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GLOBAL SOLVABILITY OF INVERSE PROBLEM FOR LINEAR KELVIN-VOIGT EQUATIONS WITH MEMORY

In this paper, the inverse problem for a linear system of Kelvin-Voigt equations with memory describing the dynamics of a viscoelastic incompressible non-newtonian fluid is considered. In the inverse problem under consideration, along with the solution (velocity and fluid pressure) of the equation, it is also required to find the unknown (intensity of the external force) on the right side, which depends only on the time variable. Definitions of weak and strong solutions are given. Weak and strong solutions of the set inverse problems satisfy the boundary condition of sliding at the boundary. The sliding boundary condition gives a mathematical and physical character to the study of a linear system of Kelvin-Voigt equations with memory. The applicability of the Faedo-Galerkin method for this type of system of equations is analyzed. With the help of the Faedo-Galerkin method, the global theorem of the existence of solutions to the presented inverse problem is proved in a weak and strong generalized sense. To prove the theorem of the existence of a solution "as a whole"in time, it is associated with obtaining a priori estimates, the constants in which depend only on the data of the problem and the magnitude of the time interval. And also the uniqueness theorem of the solutions of the considered inverse problems for a linear system of Kelvin-Voigt equations with memory is obtained and proved.
Key words: Inverse problem, Kelvin-Voigt system with memory, global existence and uniqueness.

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Жады бар Кельвин-Фойгт теңдеуі үшін кері есептің глобалды шешімділігі

Бұл жұмыста тұтқыр серпімді сығылмайтын ньютондық емес сұйықтықтардың қозғалысын сипаттайтын интегро-дифференциалдық (жады бар) Кельвин-Фойгт теңдеулер жүйесі үшін қойылған кері есеп қарастырылады. Кері есепте теңдеудің шешімі болып табылатын сұйықтың жылдамдығын және қысымын, сонымен қатар сыртқы күштердің интенсивтілігі деп аталатын уақыттан әуелді оң жағын анықтау көзделген. Мақалада жалпылама әлсіз және әлді шешімдердің анықтамасы берілді. Кері есептің жалпылама әлсіз және әлді шешімдері шекарада жүгу шекаралық шартын қанағаттандырады. Жүгу шекаралық шарты өз кезегінде интегро-дифференциалдық Кельвин-Фойгт теңдеулер жүйесін математикалық және физикалық тұрғыдан зерттеуде үлкен ғылыми қызығушылық туғызады. Жұмыста қарастырылып отырған кері есепке Фаэдо-Галеркин әдісінің қолданысы талқыланады. Фаэдо-Галеркин әдісінің көмегімен кері есептің жалпылама әлсіз және әлді шешімдерінін кез келген уакыт мезеті бойынша глобальды бар болуы туралы теорема дәлелденді. Теореманы дәлелдеу априорлық бағалаулар алуға негізделген, алынған априорлық бағалаулар есептің берілгендерінен тәуелді болып табылады. Сонымен қоса, қарастырылып отырған интегро-дифференциалдық Кельвин-Фойгт теңдеулер жүйесі үшін қойылған кері есептің жалғыздығы туралы теорема алынды және ол априорлық бағалаулар негізінде дәлелденді.
Түйін сөздер: Кері есеп, жады бар Кельвин-Фойгт жүйесі, шешімнің глобалды бар болуы және жалғыздығы.

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Глобальная разрешимость обратной задачи для линейных уравнений Кельвина-Фойгта с памятью

В данной работе рассматривается обратная задача для линейной системы уравнений Кельвина-Фойгта с памятью, описывающей динамику вязкоупругой несжимаемой неньютоновской жидкости. В рассматриваемой обратной задаче вместе с решением (скорость и давление жидкости) уравнения, требуется также найти неизвестное (интенсивность внешней силы) в правой части, которое зависит только от временной переменной. Даны определения слабых и сильных решений. Слабые и сильные решения поставленных обратных задач удовлетворяют краевым условиям проскальзывания на границе. Поставленное краевое условие придает математический и физический характер изучению линейной системы уравнений Кельвина-Фойгта с памятью. Разобрана применимость метода Фаэдо-Галеркина для данного типа системы уравнений. С помощью методом Фаэдо-Галеркина глобальная теорема существования решения рассматриваемых обратных задач доказана в слабом и сильным обобщенном смысле. Для доказательство теоремы существования решения "в целом"по времени связано с получением априорных оценок, постоянные в которых зависят только от данных задачи и величины интервала времени. А также получена теорема единственности решения рассматриваемой обратной задач для линейной системы уравнений Кельвина-Фойгта с памятью.

Ключевые слова: Обратная задача, система Кельвина-Фойгта с памятью, глобальное существование и единственность.

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^d,\ d=2,3$ with a smooth boundary $\partial\Omega$, and $Q_T=\Omega\times(0,T)$, T is a fixed positive finite constant, and $\Gamma_T=\partial\Omega\times[0,T]$. This paper deals with the recovering of a solely time-dependent source function $f(t)$ in the system of integro-differential Kelvin-Voigt equations governing flows of incompressible viscoelastic fluids. More precisely, we study the following inverse problems of determining the functions $(\mathbf{u}(x,t),p(x,t),f(t))$, from the system of equations

$$\mathbf{u}_t-\varkappa\Delta\mathbf{u}_t-\nu\Delta\mathbf{u}-\int\limits_0^t K(t-s)\Delta\mathbf{u}(\mathbf{x},s)ds-\nabla p=f(t)\mathbf{g}(\mathbf{x},t)\ \text{ in }\ Q_T, \tag{1}$$

$$\mathbf{div}\,\mathbf{u}(\mathbf{x},t)=0\ \text{ in }\ Q_T, \tag{2}$$

supplemented with the initial condition

$$\mathbf{u}(\mathbf{x},0)=\mathbf{u}_0(\mathbf{x})\text{ in }\Omega, \tag{3}$$

the sliding-boundary condition [3–5]

$$\mathbf{u}_{\mathbf{n}}(\mathbf{x},t)=\mathbf{u}\cdot\mathbf{n}=0,\quad \mathbf{curl}\,\mathbf{u}\times\mathbf{n}=0,\ (\mathbf{x},t)\in\Gamma_T \tag{4}$$

and overdeterminition condition

$$\int_\Omega \mathbf{u}\omega(\mathbf{x})d\mathbf{x}=e(t),\quad t\in[0,T], \tag{5}$$

where $\mathbf{u_n}$ is the normal component of $\mathbf{u}(\mathbf{x},t)$ on $\partial\Omega$, and $\mathbf{n}$ denotes the unit outward normal vector to $\partial\Omega$.

Here the bold letters denote vector-valued functions and $\mathbf{u}(\mathbf{x},t) = (u_1,u_2,...,u_n)$ and $p(\mathbf{x},t)$ are respectively a velocity field and a pressure, and ν and $\varkappa > 0$ are coefficients of the kinematic viscosity and relaxation of the fluids, respectively. The vector-function $\mathbf{g}(\mathbf{x},t)$ $\mathbf{u}_0(\mathbf{x})$, $\omega(\mathbf{x})$, $e(t)$ and $K(t)$ are given functions. The intensity of external force $f(t)$, a velocity field $\mathbf{u}$ and a pressure p are unknown functions.

The system of equations (1)-(2) is called a system of Kelvin-Voigt (Navier-Stokes-Voigt) equation with memory [8] or Oskolkov system [9] and it models of an incompressible viscoelastic non-Newtonian fluids. The integral term in (1) with the convolution kernel $K(t)$ is a memory term, which designs the viscoelastic property of non-Newtonian fluids. For the details on the physical background and its mathematical modeling, we refer readers to [8, 10–12].

The well-posedness of various direct problems for (1)-(2), i.e. in the case the external source term $\mathbf{F}(x,t) = f(t)\mathbf{g}(\mathbf{x},t)$ is given, have been investigated by different authors, see for instant [9, 10, 13], and references there in.

The local existence and uniqueness of solutions to the presented inverse problem (1)-(5) was established in work [1] to the case when the given data satisfy

$$\frac{\varkappa}{k_0^2}\sup_{t\in[0,T]}\|\mathbf{g}(t)\|_{2,\Omega}^2\,\|\omega\|_{\mathbf{V}^1(\Omega)}^2 \le m < 2.$$

The main goal of this paper is to investigate global in time existence and uniqueness of weak and strong solutions to the inverse problem (1)-(5) in the patricular case $\mathbf{g}(\mathbf{x},t) = \omega(x)$.

The outline of the paper is the following. In Section 2, we introduce the functional spaces and some auxiliary materials related to the boundary condition (4), reduce the investigating inverse problems to an equivalent nonlocal direct problems and define a weak and strong solutions.The uniqueness of weak and strong solutions of these inverse problems is proved in Section 3. The existence of weak solutions of inverse problems (1)-(5) is established in Section 5. Section 6 devoted to prove the global in time existence of strong solution of inverse problem.

2 Preliminaries.

In this section, we introduce the main functional spaces and some useful inequalities related to the boundary condition (4) from [3].

We denote by $\mathbf{L}^2(\Omega)$ the usual Lebesgue space of square integrable vector-valued functions on Ω, and by $\mathbf{W}^{m,2}(\Omega)$ the Sobolev space of functions in $\mathbf{L}^2(\Omega)$ whose weak derivatives of order not greater than m are in $\mathbf{L}^2(\Omega)$. The norm and inner product in $\mathbf{L}^2(\Omega)$ denoted by $\|\cdot\|_{2,\Omega}$ and $(\cdot,\cdot)_{2,\Omega}$, respectively.

We use the following functional spaces, introduced in [3] and [2]

$$\mathcal{V}(\Omega) := \{\mathbf{u} \in \mathbf{C}_0^\infty(\Omega) : \operatorname{div}\mathbf{u} = 0\},$$
$$\mathbf{H_n}(\Omega) := \text{closure of } \mathcal{V} \text{ in the norm of } \mathbf{L}^2(\Omega), \text{ and}$$
$$\mathbf{H_n^1}(\Omega) := \text{closure of } \mathcal{V} \text{ in the norm of } \mathbf{W}^{1,2}(\Omega);$$
$$\mathbf{H_n^2}(\Omega) := \mathbf{H_n^1}(\Omega) \cap \mathbf{W}^{2,2}(\Omega).$$

The weak and strong solutions of (1)-(5) understood as the sense as in the following definitions. Similarly, definitions of weak and strong solutions of other inverse problems are will defined, replacing corresponding boundary or overdetermination conditions.

Definition 1 *The functions* $(\boldsymbol{u}, f)$ *is a weak solution to the inverse problem (1)-(5), if:*

1. $\mathbf{u} \in \mathbf{L}^\infty(0,T;\mathbf{H}^1_{\mathbf{n}}(\Omega)) \cap \mathbf{L}^2(0,T;\mathbf{H}^1_{\mathbf{n}}(\Omega)),\ \mathbf{u}_t \in \mathbf{L}^2(0,T;\mathbf{H}^1_{\mathbf{n}}(\Omega)),\ f(t) \in L^2[0,T]$;

2. $\boldsymbol{u}(0) = \boldsymbol{u}_0$ *a.e. in* Ω;

3. *For every* $\boldsymbol{\varphi} \in \mathbf{H}^1_{\mathbf{n}}(\Omega)$ *and for a.a.* $t \in (0,T)$ *holds*

$$\frac{d}{dt}\Big((\mathbf{u},\varphi)_{2,\Omega} + \varkappa\,(\mathbf{curl}\,\mathbf{u},\mathbf{curl}\,\varphi)_{2,\Omega}\Big) + \nu\,(\mathbf{curl}\,\mathbf{u},\mathbf{curl}\,\varphi)_{2,\Omega} + \int\limits_0^t K(t-s)\,(\mathbf{curl}\,\mathbf{u},\mathbf{curl}\,\varphi)_{2,\Omega}\,ds = f(t)\,(\omega,\varphi)_{2,\Omega}\,. \tag{6}$$

Definition 2 *The functions* $(\boldsymbol{u}, f)$ *is a strong solution to the inverse problem (1)-(5), if:*

1. $\mathbf{u} \in \mathbf{L}^\infty(0,T;\mathbf{H}^1_{\mathbf{n}}(\Omega)\cap\mathbf{H}^2_{\mathbf{n}}(\Omega)) \cap \mathbf{L}^2(0,T;\mathbf{H}^1_{\mathbf{n}}(\Omega)\cap\mathbf{H}^2_{\mathbf{n}}(\Omega)),\ \mathbf{u}_t \in \mathbf{L}^2(0,T;\mathbf{H}^2_{\mathbf{n}}(\Omega)),\ f(t) \in L^2[0,T]$;

2. *Each equation holds in the distribution sense in the their corresponding domain.*

Assume that data of the problem satisfy the following conditions

$$\mathbf{u}_0(\mathbf{x}) \in \mathbf{H}^1_{\mathbf{n}}(\Omega); \tag{7}$$

$$\omega_0 = \|\omega(\mathbf{x})\|^2_{2,\Omega} < \infty; \tag{8}$$

$$\omega(\mathbf{x}) \in \mathbf{H}^1_{\mathbf{n}}(\Omega),\ \ e(t) \in W^1_2([0,T]); \tag{9}$$

$$(\mathbf{u}_0,\omega)_{2,\Omega} = e(0); \tag{10}$$

$$K(t) \in L^2([0,T]): \quad \|K(t)\|_{L^2([0,T])} \equiv K_0 < \infty. \tag{11}$$

3 Uniqueness of inverse problem

Theorem 1 *Let the conditions* (7)-(11) *are valid. Then a weak and all the more strong solution of the problem (1)-(5) is unique.*

Proof 1 *Let* $(\mathbf{u}_1, f_1)$ *and* $(\mathbf{u}_2, f_2)$ *be two weak solutions to the inverse problem (1)-(5) regarding to same data. Subtracting the equation (1) for* $(\mathbf{u}_2, f_2)$ *to the equation for* $(\mathbf{u}_1, f_1)$*, and taking inner product at (1) with* $\mathbf{u} := \mathbf{u}_1 - \mathbf{u}_2$ *in* $L_2(\Omega)$ *and using (5), we obtain*

$$\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{u}\|_{2,\Omega}^2 + \varkappa\|\mathbf{u}\|_{\mathbf{H}^1_{\mathbf{n}}(\Omega)}^2\right) + \nu\|\mathbf{u}\|_{\mathbf{H}^1_{\mathbf{n}}(\Omega)}^2 = -\int\limits_0^t K(t-\tau)\left(\mathbf{curl}\,\mathbf{u}(s), \mathbf{curl}\,\mathbf{u}(t)\right)_{2,\Omega} ds. \quad (12)$$

where

$$\|f(t)\|^2_{L^2[0,T]} \le \frac{1}{\omega_0}\left[\|\omega\|_{\mathbf{H}^1_{\mathbf{n}}(\Omega)}\left(\varkappa\|\mathbf{u}^n_t\|_{\mathbf{H}^1_{\mathbf{n}}(\Omega)} + \nu\|\mathbf{u}^n\|_{\mathbf{H}^1_{\mathbf{n}}(\Omega)} + K_0\left(\int\limits_0^t \|\mathbf{u}^n(s)\|^2_{\mathbf{H}^1_{\mathbf{n}}(\Omega)} ds\right)^{\frac{1}{2}}\right)\right].$$

Using the Hölder and Young inequalities estimate the terms on the right hand side of (12)*, analogical as we have got* (22) *with assumption* $\varepsilon_0 = \nu$ *and integrating by* s *from* 0 *to* $t \in [0,T]$*, we obtain the following integral inequality*

$$y(t) + \nu\|\mathbf{u}\|^2_{\mathbf{L}^2(0,T;\mathbf{H}^1_{\mathbf{n}}(\Omega))} \le C\int\limits_0^t y(s)ds \quad (13)$$

for the function $y(t) = \|\mathbf{u}\|_{2,\Omega}^2 + \varkappa\|\mathbf{u}\|^2_{\mathbf{H}^1_{\mathbf{n}}(\Omega)}$ *with* $y(0) = 0$*, where* $C = \frac{K_0^2 T}{\nu\varkappa}$ *is a positive finite number.*
Therefore, by Granwall lemma, it follows from (13) that $y(t) \equiv 0$ *for all* $t \in [0,T]$*, i.e.* $\mathbf{u}_1 \equiv \mathbf{u}_2$ *and* $f_1 = f_2$*.*

4 An equivalent nonlocal direct problems

Let us multiply the equation (1) by $\omega(\mathbf{x})$ and integrate over Ω. Integrating by parts and using (5) and the assumption (8), we define $f(t)$

$$f(t) = \frac{1}{\omega_0}\left(e'(t) + \varkappa\left(\mathbf{curl}\mathbf{u}_t, \mathbf{curl}\omega\right)_{2,\Omega} + \nu\left(\mathbf{curl}\mathbf{u}, \mathbf{curl}\omega\right)_{2,\Omega} + \int\limits_0^t K(t-s)\left(\mathbf{curl}\mathbf{u}, \mathbf{curl}\omega\right)_{2,\Omega} ds\right).$$

Substituting $f(t)$ into the system (1)-(2), we obtain the following system of nonlocal equations for unknown functions $\mathbf{u}$ and p

$$\mathbf{u}_t - \varkappa\Delta\mathbf{u}_t - \nu\Delta\mathbf{u} - \int\limits_0^t K(t-s)\Delta\mathbf{u}(\mathbf{x},s)ds - \nabla p = \frac{1}{\omega_0}\Big(e'(t) + \varkappa\left(\mathbf{curl}\,\mathbf{u}_t, \mathbf{curl}\,\omega\right)_{2,\Omega} +$$

$$\nu\left(\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\omega\right)_{2,\Omega} + \int\limits_0^t K(t-s)\left(\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\omega\right)_{2,\Omega} ds\Big)\,\omega(\mathbf{x}),\ \ \mathbf{div}\,\mathbf{u}(\mathbf{x},t) = 0.$$

(14)

The following lemma is valid.

4.1 Reducing to an equivalent nonlocal direct problems

Лемма 1 *Assume that the conditions* (8)-(10) *are fulfilled. Then the solvability of the inverse problem (1)-(5) is equivalent to the nonlocal direct problem (14),* (3), (4).

Proof of the Lemma 1 is similar to the lemmas in [1] and [7].

5 Existence of a weak solution of inverse problem (1)-(5)

By Lemma 1, we prove the existence of solutions of the nonlocal direct problem (14), (3)-(4) for the existence of solutions of inverse problems (1)-(5), respectively.

Theorem 2 *Let the conditions (7)-(11) be fulfilled. Then the nonlocal direct problem (14), (3)-*(4) *has, at least, a global weak solution* $\mathbf{u}(\mathbf{x},t)$ *in* Q_T. *Moreover, the weak solutions satisfy the estimate*

$$\|\mathbf{u}\|^2_{\mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega)\cap\mathbf{H}^1_{\mathbf{n}}(\Omega))} + \|\mathbf{u}\|^2_{\mathbf{L}^2(0,T;\mathbf{H}^1_{\mathbf{n}}(\Omega))} + \|\mathbf{u}_t\|^2_{\mathbf{L}^2(0,T;\mathbf{L}^2(\Omega)\cap\mathbf{H}^1_{\mathbf{n}}(\Omega))} \le C, \tag{15}$$

where C *is a constant depending on data of the problem.*

Proof 2 *The existence of this theorem consists of the steps: constructing Galerkin's approximations, obtain first and second energy estimates for Galerkin's approximations and passage to the limit.*

5.1 Galerkin's approximations

Let $\{\varphi_k\}_{k\in\mathbb{N}}$ be an orthonormal family in $\mathbf{L}^2(\Omega)$ formed by functions of $\mathbf{H_n}$ whose linear combinations are dense in $\mathbf{H}^1_{\mathbf{n}}(\Omega)$. Given $n\in\mathbb{N}$, let us consider the n-dimensional space $\mathbf{X}^n$ spanned by φ_k, $k=1,\dots,n$, respectively. For each $n\in\mathbb{N}$, we search for approximate solutions to the problem (14),(3)-(4) in the form

$$\mathbf{u}^n(x,t) = \sum_{j=1}^{n} c_j^n(t)\varphi_j(x), \quad \varphi_j \in \mathbf{X}^n, \tag{16}$$

where unknown coefficient $c_j^n(t)$, $j=1,...,n$ are defined as solutions of the following system of ordinary differential equations (ODE) derived from

$$\begin{gathered}\frac{d}{dt}\Big((\mathbf{u}^n,\varphi_k)_{2,\Omega} + \varkappa\,(\mathbf{curl}\,\mathbf{u}^n,\mathbf{curl}\,\varphi_k)_{2,\Omega}\Big) + \nu\,(\mathbf{curl}\,\mathbf{u}^n,\mathbf{curl}\,\varphi_k)_{2,\Omega} = \\ -\int_0^t K(t-s)\,(\mathbf{curl}\,\mathbf{u}^n,\mathbf{curl}\,\varphi_k)_{2,\Omega}\,ds + \frac{1}{\omega_0}\Big(e'(t) + \varkappa\,(\mathbf{curl}\,\mathbf{u}^n_t,\mathbf{curl}\,\omega)_{2,\Omega} + \\ \nu\,(\mathbf{curl}\,\mathbf{u}^n,\mathbf{curl}\,\omega)_{2,\Omega} + \int_0^t K(t-s)\,(\mathbf{curl}\,\mathbf{u}^n,\mathbf{curl}\,\omega)_{2,\Omega}\,ds\Big)\,(\omega,\varphi_k)_{2,\Omega}\,,\end{gathered} \tag{17}$$

for $k = 1,\ 2,\ \dots,\ n$.

The system (17) is supplemented with the following Cauchy data

$$\mathbf{u}^n(0) = \mathbf{u}_0^n \quad \text{in } \Omega. \tag{18}$$

where

$$\mathbf{u}_0^n = \sum_{j=1}^{n} (\mathbf{u}_0, \varphi_j)_{2,\Omega}\, \varphi_j$$

is sequence in $\mathbf{L}^2(\Omega) \cap \mathbf{H}^1_{\mathbf{n}}(\Omega)$ respectively such that

$$\mathbf{u}_0^n \to \mathbf{u}_0(x) \text{ strong as } n \to \infty \ \text{ in } \ \mathbf{L}^2(\Omega) \cap \mathbf{H}^1_{\mathbf{n}}(\Omega). \tag{19}$$

According to a general theory of ordinary differential equations, the system (17)-(18) has a solution $c_j^n(t)$ in $[0, t_0]$. By a priori estimates which we shall establish below, the solution can be extended to $[0, T_0] \subset [0, T]$, where $[0, T_0]$ is a maximal time interval, such that a priori estimates are hold.

Proof 3 *Multiply (17) by $c_k^n(t)$ and $\frac{d\, c_k^n(t)}{d\, t}$ and sum up the results from $k = 1$ until $k = n$, and integrate over Ω. Using the formula of integrating by parts, and adding results, we obtain*

$$\begin{gathered}\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{u}^n\|^2_{2,\Omega} + (\nu + \varkappa)\, \|\mathbf{u}^n\|^2_{\mathbf{H}^1_{\mathbf{n}}(\Omega)}\right) + \nu\, \|\mathbf{u}^n\|^2_{\mathbf{H}^1_{\mathbf{n}}(\Omega)} + \|\mathbf{u}^n_t(t)\|^2_{2,\Omega} + \varkappa\, \|\mathbf{u}^n_t(t)\|^2_{\mathbf{H}^1_{\mathbf{n}}(\Omega)} = \\ -\int_0^t K(t-s)\, (\mathbf{curl}\, \mathbf{u}^n(s), \mathbf{curl}\, \mathbf{u}^n(t))_{2,\Omega}\, ds + \Phi(\mathbf{u}^n, t) e(t) - \\ \int_0^t K(t-s)\, (\mathbf{curl}\, \mathbf{u}^n(s), \mathbf{curl}\, \mathbf{u}^n_t(t))_{2,\Omega}\, ds + \Phi(\mathbf{u}^n, t) e'(t) = \sum_{i=1}^{4} J_i,\end{gathered} \tag{20}$$

$$\begin{gathered}\textit{where}\ \ \Phi(\mathbf{u}^n, t) := e'(t) + \varkappa\, (\mathbf{curl}\, \mathbf{u}^n_t(t), \mathbf{curl}\, \omega)_{2,\Omega} + \\ \nu\, (\mathbf{curl}\, \mathbf{u}^n(t), \mathbf{curl}\, \omega)_{2,\Omega} + \int_0^t K(t-s)\, (\mathbf{curl}\, \mathbf{u}^n(t), \mathbf{curl}\, \omega)_{2,\Omega}\, ds.\end{gathered} \tag{21}$$

Next, estimate the term on the right-hand side of (20)

$$|J_1| \le \left| -\int_0^t K(t-s)\, (\mathbf{curl}\, \mathbf{u}^n(s), \mathbf{curl}\, \mathbf{u}^n(t))_{2,\Omega}\, ds \right| \le \frac{\nu}{6}\, \|\mathbf{u}^n\|^2_{\mathbf{H}^1_{\mathbf{n}}(\Omega)} + \frac{3K_0^2}{2\nu} \int_0^t \|\mathbf{u}^n(s)\|^2_{\mathbf{H}^1_{\mathbf{n}}(\Omega)}\, ds, \tag{22}$$

$$|J_2| \le \left| -\int_0^t K(t-s)\, (\mathbf{curl}\, \mathbf{u}^n(s), \mathbf{curl}\, \mathbf{u}^n_t(t))_{2,\Omega}\, ds \right| \le \frac{\varkappa}{8}\, \|\mathbf{u}^n_t\|^2_{\mathbf{H}^1_{\mathbf{n}}(\Omega)} + \frac{2K_0^2}{\varkappa} \int_0^t \|\mathbf{u}^n(s)\|^2_{\mathbf{H}^1_{\mathbf{n}}(\Omega)}\, ds,$$

(23)

$$|J_3| \le \frac{|e(t)|}{\omega_0}\left[|e'(t)| + \|\omega\|_{\mathbf{H}^1_{\mathbf{n}}(\Omega)}\left(\varkappa\|\mathbf{u}^n_t\|_{\mathbf{H}^1_{\mathbf{n}}(\Omega)} + \nu\|\mathbf{u}^n\|_{\mathbf{H}^1_{\mathbf{n}}(\Omega)} + K_0\left(\int_0^t \|\mathbf{u}^n(s)\|^2_{\mathbf{H}^1_{\mathbf{n}}(\Omega)}\,ds\right)^{\frac{1}{2}}\right)\right] \le$$

$$\frac{1}{2\omega_0}\left(|e(t)|^2 + |e'(t)|^2\right) + \frac{\varkappa}{8}\|\mathbf{u}^n_t\|^2_{\mathbf{H}^1_{\mathbf{n}}(\Omega)} + (4\varkappa + 3\nu)\frac{|e(t)|^2}{2\omega_0^2}\|\omega\|^2_{\mathbf{H}^1_{\mathbf{n}}(\Omega)} + \frac{\nu}{6}\|\mathbf{u}^n\|^2_{\mathbf{H}^1_{\mathbf{n}}(\Omega)} +$$

$$+ \frac{K_0^2}{2\omega_0^2}|e(t)|^2\|\omega\|^2_{\mathbf{H}^1_{\mathbf{n}}(\Omega)} + \frac{1}{2}\int_0^t \|\mathbf{u}^n(s)\|^2_{\mathbf{H}^1_{\mathbf{n}}(\Omega)}\,ds.$$

(24)

$$|J_4| \le \frac{|e'(t)|}{\omega_0}\left[|e'(t)| + \|\omega\|_{\mathbf{H}^1_{\mathbf{n}}(\Omega)}\left(\varkappa\|\mathbf{u}^n_t\|_{\mathbf{H}^1_{\mathbf{n}}(\Omega)} + \nu\|\mathbf{u}^n\|_{\mathbf{H}^1_{\mathbf{n}}(\Omega)} + K_0\left(\int_0^t \|\mathbf{u}^n(s)\|^2_{\mathbf{H}^1_{\mathbf{n}}(\Omega)}\,ds\right)^{\frac{1}{2}}\right)\right] \le$$

$$\frac{1}{\omega_0}|e'(t)|^2 + \frac{\varkappa}{8}\|\mathbf{u}^n_t\|^2_{\mathbf{H}^1_{\mathbf{n}}(\Omega)} + (4\varkappa + 3\nu)\frac{|e'(t)|^2}{2\omega_0^2}\|\omega\|^2_{\mathbf{H}^1_{\mathbf{n}}(\Omega)} + \frac{\nu}{6}\|\mathbf{u}^n\|^2_{\mathbf{H}^1_{\mathbf{n}}(\Omega)} +$$

$$+ \frac{K_0^2}{2\omega_0^2}|e'(t)|^2\|\omega\|^2_{\mathbf{H}^1_{\mathbf{n}}(\Omega)} + \frac{1}{2}\int_0^t \|\mathbf{u}^n(s)\|^2_{\mathbf{H}^1_{\mathbf{n}}(\Omega)}\,ds.$$

(25)

Plugging (22)-(25) *into (20), then integrating by* s *from* 0 *to* t*, we have*

$$y(t) + \nu\|\mathbf{u}^n\|^2_{L^2(0,T;\mathbf{H}^1_{\mathbf{n}}(\Omega))} + \|\mathbf{u}^n_t\|^2_{L^2(Q_T)} + \varkappa\|\mathbf{u}^n_t\|^2_{L^2(0,T;\mathbf{H}^1_{\mathbf{n}}(\Omega))} \le C_1 + C_2\int_0^t y(s)ds, \qquad (26)$$

where $y(t) = \|\mathbf{u}^n\|^2_{2,\Omega} + (\nu + \varkappa)\|\mathbf{u}^n\|^2_{\mathbf{H}^1_{\mathbf{n}}(\Omega)}$;

$$C_1 = \left(1 + \frac{4\varkappa + 3\nu + K_0^2}{\omega_0}\right)\frac{\|e(t)\|^2_{L^2[0,T]} + \|e'(t)\|^2_{L^2[0,T]}}{\omega_0}\|\omega\|^2_{\mathbf{H}^1_{\mathbf{n}}(\Omega)} + \frac{2\|e'(t)\|^2_{L^2[0,T]}}{\omega_0} +$$

$$+ \|\mathbf{u}^n_0\|^2_{2,\Omega} + (\nu + \varkappa)\|\mathbf{u}^n_0\|^2_{\mathbf{H}^1_{\mathbf{n}}(\Omega)}; \quad C_2 = \left(\frac{3K_0^2}{\nu} + \frac{4K_0^2}{\varkappa} + 2\right)\frac{T}{\nu + \varkappa}.$$

Apply classical Grönwall's lemma to (26)*, we have*

$$y(t) \le C_1 e^{C_2 T} < \infty \qquad (27)$$

Applying the estimate (27) to the right hand side of (26) and taking the supremum by $t \in [0, T]$*, we obtain the following estimate*

$$\|\mathbf{u}^n\|_{\mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega)\cap\mathbf{H}^1_{\mathbf{n}}(\Omega))} + \|\mathbf{u}^n\|^2_{L^2(0,T;\mathbf{H}^1_{\mathbf{n}}(\Omega))} + \|\mathbf{u}^n_t\|^2_{L^2(0,T;\mathbf{L}^2(\Omega)\cap\mathbf{H}^1_{\mathbf{n}}(\Omega))} \le M_1 = M_1(C_1, C_2, T, \varkappa). \qquad (28)$$

6 Existence of a strong solutions of inverse problem (1)-(5)

Theorem 3 *Assume that all conditions of Theorem 2 be fulfilled and*

$$\mathbf{u}_0(\mathbf{x}) \in \mathbf{H}^1_{\mathbf{n}}(\Omega) \cap \mathbf{H}^2_{\mathbf{n}}(\Omega)$$

holds.. The direct problem (14), (3), (4) *has global in time a unique srtong solution* $\mathbf{u}(\mathbf{x}, t)$ *in* Q_T*, and for a strong solution the following estimate is hold for all* $t \in (0, T]$

$$\|\mathbf{u}\|^2_{\mathbf{L}^\infty(0,T;\mathbf{H}^1_{\mathbf{n}}(\Omega)\cap\mathbf{H}^2_{\mathbf{n}}(\Omega))} + \|\mathbf{u}_t\|^2_{\mathbf{L}^2(0,T;\mathbf{H}^1_{\mathbf{n}}(\Omega)\cap\mathbf{H}^2_{\mathbf{n}}(\Omega))} \leq C < \infty. \tag{29}$$

where C *is positive constant depending on data of the problem.*

Proof 4 *We prove the existence of a strong solutions to these problems by using the special basis, associated to the eigenfunctions of the Stokes operator*

$$\mathbb{A} : \mathbf{V}^2(\Omega) \to \mathbf{V}(\Omega), \tag{30}$$

and

$$\mathbb{A}\varphi_k := -\Delta\varphi_k = \lambda_k\ \varphi_k, \quad \varphi_k \in \mathbf{H}^1_{\mathbf{n}}(\Omega) \cap \mathbf{H}^2_{\mathbf{n}}(\Omega) \tag{31}$$

in the case (4)*. The latter is due to the fact (see [14])*

$$(\Delta\varphi, \nabla p)_{2,\Omega} = 0 \text{ for any } \varphi \in \mathbf{H}^1_{\mathbf{n}} \cap \mathbf{H}^2_{\mathbf{n}}(\Omega),\ p \in W^{1,2}(\Omega), \text{ and } \mathbf{L}^2(\Omega) = \mathbf{H}_{\mathbf{n}}(\Omega) \oplus \mathbf{G}(\Omega).$$

It is known from [14] and [15], that the system $\{\varphi_k\}_{k\in\infty}$ *of eigenfunctions of spectral problem* (31) *are orthogonal in* $\mathbf{H}_{\mathbf{n}}$ *and an orthonormal basis in the space* $\mathbf{H}^1_{\mathbf{n}}(\Omega) \cap \mathbf{H}^2_{\mathbf{n}}(\Omega)$*, respectively.*

Let us first consider the (1)-(5)*. In this case, all first and second estimates are true for strong solution. Thus, in order to complete the proof this theorem, it is sufficient to get more strong estimates, i.e. estimate* $\Delta\mathbf{u}^n$ *and* $\Delta\mathbf{u}^n_t$*.*

In this case, the equation (17) can be written as the form

$$\begin{gathered}\frac{d}{dt}\Big((\mathbf{u}^n, \varphi_k)_{2,\Omega} + \varkappa\,(\Delta\mathbf{u}^n, \varphi_k)_{2,\Omega}\Big) + \nu\,(\Delta\mathbf{u}^n, \varphi_k)_{2,\Omega} = -\int\limits_0^t K(t-s)\,(\Delta\mathbf{u}^n, \varphi_k)_{2,\Omega}\,ds + \\ \frac{1}{g_0(t)}\Big(e'(t) + \varkappa\,(\mathbf{curl}\,\mathbf{u}^n_t(t), \mathbf{curl}\,\omega)_{2,\Omega} + \nu\,(\mathbf{curl}\,\mathbf{u}^n(t), \mathbf{curl}\,\omega)_{2,\Omega} + \\ + \int\limits_0^t K(t-s)\,(\mathbf{curl}\,\mathbf{u}^n(t), \mathbf{curl}\,\omega)_{2,\Omega}\,ds\Big)\,(\omega, \varphi_k)_{2,\Omega}\,,\end{gathered} \tag{32}$$

and multiply the k-th equation of (32) by $-\lambda_k c^n_k(t)$ *and* $-\lambda_k \frac{dc^n_k(t)}{dt}$*, and sum up respect to k from* 1 *to* n *and adding results, taking in account the Stokes operator (30), we obtain*

$$\frac{\nu+\varkappa}{2}\frac{d}{dt}\,\|\mathbb{A}\mathbf{u}^n\|^2_{2,\Omega} + \nu\|\mathbb{A}\mathbf{u}^n\|^2_{2,\Omega} + \varkappa\,\|\mathbb{A}\mathbf{u}^n_t\|^2_{2,\Omega} = I_1 + I_2. \tag{33}$$

where $I_1 \equiv (\mathbf{u}_t^n, \mathbb{A}\mathbf{u}^n)_{2,\Omega} - \int\limits_0^t K(t-s)\left(\mathbb{A}\mathbf{u}^n(s), \mathbb{A}\mathbf{u}^n(t)\right)_{2,\Omega} ds + \Phi(\mathbf{u}^n, t)\left(\omega, \mathbb{A}\mathbf{u}^n\right)_{2,\Omega},$

$I_2 \equiv (\mathbf{u}_t^n, \mathbb{A}\mathbf{u}_t^n)_{2,\Omega} - \int\limits_0^t K(t-s)\left(\mathbb{A}\mathbf{u}^n(s), \mathbb{A}\mathbf{u}_t^n(t)\right)_{2,\Omega} ds + \Phi(\mathbf{u}^n, t)\left(\omega, \mathbb{A}\mathbf{u}_t^n\right)_{2,\Omega}$

and $\Phi(\mathbf{u}^n, t)$ *defined by* (21) *and it can be estimated as follow*

$$|\Phi|^2 \le \frac{3}{k_0^2}\left[|e'(t)|^2 + \nu\,\|\mathbf{u}^n\|^2_{\mathbf{H}^1_{\mathbf{n}}(\Omega)}\|\omega\|^2_{\mathbf{H}^1_{\mathbf{n}}(\Omega)} + K_0^2\,\|\omega\|^2_{\mathbf{H}^1_{\mathbf{n}}(\Omega)}\int\limits_0^t \|\mathbf{u}^n(s)\|^2_{\mathbf{H}^1_{\mathbf{n}}(\Omega)} ds\right] \le \frac{3}{k_0^2}\left[|e'(t)|^2 + \nu M_1\,\|\omega\|^2_{\mathbf{H}^1_{\mathbf{n}}(\Omega)} + M_1 K_0^2\,\|\omega\|^2_{\mathbf{H}^1_{\mathbf{n}}(\Omega)}\right]. \tag{34}$$

With the help of Hölder, Young inequalities and (34), we estimate the right hand side of (33)

$$|I_1| \le \|\mathbb{A}\mathbf{u}^n\|_{2,\Omega}\left[\|\mathbf{u}_t^n\|_{2,\Omega} + |\Phi(\mathbf{u}^n, t)|\,\|\omega\|_{2,\Omega} + \int\limits_0^t |K(t-s)|\,\|\mathbb{A}\mathbf{u}^n(s)\|_{2,\Omega} ds\right] \le \frac{\nu}{2}\|\mathbb{A}\mathbf{u}^n(t)\|^2_{2,\Omega} + \frac{1}{2\nu}\left[\|\mathbf{u}_t^n\|^2_{2,\Omega} + |\Phi(\mathbf{u}^n, t)|^2\,\|\omega\|^2_{2,\Omega} + K_0^2\int\limits_0^t \|\mathbb{A}\mathbf{u}^n(s)\|^2_{2,\Omega} ds\right], \tag{35}$$

$$|I_2| \le \frac{\varkappa}{2}\|\mathbb{A}\mathbf{u}_t^n(t)\|^2_{2,\Omega} + \frac{1}{2\varkappa}\|\mathbf{u}_t^n\|^2_{2,\Omega} + |\Phi(\mathbf{u}^n, t)|^2\,\|\omega\|^2_{2,\Omega} + K_0^2\int\limits_0^t \|\mathbb{A}\mathbf{u}^n(s)\|^2_{2,\Omega} ds, \tag{36}$$

Plugging (35), (36) into (33) and integrating by s *from* 0 *to* $t \in [0, T]$, *we have*

$$(\nu+\varkappa)\,\|\mathbb{A}\mathbf{u}^n\|^2_{2,\Omega} + \nu\int\limits_0^t \|\mathbb{A}\mathbf{u}^n\|^2_{2,\Omega} ds + \varkappa\int\limits_0^t \|\mathbb{A}\mathbf{u}_t^n\|^2_{2,\Omega} ds \le (\nu+\varkappa)\,\|\mathbb{A}\mathbf{u}_0\|^2_{2,\Omega} + \left(\frac{1}{\nu} + \frac{1}{\varkappa}\right)\left[\int\limits_0^t \|\mathbf{u}_t^n\|^2_{2,\Omega} ds + \|\omega\|^2_{2,\Omega}\int\limits_0^t |\Phi(\mathbf{u}^n, s)|^2 ds + K_0^2 T\int\limits_0^t \|\mathbb{A}\mathbf{u}^n(s)\|^2_{2,\Omega} ds\right]. \tag{37}$$

Applying the already obtained estimates for $\Phi(\mathbf{u}^n, t)$ *and* $\mathbf{u}_t^n$, *we obtain*

$$(\nu+\varkappa)\,\|\mathbb{A}\mathbf{u}^n\|^2_{2,\Omega} + \int\limits_0^t \left(\nu\,\|\mathbb{A}\mathbf{u}^n\|^2_{2,\Omega} + \varkappa\,\|\mathbb{A}\mathbf{u}_t^n\|^2_{2,\Omega}\right) ds \le C_3 + C_4\int\limits_0^t \|\mathbb{A}\mathbf{u}^n(s)\|^2_{2,\Omega} ds, \tag{38}$$

where

$$C_3 := (\nu+\varkappa)\,\|\mathbb{A}\mathbf{u}_0\|^2_{2,\Omega} + \left(\frac{1}{\nu} + \frac{1}{\varkappa}\right)\left[\int\limits_0^t \|\mathbf{u}_t^n\|^2_{2,\Omega} ds + \|\omega\|^2_{2,\Omega}\int\limits_0^t |\Phi(\mathbf{u}^n, s)|^2 ds\right];\ C_4 := \frac{K_0^2 T}{\nu\varkappa}.$$

Omitting second term on the right hand side of (38) and appling Grönwall's lemma, result that

$$\|\mathbb{A}\mathbf{u}^n\|^2_{2,\Omega} \le C_3 e^{C_4 T}. \tag{39}$$

Taking the suptemum from both sides of (38) by $t \in [0,T]$ and using the estimate (39) we obtain

$$\|\mathbf{u}\|^2_{\mathbf{L}^\infty(0,T;\mathbf{H}^2_{\mathbf{n}}(\Omega))} + \|\mathbf{u}_t\|^2_{\mathbf{L}^2(0,T;\mathbf{H}^2_{\mathbf{n}}(\Omega))} \le M_2 = M_2(\nu, \varkappa, T, C_3, C_4) < \infty. \tag{40}$$

The passage to the limit is proved in a similar way as in [1].

7 Conclusion

In the paper, the space of a weak and strong generalized solutions of inverse problem for the integro-differential Kelvin-Voigt equation is defined. Under suitable conditions on the data of the problem, the global existence and uniqueness theorems are obtained and proved.

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