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A CASE OF IMPULSIVE SINGULARITY

The paper considers an impulsive system with singularities. Different types of problems with singular perturbations have been discussed in many books. In Bainov and Kovachev's book [4]several articles cited therein consider impulse systems with small parameter involving only differential equations. The parameter is not in the impulsive equation of the systems. In our present the small parameter is inserted into the impulse equation. This is the principal novelty of our study. Furthermore, for the impulsive function, we found a condition that prevents the impulsive function to blow up as the parameter tends to zero. So we have significantly extended the singularity concept for discontinuous dynamics.

The singularity of the impulsive part of the system can be treated in the manner of perturbation theory methods. This article is a continuation of [1] work. In our present research, we apply the method of the paper [1]. Our goal is to construct an approximation with higher accuracy and to obtain the complete asymptotic expansion. We construct a uniform asymptotic approximation of the solution that is valid in the entire close interval by using the method of boundary functions [22]. An illustrative example using numerical simulations is given to support the theoretical results.
Key words: Impulsive systems, Differential equations with singular impulses, the Vasil'eva theorem, the method of boundary functions.

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Импульсті сингулярлық жағдай

Бұл жұмыста сингулярлы импульстік жүйе қарастырылады. Көптеген кітаптарда әр түрлі сингулярлы ауытқыған есептер талқыланған. Байнов пен Ковачевтің кітабында [4] және ол кітаптағы сілтеме жасалынған мақалаларда импульсті сингулярлы дифференциалдық теңдеулер жүйесі қарастырылған. Бірақ ол жұмыстарда кіші параметр импульстік теңдеулерде емес, тек дифференциалдық теңдеулерінде қатысады. Ал бұл жұмыста импульсті теңдеугеде кіші параметр енгізілді. Бұл зерттеудің негізгі жаңалығы болып табылады. Сонымен қатар импульстік функция үшін кіші параметрі нөлге ұмтылған кезде импульстік функцияның шексіздікке кетуін болдырмайтын қосымша шарт табылды. Осылайша, үзіліссіз динамика үшін сингулярлық ұғымы айтарлықтай кеңейтілді.

Жүйенің импульстік бөлігіндегі сингулярлықты ауытқу теориясының әдістерін қолдану арқылы қарастыруға болады. Бұл мақала [1] жұмыстың жалғасы болып табылады. Осы зерттеудеде [1] жұмыста сипатталған әдіс қолданылады. Жұмыстың мақсаты- жуықтауды жоғары дәлдікпен құру және толық асимптотикалық жіктелуді алу. Жұмыста шешімнің бірқалыпты асимптотикалық жуықтауы шекаралық функциялар әдісін [22] қолдана отырып, толық кесіндіде анықталды. Теориялық нәтижені растау үшін графикалық көрнекілікпен нақты мысал келтірілді.

Түйін сөздер: Импульстік жүйелер, сингулярлы импульстік дифференциалдық теңдеулер, Васильева теоремасы, шекаралық функциялар әдісі.

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Случай импульсивной сингулярности

В статье рассматривается импульсная система с сингулярностью. Различные типы задач с сингулярными возмущениями обсуждались во многих книгах. В книге Байнова и Ковачева [4] и нескольких статьях, цитируемых в книге рассматривались сингулярные импульсные системы с малым параметром, присутствующим только в дифференциальных уравнениях этих систем, но не в импульсных уравнениях. Мы же вводим малый параметр в уравнение с импульсом. Это является принципиальной новизной нашего исследования. Более того, для импульсной функции мы нашли условие, которое предотвращает коллапс импульсной функции при уменьшении параметра до нуля. Таким образом, мы значительно расширили концепцию сингулярности для разрывной динамики.

Сингулярность в импульсной части системы может быть рассмотрена с помощью методов теории возмущений. Эта статья является продолжением работы [1]. В настоящем исследовании применяется метод, описанный в этой статье. Наша цель - построить аппроксимации высокого порядка и получить полное асимптотическое разложение. Мы нашли равномерную асимптотическую аппроксимацию решения на всем замкнутом рассматриваемом интервале используя метод граничных функций [22]. Для подтверждения теоретического результата приведен численный пример с моделированием.

Ключевые слова: Импульсные системы, дифференциальные уравнения с сингулярными импульсами, теорема Васильевой, метод граничных функций.

1 Introduction

Singularly perturbed equations are frequently applied as mathematical models describing processes in physics, chemical kinetics [19], mathematical biology [14,18], fluid dynamics [11] and they frequently appear when applied engineering and technology problems are being investigated [11,13,14,16,18]. Since these problems depend on small parameters, the solutions show non-uniform behavior over time as the parameters tend to zero. Asymptotic methods of approximation of solutions for singularly perturbed differential equations are currently actively investigated by many authors. Efficient asymptotic methods are developed for a fairly broad class of singularly perturbed problems. These methods enable one to construct uniform approximations with any required accuracy. One of classical asymptotic methods is the method of boundary functions [23]. One can apply this method for solving a singularly perturbed problem, if in a part of its domain the condition of the well-known Tikhonov theorem is valid. The present paper, we shall imply the method for analysis of an impulsive systems. Impulse effects exist in various evolutionary processes that exhibit abrupt changes in states [2–4]. Impulse effects are common in many systems in addition to singular perturbations [8,9,20].

Various types of singular perturbation problems have been described in many books [7, 17, 23]. Consider the following model of singularly perturbed differential equation

$$\begin{aligned} \varepsilon \dot{z} &= f(z, y, t), \\ \dot{y} &= g(z, y, t) \end{aligned} \tag{1}$$

where ε is a small positive real number. In literature, the results based on the system is known as Tikhonov Theorem [17, 21]. In [5], authors consider systems with a singularity, which appears through moments of impacts. More precisely, the impact moments are singular if they are infinite and there exist accumulation points for the moments. Since there exist an infinite number of discontinuity moments in a finite time, the possibility of the blow up of solutions occurs here.

Akhmet and Çağ [1] first time in literature considered differential equations when impulses are also singular beside the differential equation. They presented the following problem

$$\begin{aligned} &\varepsilon \dot{z} = F(z), \\ &\varepsilon \Delta z|_{t=\theta_i} = I(z, \varepsilon), \end{aligned} \tag{2}$$

with $z(0, \varepsilon) = z_0$, where $z \in \mathbb{R}^m$, $t \in [0, T]$, $F(z)$ is a continuously differentiable function on D and $I(z, \varepsilon)$ is a continuous function for $(z, \varepsilon) \in D \times [0, 1], D$ is the domain $D = \{0 \leq t \leq T, \| z \| < d\}, \theta_i$ are defined above. If $\varepsilon = 0$ in (2), then $0 = F(z) = I(z; 0)$. It is a degenerate system because its order is less than that of (2). Consider an isolated real root $z = \varphi$ with $F(z) = 0$ and $I(z; 0) = 0$. Additionally, for the impulsive function they following condition if used

$$\lim_{(z,\varepsilon)\to(\varphi,0)} \frac{I(z, \varepsilon)}{\varepsilon} = 0 \tag{$*$}$$

which prevents impulsive function to blow up as the parameter ε tends to zero.

The main novelty of the paper [1] is the extension of Tikhonov's theorem so that the system (3) has the small parameter of the impulse function and the instant of discontinuity is different for each dependent variables. The singularity of the impulse part of the system can be handled using methods of perturbation theory. In our present research, we apply the ideas of the paper [1].

Our discussion will be centered on the following system:

$$\varepsilon \dot{z} = f(z, y, t), \qquad \dot{y} = g(z, y, t), \tag{3a}$$

$$\varepsilon \Delta z|_{t=\theta_i} = I(z, y, \varepsilon), \quad \Delta y|_{t=\eta_j} = J(z, y), \tag{3b}$$

where z, F and I are m-dimensional vector valued functions, y, f and J are n-dimensional vector valued functions, $0 < \theta_1 < \theta_2 < ... < \theta_p < T, \theta_i, i = 1, 2, ..., p$, and $\eta_j, j = 1, 2, ..., k$, are distinct discontinuity moments in $(0, T)$.

Impulsive system consists of differential equations (3a) and impulsive equations (3b). In the book [7] and papers [8, 9] impulsive systems are considered. But only the differential equation in [7] is singularly perturbed. Akhmet and Çağ were the first who inserted a small parameter into the impulsive part of the singular equation. In the paper [1], the authors investigated the behavior of solutions of a singularly perturbed system and considered two cases of singularity with single and multi-layers that depend on the condition $(**)$. Akhmet and Çağ showed that the transition to the limit for $y(t,\varepsilon)$ is uniform in the entire interval of $0 \le t \le T$, while the transition to the limit for $z(t,\varepsilon)$ isn't uniform in the entire interval of $0 \le t \le T$, but only in the subintervals $\delta \le t \le \theta_i, i = 1, 2, ..., p$ for $\delta > 0$, outside the boundary layers.

This article is a continuation of [1] work. We consider the case of differential equations with singular impulses. We construct a uniform asymptotic approximation of the solution that is valid in the whole interval $0 \le t \le T$ using the method of boundary functions. However, the theorems of the article [1] do not give the order of accuracy of the asymptotic approximation $\overline{y}(t)$ for the solution $y(t,\varepsilon)$ in $0 \le t \le T$ and that of $\overline{z}(t)$ for $z(t,\varepsilon)$ outside the boundary layer. Our goal is to construct an approximation with higher accuracy and, if possible, the complete asymptotic expansion for the solution of the problem (4), (5). The method for constructing asymptotic expansions of the type (10) for solutions of a number of singularly perturbed systems is called the method of boundary functions. Additionally, the following conditions are required

$$\lim_{(z,y,\varepsilon)\to(\varphi,\overline{y},0)} \frac{I(z,y,\varepsilon)}{\varepsilon} = I_0 \neq 0 \tag{**}$$

for the impulsive function, where $\overline{y} = \overline{y}(\theta_i)$ are the values for each impulse moment at the points $t = \theta_i, i = 1, 2, ..., p$.

2 Main Result

In this study, we consider the homogeneous linear differential system where impulses are singularly perturbed. The following system is our focus of discussion

$$\begin{aligned} &\varepsilon\frac{dz}{dt} = az + by, \quad \varepsilon\Delta z|_{t=\theta_i} = z + \frac{b}{a}y + \varepsilon z + 2\varepsilon y, \\ &\frac{dy}{dt} = cz + dy, \quad \Delta y|_{t=\theta_i} = z + y, \end{aligned} \tag{4}$$

with initial condition

$$z(0,\varepsilon) = z^0, \quad y(0,\varepsilon) = y^0. \tag{5}$$

where ε is small positive real number, $a < 0, b, c, d - constants$, $b \neq 2a$, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$, $0 < \theta_1 < \theta_2 < ... < \theta_p < T, \theta_i, i = 1, 2, ..., p$, are distinct discontinuity moments in $(0, T)$.

In system (4), we take $\varepsilon = 0$, to obtain

$$\begin{aligned} &0 = a\overline{z} + b\overline{y}, \quad \frac{d\overline{y}}{dt} = c\overline{z} + d\overline{y}, \\ &\Delta\overline{y}|_{t=\theta_i} = \overline{z} + \overline{y}. \end{aligned} \tag{6}$$

We find the root $\overline{z} = \varphi = -\frac{b}{a}\overline{y}$ and substitute it into (6) with the initial value (5) to obtain

$$\begin{aligned} &\frac{d\overline{y}}{dt} = \frac{ad - cb}{a}\overline{y}, \quad \Delta\overline{y}|_{t=\theta_i} = (1 - \frac{b}{a})\overline{y}, \\ &\overline{y}(0) = y^0. \end{aligned} \tag{7}$$

Solving this problem, we find

$$\overline{y}(\theta_i) = y^0(2 - \frac{b}{a})^{i-1}(\exp(\frac{ad-cb}{a}))^{\theta_i} \tag{8}$$

and eventually

$$\overline{z}(\theta_i) = -\frac{by^0}{a}(2 - \frac{b}{a})^{i-1}(\exp(\frac{ad-cb}{a}))^{\theta_i}. \tag{9}$$

Now let us check the condition $(**)$. Then one can verify that condition $(**)$ is correct, since

$$\lim_{(z,y,\varepsilon)\to(\varphi,\overline{y},0)} \frac{z + \frac{b}{a}y + \varepsilon z + 2\varepsilon y}{\varepsilon} = \lim_{(z,y,\varepsilon)\to(\varphi,\overline{y},0)} (2-\frac{b}{a})\overline{y}(\theta_i) = y^0(2-\frac{b}{a})^i(\exp(\frac{ad-cb}{a}))^{\theta_i} \neq 0.$$

We will seek the asymptotic expansion of the solution $z(t,\varepsilon), y(t,\varepsilon)$ of problem (4)-(5) in the form

$$\begin{aligned} z(t,\varepsilon) &= \overline{z}(t,\varepsilon) + \omega^{(i)}(\tau_i,\varepsilon), \\ y(t,\varepsilon) &= \overline{y}(t,\varepsilon) + \varepsilon\nu^{(i)}(\tau_i,\varepsilon), \quad \tau_i = \frac{t-\theta_i}{\varepsilon}, \end{aligned} \tag{10}$$

where

$$\begin{aligned} \overline{z}(t,\varepsilon) &= \sum_{k=0}^{\infty} \varepsilon^k \overline{z}_k(t), \qquad \overline{y}(t,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \overline{y}_k(t), \\ \omega^{(i)}(\tau_i,\varepsilon) &= \sum_{k=0}^{\infty} \varepsilon^k \omega_k^{(i)}(\tau_i), \quad \nu^{(i)}(\tau_i,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \nu_k^{(i)}(\tau_i). \end{aligned} \tag{11}$$

The coefficients $\omega_k^{(i)}(\tau_i)$ and $\nu_k^{(i)}(\tau_i)$ in the expansions (11) are called boundary functions and we impose the additional condition on them:

$$\omega_k^{(i)}(\infty) = 0, \ \nu_k^{(i)}(\infty) = 0 \ \ (i = \overline{0,p}). \tag{12}$$

Substituting (10) into (4) and comparing functions of t and τ_i separately, we obtain two systems of equations

$$\begin{aligned} \varepsilon\overline{z}' &= a\overline{z} + b\overline{y}, \\ \overline{y}' &= c\overline{z} + d\overline{y}, \end{aligned} \tag{13}$$

and

$$\begin{aligned} \varepsilon\dot{\omega}^{(i)}(\tau_i,\varepsilon) &= a\omega^{(i)}(\tau_i,\varepsilon) + \varepsilon \cdot b\nu^{(i)}(\tau_i,\varepsilon), \\ \dot{\nu}^{(i)}(\tau_i,\varepsilon) &= c\omega^{(i)}(\tau_i,\varepsilon) + \varepsilon \cdot d\nu^{(i)}(\tau_i,\varepsilon). \end{aligned} \tag{14}$$

The next step is to expand all functions appearing in (13) and (14) into powers of ε and compare terms of the same order in ε. As a result, we obtain the systems

$$\begin{aligned} \varepsilon^0 : 0 &= a\overline{z}_0(t) + b\overline{y}_0(t), \\ \overline{y}_0'(t) &= c\overline{z}_0(t) + d\overline{y}_0(t), \end{aligned} \tag{15}$$

$$\varepsilon^k : \overline{z}'_{k-1}(t) = a\overline{z}_k(t) + b\overline{y}_k(t), \\ \overline{y}'_k(t) = c\overline{z}_k(t) + d\overline{y}_k(t), \tag{16}$$

and

$$\varepsilon^0 : \dot{\omega}_0^{(i)}(\tau_i) = a\omega_0^{(i)}(\tau_i), \\ \dot{\nu}_0^{(i)}(\tau_i) = c\omega_0^{(i)}(\tau_i), \tag{17}$$

$$\varepsilon^k : \dot{\omega}_k^{(i)}(\tau_i) = a\omega_k^{(i)}(\tau_i) + b\nu_{k-1}^{(i)}(\tau_i), \\ \dot{\nu}_k^{(i)}(\tau_i) = c\omega_k^{(i)}(\tau_i) + d\nu_{k-1}^{(i)}(\tau_i). \tag{18}$$

Consider the first interval $t \in [0, \theta_1]$. In order to determine the terms of expansion (10) from the obtained equations, it is necessary to set the initial conditions. To accomplish this, we substitute the series (10) into the initial value (5):

$$\overline{z}_0(0) + \varepsilon\overline{z}_1(0) + ... + \omega_0^{(0)}(0) + \varepsilon\omega_1^{(0)}(0) + ... = z^0, \\ \overline{y}_0(0) + \varepsilon\overline{y}_1(0) + ... + \varepsilon\nu_0^{(0)}(0) + \varepsilon^2\nu_1^{(0)}(0) + ... = y^0. \tag{19}$$

We equate the coefficients according to the powers of ε in both parts of the equalities.

$$\varepsilon^0 : \overline{z}_0(0) + \omega_0^{(0)}(0) = z^0, \\ \overline{y}_0(0) = y^0, \tag{20}$$

$$\varepsilon^k : \overline{z}_k(0) + \omega_k^{(0)}(0) = 0, \\ \overline{y}_k(0) + \nu_{k-1}^{(0)}(0) = 0. \tag{21}$$

To determine the approximation of order zero $\overline{z}_0(t)$ and $\overline{y}_0(t)$, we obtain the systems

$$\varepsilon^0 : 0 = a\overline{z}_0(t) + b\overline{y}_0(t), \\ \overline{y}'_0(t) = c\overline{z}_0(t) + d\overline{y}_0(t), \quad \overline{y}_0(0) = y^0.$$

To find $\omega_0^{(0)}(\tau_0)$, we must solve the equation

$$\dot{\omega}_0^{(0)}(\tau_0) = a\omega_0^{(0)}(\tau_0)$$

with this initial condition

$$\omega_0^{(0)}(0) = z^0 - \overline{z}_0(0).$$

From the second equation (17) and (12), we obtain

$$\nu_0^{(0)}(0) = \frac{c}{a}(z^0 - \overline{z}_0(0)). \tag{22}$$

It remains to solve equation

$$\dot{\nu}_0^{(0)}(\tau_0) = c\omega_0^{(0)}(\tau_0)$$

with the initial condition (22).

To determine the coefficients at $\varepsilon^k (k \geq 1)$, we obtain the systems

$$\begin{aligned} \varepsilon^k :& \overline{z}'_{k-1}(t) = a\overline{z}_k(t) + b\overline{y}_k(t), \\ & \overline{y}'_k(t) = c\overline{z}_k(t) + d\overline{y}_k(t), \quad \overline{y}_k(0) = -\nu_{k-1}^{(0)}(0). \end{aligned}$$

Solving the first equation (18)

$$\dot{\omega}_k^{(0)}(\tau_0) = a\omega_k^{(0)}(\tau_0) + b\nu_{k-1}^{(0)}(\tau_0)$$

with the initial condition

$$\omega_k^{(0)}(0) = -\overline{z}_k(0), \tag{23}$$

we find $\omega_k^{(0)}(\tau_0)$. Then, from the second equation (18), (12) we find the initial conditions

$$\nu_k^{(0)}(0) = \frac{c}{a}\omega_k^{(0)}(0) + \frac{bc - ad}{a}\int_0^\infty \nu_{k-1}^{(0)}(s)ds, \tag{24}$$

where $\omega_k^{(0)}(0)$ is expressed by the formula (23). Using the initial conditions (24) and equation

$$\dot{\nu}_k^{(0)}(\tau_0) = c\omega_k^{(0)}(\tau_0) + d\nu_{k-1}^{(0)}(\tau_0),$$

we find $\nu_k^{(0)}(\tau_0)$.

Now consider the next interval $t \in (\theta_i, \theta_{i+1}], i = 1, 2, 3, ...p$. For this interval, the initial values are $z(\theta_i+, \varepsilon)$ and $y(\theta_i+, \varepsilon)$. We substitute the series (10) into the impulsive equation (4)

$$\varepsilon(\overline{z}(\theta_i+, \varepsilon) + \omega^{(i)}(0, \varepsilon) - \overline{z}(\theta_i, \varepsilon) - \omega^{(i-1)}(\frac{\theta_i - \theta_{i-1}}{\varepsilon}, \varepsilon)) = \overline{z}(\theta_i, \varepsilon) + \omega^{(i-1)}(\frac{\theta_i - \theta_{i-1}}{\varepsilon}, \varepsilon) +$$

$$+\frac{b}{a}(\overline{y}(\theta_i, \varepsilon) + \varepsilon\nu^{(i-1)}(\frac{\theta_i - \theta_{i-1}}{\varepsilon}, \varepsilon)) + \varepsilon(\overline{z}(\theta_i, \varepsilon) + \omega^{(i-1)}(\frac{\theta_i - \theta_{i-1}}{\varepsilon}, \varepsilon)) + 2\varepsilon(\overline{y}(\theta_i, \varepsilon) + \varepsilon\nu^{(i-1)}(\frac{\theta_i - \theta_{i-1}}{\varepsilon}, \varepsilon)),$$

$$\overline{y}(\theta_i+, \varepsilon) + \varepsilon\nu^{(i)}(0, \varepsilon) - \overline{y}(\theta_i, \varepsilon) - \varepsilon\nu^{(i-1)}(\frac{\theta_i - \theta_{i-1}}{\varepsilon}, \varepsilon) =$$

$$= \overline{z}(\theta_i, \varepsilon) + \omega^{(i-1)}(\frac{\theta_i - \theta_{i-1}}{\varepsilon}, \varepsilon) + \overline{y}(\theta_i, \varepsilon) + \varepsilon\nu^{(i-1)}(\frac{\theta_i - \theta_{i-1}}{\varepsilon}, \varepsilon).$$

We equate the coefficients according to the powers of ε, taking account of (11)

$$\varepsilon^0 : 0 = \overline{z}_0(\theta_i) + \frac{b}{a}\overline{y}_0(\theta_i), \tag{25}$$

$$\begin{aligned} \varepsilon^1 :& \omega_0^{(i)}(0) = \overline{z}_1(\theta_i) + \frac{b}{a}\overline{y}_1(\theta_i) + \overline{z}_0(\theta_i) + 2\overline{y}_0(\theta_i) - \Delta\overline{z}_0|_{t=\theta_i}, \\ & \Delta\overline{y}_0|_{t=\theta_i} = \overline{z}_0(\theta_i) + \overline{y}_0(\theta_i), \end{aligned} \tag{26}$$

$$\begin{aligned}\varepsilon^k :&\omega_k^{(i)}(0) = \overline{z}_{k+1}(\theta_i) + \frac{b}{a}\overline{y}_{k+1}(\theta_i) + \overline{z}_k(\theta_i) + 2\overline{y}_k(\theta_i) - \Delta\overline{z}_k|_{t=\theta_i},\\ &\Delta\overline{y}_k|_{t=\theta_i} = \overline{z}_k(\theta_i) + \overline{y}_k(\theta_i) - \nu_{k-1}^{(i)}(0).\end{aligned} \tag{27}$$

To determine the approximation of order zero $\overline{z}_0(t)$ and $\overline{y}_0(t)$, we obtain the systems

$$\begin{aligned}\varepsilon^0 :&0 = a\overline{z}_0(t) + b\overline{y}_0(t),\\ &\overline{y}_0'(t) = c\overline{z}_0(t) + d\overline{y}_0(t), \quad \Delta\overline{y}_0|_{t=\theta_i} = \overline{z}_0(\theta_i) + \overline{y}_0(\theta_i).\end{aligned}$$

To find $\omega_0^{(i)}(\tau_i)$, we must solve the equation

$$\dot{\omega}_0^{(i)}(\tau_i) = a\omega_0^{(i)}(\tau_i)$$

with this initial condition

$$\omega_0^{(i)}(0) = \frac{1}{a}\overline{z}_0'(\theta_i) + \overline{z}_0(\theta_i) + 2\overline{y}_0(\theta_i) - \Delta\overline{z}_0|_{t=\theta_i}, \tag{28}$$

where $\frac{1}{a}\overline{z}_0'(\theta_i) = \overline{z}_1(\theta_i) + \frac{b}{a}\overline{y}_1(\theta_i)$. From the second equation (17) and (12), we obtain

$$\nu_0^{(i)}(0) = \frac{c}{a}\omega_0^{(i)}(0), \tag{29}$$

where $\omega_0^{(i)}(0)$ is expressed by the formula (28). It remains to solve equation

$$\dot{\nu}_0^{(i)}(\tau_i) = c\omega_0^{(i)}(\tau_i)$$

with the initial conditions (29).

To determine the coefficients at $\varepsilon^k (k \geq 1)$, we obtain the systems

$$\begin{aligned}\varepsilon^k :&\overline{z}_{k-1}'(t) = a\overline{z}_k(t) + b\overline{y}_k(t),\\ &\overline{y}_k'(t) = c\overline{z}_k(t) + d\overline{y}_k(t), \quad \Delta\overline{y}_k|_{t=\theta_i} = \overline{z}_k(\theta_i) + \overline{y}_k(\theta_i) - \nu_{k-1}^{(i)}(0).\end{aligned}$$

Solving the first equation (18)

$$\dot{\omega}_k^{(i)}(\tau_i) = a\omega_k^{(i)}(\tau_i) + b\nu_{k-1}^{(i)}(\tau_i)$$

with the initial condition

$$\omega_k^{(i)}(0) = \frac{1}{a}\overline{z}_k'(\theta_i) + \overline{z}_k(\theta_i) + 2\overline{y}_k(\theta_i) - \Delta\overline{z}_k|_{t=\theta_i}, \tag{30}$$

we find $\omega_k^{(i)}(\tau_i)$, where $\frac{1}{a}\overline{z}_k'(\theta_i) = \overline{z}_{k+1}(\theta_i) + \frac{b}{a}\overline{y}_{k+1}(\theta_i)$. Then from the second equation (18) and (12), we find the initial conditions

$$\nu_k^{(i)}(0) = \frac{c}{a}\omega_k^{(i)}(0) + \frac{bc - ad}{a}\int_0^\infty \nu_{k-1}^{(i)}(s)ds, \tag{31}$$

where $\omega_k^{(0)}(0)$ is expressed by the formula (30). Using the initial conditions (31) and equation

$$\dot{\nu}_k^{(i)}(\tau_i) = c\omega_k^{(i)}(\tau_i) + d\nu_{k-1}^{(i)}(\tau_i),$$

we find $\nu_k^{(i)}(\tau_i)$. Thus, under the assumptions made, the terms of the series (10) $\overline{z}_k(t), \overline{z}_k(t), \omega_k^{(i)}(\tau_i), \nu_k^{(i)}(\tau_i), i = \overline{1,p}$ can be determined up to and including $k = n$.

Теорема 1 *There exist positive constants ε_0 and c such that for $\varepsilon \in (0, \varepsilon_0]$ there exists a unique solution $z(t,\varepsilon), y(t,\varepsilon)$ of problem (4),(5) on the segment [0,T], which satisfies the inequality*

$$\begin{aligned} &|z(t,\varepsilon) - Z_n(t,\varepsilon)| \le c\varepsilon^{n+1}, \quad 0 \le t \le T, \\ &|y(t,\varepsilon) - Y_n(t,\varepsilon)| \le c\varepsilon^{n+1}, \quad 0 \le t \le T, \end{aligned} \tag{32}$$

where

$$\begin{aligned} Z_n(t,\varepsilon) &= \sum_{k=0}^{n} \varepsilon^k \overline{z}_k(t) + \sum_{k=0}^{n} \varepsilon^k \omega_k^{(i)}(\tau_i), \\ Y_n(t,\varepsilon) &= \sum_{k=0}^{n} \varepsilon^k \overline{y}_k(t) + \varepsilon \sum_{k=0}^{n} \varepsilon^k \nu_k^{(i)}(\tau_i). \end{aligned} \tag{33}$$

3 Example

Consider the system

$$\begin{aligned} &\varepsilon \dot{z} = -4z + 3y, \qquad \varepsilon \Delta z|_{t=\theta_i} = z - \frac{3}{4}y + \varepsilon z + 2\varepsilon y, \\ &\dot{y} = y - 3z, \qquad \Delta y|_{t=\theta_i} = z + y, \end{aligned} \tag{34}$$

with initial conditions

$$z(0,\varepsilon) = 1, \quad y(0,\varepsilon) = 2. \tag{35}$$

where $a = -4 < 0$, $\theta_i = i, i = 1,2,3$. Let us take $\varepsilon = 0$ in this problem. Then, the first equation becomes $-4\overline{z} + 3\overline{y} = 0$. We find $\overline{z} = \varphi = \frac{3}{4}\overline{y}$ and substitute it into (34) with the initial value (35), to obtain

$$\begin{aligned} &\overline{y}' = -\frac{5}{4}\overline{y}, \quad \Delta \overline{y}|_{t=\theta_i} = \frac{7}{4}\overline{y}, \\ &\overline{y}(0) = 2. \end{aligned}$$

Using the formulas (8) and (9), we find $\overline{y}(\theta_i) = 2(\frac{11}{4})^{i-1}(\exp(-\frac{5}{4}))^{\theta_i}$ and $\overline{z}(\theta_i) = \frac{3}{2}(\frac{11}{4})^{i-1}(\exp(-\frac{5}{4}))^{\theta_i}$, respectively. We check the condition

$$\frac{\partial}{\partial z}(-4z + 3y) = -4 < 0.$$

Therefore, $\overline{z} = \varphi = \dfrac{3}{4}\overline{y}$ is uniformly asymptotically stable. Now let's check the condition $(**)$. Then

$$\lim_{(z,y,\varepsilon)\to(\varphi,\overline{y},0)} \frac{z - \frac{3}{4}y + \varepsilon z + 2\varepsilon y}{\varepsilon} = \frac{11}{4}\overline{y}(\theta_i) = 2(\frac{11}{4})^i(\exp(-\frac{5}{4}))^{\theta_i} \neq 0.$$

Then we can determine the terms of the asymptotic in a similar way. The solution $z(t,\varepsilon)$ of system (34) with an initial value of (35) has multi-layers at $t = 0$ and $t = \theta_i+, i = 1, 2, 3$. Obviously, in Figure 1, it can be seen that multi-layers occur.

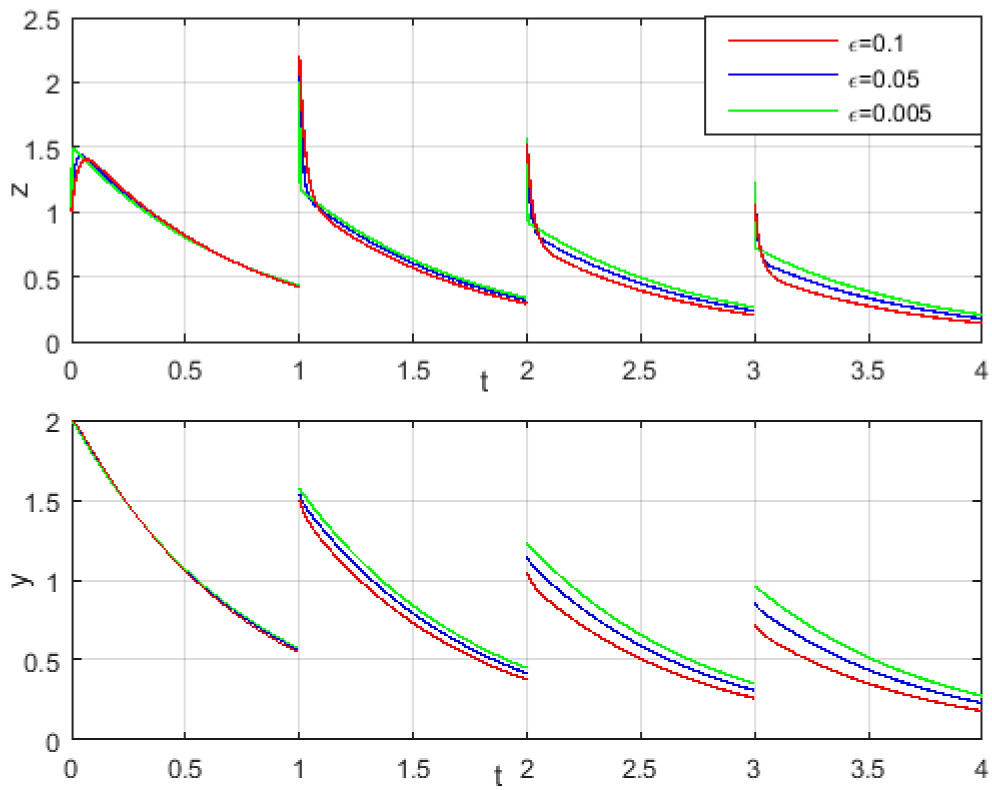

Figure 1: Red, blue, and green lines represent the coordinates of system (34) with initial values $z(0,\varepsilon) = 1$ and $y(0,\varepsilon) = 2$ for various values of $\varepsilon : 0.1, 0.05, 0.005$, respectively.

4 Conclusion

This article is devoted to the study of a new type of singular impulsive differential equation model. The method of boundary functions is used to construct the desired asymptotic solutions. We constructed the asymptotic expansion of solutions with an arbitrary degree of accuracy with respect to a small parameter. An illustrative example using simulations is given to support the theoretical results.

In several books [7–9] considered impulsive systems with small parameter involved only in the differential equations of the systems but not in their impulsive equations. We introduce a small parameter into the impulse equation. So we have significantly extended the singularity concept for discontinuous dynamics.

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References

[1] Akhmet M., Çağ S. Tikhonov theorem for differential equations with singular impulses. Discontinuity, Nonlinearity, and Complexity 2018; 7(3): 291-303.

[2] Akhmet M. Principles of Discontinuous Dynamical Systems. New York: Springer, 2010.

[3] Akhmet M. Nonlinear Hybrid Continuous/Discrete-Time Models. Paris: Atlantis Press, 2011.

[4] Akhmet M., Fen MO. Replication of Chaos in Neural Networks, Economics and Physics. Nonlinear Physical Science. Berlin Heidelberg: Springer, 2015.

[5] Akhmet, M., Cag, S. Chattering as a Singular Problem, Nonlinear Dynamics 2017; 90(4): 2797–2812.

[6] Akhmet M., Dauylbayev M., Mirzakulova A. A singularly perturbed differential equation with piecewise constant argument of generalized type, Turkish Journal of Mathematics 2018; 42(4): 1680-1685.

[7] Bainov D, Covachev V. Impulsive Differential Equations with a Small Parameter. World Scientific, 1994.

[8] Chen WH, Chen F, Lu X. Exponential stability of a class of singularly perturbed stochastic time-delay systems with impulse effect. Nonlinear Anal- Real 2010; 11(5):3463 3478.

[9] Chen WH, Wei D, Lu X. Exponential stability of a class of nonlinear singularly perturbed systems with delayed impulses. J Franklin I 2013; 350(9):2678 2709.

[10] Chen WH, Yuan G, Zheng WX. Robust stability of singularly perturbed impulsive systems under nonlinear perturbation. IEEE T Automat Contrt 2013; 58(1):168 174.

[11] Damiano ER, Rabbitt RD. A singular perturbation model of uid dynamics in the vestibular semicircular canal and ampulla. J Fluid Mech 1996; 307:333 372.

[12] Donchev T, Slavov I. Tikhonov's theorem for functional-differential inclusions. Annuaire Univ Sofia Fac Math Inform 1995; 89:6978.

[13] Gondal I. On the application of singular perturbation techniques to nuclear engineering control problems. IEEE T Nucl Sci 1988; 35:1080 1085.

[14] Hek G. Geometric singular perturbation theory in biological practice. J Math Biol 2010; 60:347386.

[15] Kadalbajoo MK, Patidar KC. Singularly perturbed problems in partial differential equations: a survey. Appl Math Comput 2003; 134(23):371 429.

[16] Kokotovic PV. Applications of singular perturbation techniques to control problems. SIAM Rev 1984; 26:501 550.

[17] O'Malley REJ. Singular Perturbation Methods for Ordinary Differential Equations. Applied Mathematical Sciences. New York: Springer, 1991.

[18] Owen MR, Lewis MA. How predation can slow, stop, or reverse a prey invasion. B Math Biol 2001; 63:655 684.

[19] Segel LA, Slemrod M. The quasi-steady state assumption: a case study in perturbation. SIAM Rev 1989; 31:446 477.

[20] Simeonov P, Bainov D. Stability of the solutions of singularly perturbed systems with impulse effect. J Math Anal Appl 1988; 136(2):575 588.

[21] Tikhonov AN. Systems of differential equations containing small parameters in the derivatives. Matematicheskii sbornik 1952; 73(3):575 586.

[22] Vasil'eva A. B., Butuzov V. F. Asymptotic Expansion of Solutions of Singularly Perturbed Equations. Nauka 1973. (in Russian)

[23] Vasil'eva A, Butuzov V, Kalachev L. The Boundary Function Method for Singular Perturbation Problems. Society for Industrial and Applied Mathematics, 1995.

[24] Veliov V. A generalization of the Tikhonov theorem for singularly perturbed differential inclusions. J Dyn Control Syst 1997; 3(3):291 319.