We consider the explicit expression of $R_{b,i}^{j}$ when proving the lemma. Let's take the expression

$$\begin{aligned} R_{b,i}^{j} &= \tau^{-1}\int_{t_{j-1}}^{t_{j+1}}\left\{\hbar_{i}^{-1}\int_{x_{i-1}}^{x_{i+1}} d\xi \frac{\partial^{2}\left[b\left(\xi,t_{j}\right)u\left(x,t_{j}\right)\right]}{\partial x^{2}}\right. \\ &\times\int_{x_{i-1}}^{x_{i+1}}\left[T_{1}\left(x-\xi\right)\varphi_{i}\left(x\right)dx - T_{1}\left(x_{i}-\xi\right)\right]\chi_{j}(t)dt \\ &\left.+\tau^{-1}\int_{t_{j-1}}^{t_{j+1}} d\eta \frac{\partial^{2}}{\partial t^{2}}\left[b\left(x_{i},\eta\right)u\left(x_{i},t\right)\right]\int_{t_{j-1}}^{t_{j+1}}\left(T_{1}\left(t-\eta\right)\chi_{j}\left(t\right)dt - T_{1}\left(t_{j}-\eta\right)\right)\right\}. \end{aligned}$$

By using the

$$\left|\frac{\partial^{k+s} u}{\partial t^{k}\partial x^{s}}\right| \leq C\left\{1 + \varepsilon^{-\frac{s}{2}}\left(e^{-x\sqrt{\frac{a(0)}{\varepsilon}}} + e^{-(l-x)\sqrt{\frac{a(l)}{\varepsilon}}}\right)\right\},$$

inequality, it stands to reason that

$$\begin{aligned} \left\|R_{b,i}^{j}\right\| &\leq \hbar_{i}^{-1}\int_{x_{i-1}}^{x_{i+1}} d\xi \max_{x}\left|b\left(\xi,t_{j}\right)\right|\left|\frac{\partial^{2} u\left(\xi,t_{j}\right)}{\partial x^{2}}\right| \\ &\times\left[\int_{x_{i-1}}^{x_{i+1}} T_{1}\left(x-\xi\right)\varphi_{i}\left(x\right)dx - T_{1}\left(x_{i}-\xi\right)\right] \\ &+\tau^{-1}\int_{t_{j-1}}^{t_{j+1}} d\eta \max_{t}\left|b\left(x,\eta\right)\right|\left|\frac{\partial^{2} u\left(x_{i},\eta\right)}{\partial t^{2}}\right| \\ &\times\left[\int_{t_{j-1}}^{t_{j+1}} T_{1}\left(t-\eta\right)\chi_{j}\left(t\right)dt - T_{1}\left(t_{j}-\eta\right)\right] \\ &\leq C\left\{\tau^{2} + h_{i}^{2} + \varepsilon^{-1}h_{i}\int_{x_{i-1}}^{x_{i+1}}\left(e^{-x\sqrt{\frac{\alpha}{\varepsilon}}} + e^{-(l-x)\sqrt{\frac{\alpha}{\varepsilon}}}\right)dx\right\}. \end{aligned}$$

The Lemma's proof is finished with this.

**Lemma 4** *The following estimate is accurate given the data from Lemma (t:1) for the remainder term* $R_i^j$ :

$$\|R\| \leq C\left(\tau^2 + N_l^{-2}\ln N_l\right).$$

We come to an inequality of the remainder terms

$$\|R\| \leq \left\|R^{(0)}\right\|_{L\left(\omega^+_{N_l}\right)} + \left\|R^{(1)}\right\|_{L\left(\omega_{N_l}\right)} + \left\|R^j_{c,i}\right\|_{L\left(\omega_{N_l}\right)},$$

and by using (**??**) inequality, the inequality is shown below

$$\|R\| \leq C\left\{\tau^2 + h_i^2 + \varepsilon^{-1}h_i\int_{x_{i-1}}^{x_{i+1}}\left(e^{-x\sqrt{\frac{\alpha}{\varepsilon}}} + e^{-(l-x)\sqrt{\frac{\alpha}{\varepsilon}}}\right)dx\right\}. \tag{4.7}$$

In the first case, consider $\sigma_1 = l/4$ and $\sigma_2 = l - \sigma_1 = 3l/4$. We obtain $l/4 < \alpha_0^{-1}\varepsilon\ln N_l$ and $3l/4 > l - \alpha_0^{-1}\varepsilon\ln N_l$ by attention to $h^{(1)} = h^{(2)} = h^{(3)} = lN_l^{-1}$. Hereby, since

$$\varepsilon^{-1}h^{(1)}\int_{x_{i-1}}^{x_{i+1}} e^{-x\sqrt{\frac{\alpha}{\varepsilon}}}dx \leq \varepsilon^{-1}\left(h^{(1)}\right)^2 < \frac{4l\alpha_0^{-1}\ln N_l}{N_l^2} = 4l\alpha_0^{-1}N_l^{-2}\ln N_l,$$

$$\varepsilon^{-1}h^{(3)}\int_{x_{i-1}}^{x_{i+1}} e^{-(l-x)\sqrt{\frac{\alpha}{\varepsilon}}}dx \leq \varepsilon^{-1}\left(h^{(3)}\right)^2 < \frac{4l\alpha_0^{-1}\ln N_l}{N_l^2} = 4l\alpha_0^{-1}N_l^{-2}\ln N_l,$$

$$\|R\| \leq C\left\{\tau^2 + N_l^{-2}\ln N_l\right\}, \qquad 1 \leq i \leq N_l.$$

We now consider the case $\sigma_1 = \alpha_0^{-1}\varepsilon\ln N_l$ and estimate $R$ on $[0,\sigma]$, $[\sigma, l-\sigma]$ and $[l-\sigma, l]$ separately. In the layer region $[0,\sigma]$ the inequality (4.7) reduces to

$$\begin{aligned}\|R\| &\leq C\left\{\tau^2 + \left(1+\varepsilon^{-1}\right)\left(h^{(1)}\right)^2\right\}\\ &\leq C\left\{\tau^2 + \left(1+\varepsilon^{-1}\right)\frac{\alpha_0^{-2}\varepsilon^2\ln^2 N_l}{N_l^2/16}\right\}, \qquad 1 \leq i \leq N_l/4.\end{aligned}$$

Hence

$$\|R\| \leq C\left\{\tau^2 + N_l^{-2}\ln^2 N_l\right\}, \quad 1 \leq i \leq N_l/4.$$

Estimate $R$ for $3N_l/4 + 1 \leq i \leq N_l$ is in same method. It remains to estimate $R$ for $N_l/4 + 1 \leq i \leq 3N_l/4$. In this case we are able to write (4.7) as

$$\begin{aligned}\|R\| &\leq C\left\{\tau^2 + \left(1+\varepsilon^{-1}\right)\left(h^{(2)}\right)^2\right\}\\ &\leq C\left\{\tau^2 + \left(h^{(2)}\right)^2 + h^{(2)}\alpha_0^{-1}\left(e^{-x_{i-1}\sqrt{\frac{\alpha}{\varepsilon}}} - e^{-x_{i+1}\sqrt{\frac{\alpha}{\varepsilon}}}\right)\right\},\\ &\quad N_l/4 + 1 \leq i \leq 3N_l/4.\end{aligned} \tag{4.8}$$

Since $x_i = \sigma_1 + (i - 1 - \frac{N}{4})h^{(2)}$,

$$\begin{aligned}&e^{-\left(\alpha_0^{-1}\varepsilon\ln N_l + (i-2-\frac{N}{4})h^{(2)}\right)\sqrt{\frac{\alpha}{\varepsilon}}} - e^{-\left(\alpha_0^{-1}\varepsilon\ln N_l + (i-\frac{N}{4})h^{(2)}\right)\sqrt{\frac{\alpha}{\varepsilon}}}\\ &\quad\leq \frac{1}{N_l}e^{-(i-2-\frac{N}{4})h^{(2)}\sqrt{\frac{\alpha}{\varepsilon}}}\left(1 - e^{-2h^{(2)}\sqrt{\frac{\alpha}{\varepsilon}}}\right) < CN_l^{-1},\end{aligned}$$

and this together with (4.8) to give the bound

$$\|R\| \leq C\left\{\tau^2 + N_l^{-2}\right\},$$

This completes the proof of Lemma.

Lemma (1)-(4) grant the main conclusions of the paper.

**Theorem 3** *Let u represent the solution to (1.1)-(1.4) and y represent the solution to (3.5)-(3.8). Consequently, the following estimate is true under the Lemma (1)-(4) hypotheses:*

$$\|y-u\|_{1,\omega_h} \leq C\left(\tau^2 + N_l^{-2}\ln^2 N_l\right).$$

## 5 Numerical Results

In this section, we confirm experimentally the theoretical outcomes obtained in the previous section. The following test problem introduces error approximations and convergence rates for the finite difference method. Pay attention to the problem

$$\begin{aligned} &-\varepsilon\frac{\partial^4 u}{\partial t^2\partial x^2} + (1+tanh(x))\frac{\partial^2 u}{\partial t^2} - \varepsilon\frac{\partial^2 u}{\partial x^2} + ((1-x)sinh(xt))\,u + t^4 u\,(x,t-r) \\ &\qquad = exp(-t)tanh(t)(1+(1-x)x+sinh(x)),\ \ (x,t)\in(0,1)\times(0,r]\,, \end{aligned} \tag{5.1}$$

$$u(x,t) = \zeta\,(x,t) = exp(-t)tanh(xt),\ \ (x,t)\in\bar{\Omega}\times[-r,0]\,, \tag{5.2}$$

$$\tag{5.3}$$

$$\frac{\partial u}{\partial t} = \psi\,(x) = cosh(x),\ \ x\in(0,1)\,, \tag{5.4}$$

$$\tag{5.5}$$

$$u(0,t) = u(l,t) = 0,\ \ t\in(0,1]\,, \tag{5.6}$$

where $x\in(0,1)$ and $t\in(0,1]$. The difference scheme (3.5)-(3.8) can be rewritten as

$$A_i y_{i-1}^{j+1} - C_i y_i^{j+1} + B_i y_{i+1}^{j+1} = -F_i, \quad (i=2,3,...,N_l-1, \quad j=2,3,...,M-1)\,, \tag{5.7}$$

where

$$\begin{aligned} A_i &= -\varepsilon h_i^{-1}\hbar_i^{-1}\tau^{-2}, \quad B_i = -\varepsilon h_{i+1}^{-1}\hbar_i^{-1}\tau^{-2},\ C_i = -\tau^{-2}\left(2\varepsilon h_{i+1}^{-1}h_i^{-1} + a_i\right), \\ F_i &= -\Big(-\left(\varepsilon h_i^{-1}\hbar_i^{-1}(2\tau^{-2}-1)\right)y_{i-1}^j \\ &\quad -\left(2\varepsilon h_{i+1}^{-1}h_i^{-1}(1-2\tau^{-2}) - 2a_i\tau^{-2} + b_i^j\right)y_i^j \\ &\quad -\left(\varepsilon h_{i+1}^{-1}\hbar_i^{-1}(2\tau^{-2}-1)\right)y_{i+1}^j + \left(\varepsilon h_i^{-1}\hbar_i^{-1}\tau^{-2}\right)y_{i-1}^{j-1} \\ &\quad -\tau^{-2}\left(2\varepsilon h_i^{-1}h_{i+1}^{-1} + a_i\right)y_i^{j-1} + \left(\varepsilon h_{i+1}^{-1}\hbar_i^{-1}\tau^{-2}\right)y_{i+1}^{j-1} \\ &\quad + f_i^j - c^j y_i^{j-M_0}\Big), \end{aligned}$$

as well as from (1.3) Difference schemes are for initial conditions is

$$D_i^* y_{i-1}^1 - F_i^* y_i^1 + E_i^* y_{i+1}^1 = -\Phi_i^*, \quad (i=2,3,...,N_l-1, \quad j=2,3,...,M-1)\,, \tag{5.8}$$

where

$$
\begin{aligned}
D_i^* &= -\varepsilon h_i^{-1}\hbar_i^{-1}\tau^{-1}, \quad E_i^* = -\varepsilon h_{i+1}^{-1}\hbar_i^{-1}\tau^{-1}, \; F_i^* = -2\varepsilon h_{i+1}^{-1}h_i^{-1}\tau^{-1} - a_i\tau^{-1}, \\
\Phi_i^* &= -\left(\left(-\varepsilon h_i^{-1}\hbar_i^{-1}\tau^{-1}\right)y_{i-1}^0 + \left(2\varepsilon h_{i+1}^{-1}\hbar_i^{-1}\tau^{-1} + a_i\tau^{-1}\right)y_i^0\right. \\
&\quad -\left(\varepsilon h_{i+1}^{-1}\hbar_i^{-1}\tau^{-1}\right)y_{i+1}^0 - \left(\varepsilon h_i^{-1}\hbar_i^{-1}\right)\psi_{i-1} + \left(2\varepsilon h_{i+1}^{-1}h_i^{-1} + a_i\right)\psi_i \\
&\quad -\left(\varepsilon h_{i+1}^{-1}\hbar_i^{-1}\right)\psi_{i+1} + \left(\frac{\varepsilon\tau}{2h_i\hbar_i}\right)\zeta_{i-1}^0 - \left(\frac{\varepsilon\tau}{2h_ih_{i+1}} + \frac{\tau}{2}b_i^0\right)\zeta_i^0 \\
&\quad \left. + \left(\frac{\varepsilon\tau}{2h_{i+1}\hbar_i}\right)\zeta_{i+1}^0 - c^0 y_i^{-M_0} + \frac{\tau}{2}F_i^0\right), \qquad (M_0 = \tau M/l).
\end{aligned}
$$

By using Thomas Algorithm, this provides us the tridiagonal system. The following factorization process, which is incredibly quick, may be used to solve the systems (5.7) and (5.8):

$$
\begin{aligned}
& \alpha_1 = 0, \quad \beta_1 = 0, \\
\alpha_{i+1} &= \frac{B_i}{C_i - A_i\alpha_i}, \quad \beta_{i+1} = \frac{F_i + A_i\beta_i}{C_i - A_i\alpha_i}, \\
& y_0^{j+1} = 0, \; y_{N_l}^{j+1} = 0,
\end{aligned}
$$

$$
y_i^{j+1} = \alpha_{i+1}y_{i+1}^{j+1} + \beta_{i+1}, \quad (i = N_l, N_l - 1, ..., 1),
$$

and

$$
\begin{aligned}
& \alpha_1^* = 0, \quad \beta_1^* = 0, \\
\alpha_{i+1}^* &= \frac{E_i^*}{F_i^* - D_i^*\alpha_i}, \quad \beta_{i+1}^* = \frac{\Phi_i^* + D_i^*\beta_i}{F_i^* - D_i^*\alpha_i}, \\
& y_0^1 = 0, \; y_{N_l}^1 = 0,
\end{aligned}
$$

$$
y_i^1 = \alpha_{i+1}^* y_{i+1}^1 + \beta_{i+1}^*, \quad (i = N_l, N_l - 1, ..., 1),
$$

that using recently correlated data, we estimate the value of $y_i^1$ $(i = 0, 2, ..., N_l)$. It is simple to confirm that

$$
\begin{aligned}
A_i &> 0, \quad B_i > 0, \quad D_i = C_i - A_i - B_i \geq 0, && (5.9) \\
& |A_i| \leq |B_i|, \qquad i = 1, 2, ..., N_l - 1, && (5.10)
\end{aligned}
$$

and

$$
\begin{aligned}
D_i^* &> 0, \quad E_i^* > 0, \quad G_i^* = F_i^* - D_i^* - E_i^* \geq 0, && (5.11) \\
& |D_i^*| \leq |E_i^*|, \qquad i = 1, 2, ..., N_l - 1. && (5.12)
\end{aligned}
$$

The invariant imbedding algorithm is stable and can be demonstrated to hold under the conditions given in (5.9)-(5.12). We define the quantities in order to test the order of uniform convergence with

$$
e^{N_l} = \max_{0 \leq i < N_l} \left| {y_i}^{N_l} - y_{2i}^{2N_l} \right|.
$$

We compute an experimental order of uniform convergence as follows,

$$p = \frac{\ln\left(e^{N_l}/e^{2N_l}\right)}{\ln 2}.$$

where $e^{N_l}$ is the maximum difference between the solutions on two succeeding meshes.

The following tables present the computed results in tabular form. The difference scheme's computational solutions are shown to be uniformly convergent with order two regard to $h$ in Tables 1.1 and 1.2. We use our Method to solve these problems with $N_l = 2^i$ , $(i = 3, ..., 7)$ for various values of $\varepsilon$ $(\varepsilon = 10^{-w},\ w = 2, 3, ..., 7)$.

Table 1.1. Computed convergence rates $p$ on $\omega_{N_l} \times \omega_{N_T}$ for $\varepsilon = 10^{-w}$ $(w = 2, 3, ..., 7)$. $r_k$ $(k = 0, 1)$ is the maximum error.

| $\varepsilon$ | | $N_l$ | | | |
|---|---|---|---|---|---|
| | | 8 | 16 | 32 | 64 |
| $10^{-2}$ | $r_0$ | 0.00640507 | 0.00258361 | 0.00074854 | 0.00018385 |
| | $r_1$ | 0.00258361 | 0.00074854 | 0.00018385 | 0.00004202 |
| | $p$ | 1.30982464 | 1.78723672 | 2.02554265 | 2.12939634 |
| $10^{-3}$ | $r_0$ | 0.01267443 | 0.00927360 | 0.00464861 | 0.00162134 |
| | $r_1$ | 0.00927360 | 0.00464861 | 0.00162134 | 0.00045180 |
| | $p$ | 0.45071949 | 0.99632945 | 1.51961721 | 1.97959627 |
| $10^{-4}$ | $r_0$ | 0.01389840 | 0.01271322 | 0.00904426 | 0.00492418 |
| | $r_1$ | 0.01271322 | 0.00904426 | 0.00492418 | 0.00197943 |
| | $p$ | 0.12858930 | 0.49125537 | 0.87712056 | 1.31479549 |
| $10^{-5}$ | $r_0$ | 0.01368416 | 0.01371739 | 0.01094833 | 0.00716860 |
| | $r_1$ | 0.01371739 | 0.01094833 | 0.00716860 | 0.00394297 |
| | $p$ | −0.00349907 | 0.32529456 | 0.61094806 | 0.86240923 |
| $10^{-6}$ | $r_0$ | 0.01317381 | 0.01405847 | 0.01193968 | 0.00831550 |
| | $r_1$ | 0.01405847 | 0.01193968 | 0.00831550 | 0.00499958 |
| | $p$ | −0.09376640 | 0.23567540 | 0.52188957 | 0.73399692 |
| $10^{-7}$ | $r_0$ | 0.01259547 | 0.01414086 | 0.01262576 | 0.00916503 |
| | $r_1$ | 0.01414086 | 0.01262576 | 0.00916503 | 0.00569825 |
| | $p$ | −0.16696470 | 0.16350026 | 0.46215819 | 0.68562197 |

Table 1.2. Computed convergence rates $p$ on $\omega_{N_l} \times \omega_{N_T}$ for $\varepsilon = 10^{-w}$ $(w = 2, 3, ..., 7)$. $r_k$ $(k = 0, 1)$ is the maximum error.

| $\varepsilon$ | | $N_l$ | | | |
|---|---|---|---|---|---|
| | | 128 | 256 | 512 | 1024 |
| $10^{-2}$ | $r_0$ | 0.00004264 | 0.00001222 | 0.00000315 | 0.00000080 |
| | $r_1$ | 0.00001130 | 0.00000315 | 0.00000080 | 0.00000020 |
| | $p$ | 1.91587226 | 1.95578248 | 1.98264458 | 1.99502989 |
| $10^{-3}$ | $r_0$ | 0.00045180 | 0.00011456 | 0.00002785 | 0.00000660 |
| | $r_1$ | 0.00011456 | 0.00002785 | 0.00000660 | 0.00000155 |
| | $p$ | 1.97959627 | 2.04030234 | 2.07800764 | 2.08926215 |
| $10^{-4}$ | $r_0$ | 0.00197943 | 0.00060906 | 0.00016566 | 0.00004389 |
| | $r_1$ | 0.00060906 | 0.00016529 | 0.00004254 | 0.00001108 |
| | $p$ | 1.70044202 | 1.88159512 | 1.96142581 | 1.98673141 |
| $10^{-5}$ | $r_0$ | 0.00394297 | 0.00211618 | 0.00155926 | 0.00047779 |
| | $r_1$ | 0.00176957 | 0.00069383 | 0.00046194 | 0.00012712 |
| | $p$ | 1.15587980 | 1.60880456 | 1.75509398 | 1.91016961 |
| $10^{-6}$ | $r_0$ | 0.00499958 | 0.00269679 | 0.00165923 | 0.00228391 |
| | $r_1$ | 0.00269679 | 0.00129466 | 0.00051680 | 0.00072718 |
| | $p$ | 0.89056492 | 1.05866635 | 1.68283185 | 1.65110973 |
| $10^{-7}$ | $r_0$ | 0.00569825 | 0.00320253 | 0.00168619 | 0.00093345 |
| | $r_1$ | 0.00320253 | 0.00168619 | 0.00083514 | 0.00037102 |
| | $p$ | 0.83130542 | 0.92544530 | 1.01367404 | 1.33105726 |

In Fig.1, the numerical solutions to the test problems for $\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-5}$ are displayed. In particular, we can see boundary layers in the right-hand figure at $x = 0$ and $x = 1$ for $\varepsilon = 10^{-7}$.

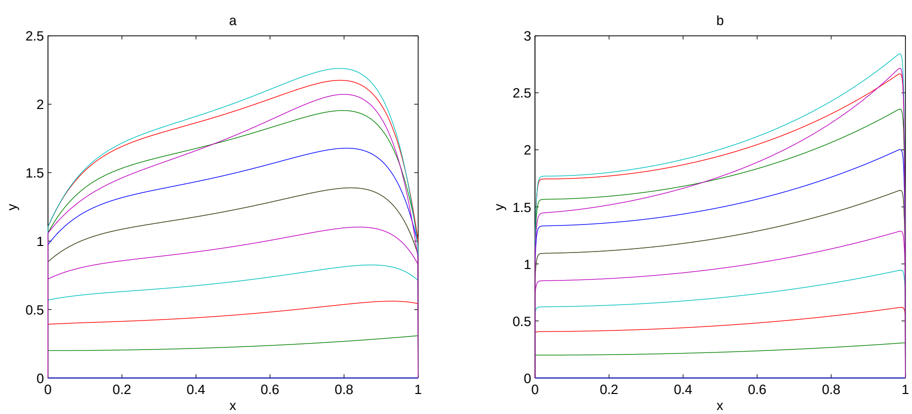


Fig.1. Solution of test problem for $\varepsilon = 10^{-2}$, $l = 1$, $r = 1$, $T = 2$. b) Solution of test problem for $\varepsilon = 10^{-5}$, $l = 1$, $r = 1$, $T = 2$.

## 6 Conclusion

In this study, we developed a difference scheme for the linear singularly perturbed Sobolev delay initial-boundary value problems on the Shishkin mesh. The method was based on an classical difference scheme on Shishkin mesh. As results from the method, $O\left(\tau^2 + N_l^{-2}\ln^2 N_l\right)$ order convergence with respect to space and time variable in the discrete maximum norm was obtained. However, using the method, a numerical example were solved and the obtained results were displayed in Tables 1.1 and 1.2. In Fig.1, the numerical solutions to the test problems for $\varepsilon = 10^{-2}$, $\varepsilon = 10^{-5}$, $l = 1$, $r = 1$ and $T = 2$ was displayed. The results in the table are obtained by keeping time variable $t$ constant. These results show the efficiency and accuracy of the our method. It shows that the theoretical results from this example and the results obtained are supported.

## Список литературы


[1] G. M. Amiraliyev and Ya. Mamedov, 1995, Difference Schemes on the Uniform Mesh for Singular Perturbed Pseudo-Parabolic Equation, *Turkish J. of Math.*, 19; p. 207-222.

[2] G. M. Amiraliyev, 1990, Difference Method for the Solution of One Problem of the Theory of Dispersive Waves, *Differential Equations*, 26, 2146-2154. (Russian)

[3] G. M. Amiraliyev, 1987, Investigation of the Difference Schemes for the Quasi-Linear Sobolev Equations, *Differential Equations*, V.23, No. 8, 1453-1455. (Russian)

[4] A.R. Ansari, S.A. Bakr, G.I. Shishkin, 2007, A parameter-robust finite difference method for singularly perturbed delay parabolic partial differential equations, *J. of Comput. and App. Math.*, 205, No.1 , 552-566.

[5] A. Barati Chiyaneh, H. Duru, 2019, ON Adaptive Mesh for the Initial-Boundary Value Singularly Perturbed Delay Sobolev Problems, Numerical Methods for Partial Differential Equations, DOI: 10.1002/num.22417.

[6] A. Barati Chiyaneh, H. Duru, 2019, Uniform Difference Method for Singularly Perturbed Delay Sobolev Problems on a Piecewise Uniform Mesh, Quaestiones Mathematicae, DOI:10.2989/16073606.2019.1653395.

[7] R. K. Bullough and P. J. Caudrey, 1980, *Solitons*, Springer-Verlag, New York, 1-13.

[8] G. V. Demidenko and S. V. Uspenskii, 1998, Equations and systems unsolved with respect to the highest derivative, Nauchnaya Kniga, Novosibirsk (Russian).

[9] E. P. Doolan, J. J. Miller, W. H. A. Schilders, 1980, *Uniform Numerical Methods for Problems with Initial and Boundary Layers*, Boole Press. Dublin.

[10] H. Duru, 2004, Difference schemes for the singularly perturbed Sobolev periodic boundary problem, *App. Math. And Comput.* 149: 187–201.

[11] I. E. Egorov, S. G. Pyatkov, and S. V. Popov, 2000, Non-classical differential-operator equations, Nauka, Novosibirsk (Russian).

[12] R. E. Ewing, 1978, Time Stepping Galerkin Methods for Nonlinear Sobolev Partial Differential Equations, *SIAM J. Numer Anal.* 15, 1125-1150.

[13] A. Favini and A. Yagi, 1999, Degenerate differential equations in Banach spaces, Marcel Dekker, New York.

[14] W. E. Ford, T. W. Ting, 1974, Uniform Error Estimates for Difference Approximations to Nonlinear Pseudo-Parabolic Partial Differential Equations, *SIAM J. Numer. Anal.* 11, 155-169.

[15] H. Gajewski, K. Groeger, and K. Zacharias, 1974, Nichtlineare Operatorgleichungen und Operator differentialgleichungen, Mathematische Lehrbücher und Monographien, Band 38, Akademie-Verlag, Berlin; Russian transl., Mir, Moscow 1978.

[16] H. Ikezi, 1978, *Experiments on Solitons in Plasmas, Solitons in Action*, Ed. K. Lonngren and A. Scott, Academic Press, 152-170.

[17] M. K. Kadalbajoo and Y. N. Reddy, 1989, Asymptotic and Numerical Analysis of Singular Perturbation Problems, *A survey, App. Math. And Comput.* 30:223-259.

[18] J. L. Langnese, 1972, General Boundary-Value Problems for Differential Equations of Sobolev Type, *SIAM J. Math. Anal.*, v.3, 105-119.

[19] V. I. Lebedev, 1957, The Method of Difference for the Equations of Sobolev Type, *Dokl. Acad. Sci. USSR,* V.114, No. 6, 1166-1169.

[20] K. E. Lonngren, 1978, *Observation of Solitons on Nonlinear Dispersive Transmission Lines, Soliton in Action*, Academic Press. 127-148.

[21] A. A. Samarskii, 1983, *Theory of Difference Schemes*, $2^{nd}$Ed., Nauka, Moscow.

[22] C. L. Sobolev, 1954, On a New Problems of Mathematical Physics, *Izv. Acad. Sci. USSR*, Mathematics, 18, No.1, 3-50.