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DISCONTINUOUS COMPARTMENTAL PERIODIC POISSON STABLE FUNCTIONS

Among recurrent functions the most sophisticated are Poisson stable functions. For discontinuous functions, there are very few results, for the stability. Discontinuous compartmental Poisson stable functions are in the focus of this research. As the discontinuity points of the functions, a special time sequences, Poisson sequences, are considered. It the first time, the discontinuous functions of two compartments, periodic and Poisson stable, are investigated. To combine periodicity and Poisson stability, in the case of continuous functions, a convergence sequence with a special kappa property was used [1, 2]. For discontinuous functions, this property is not enough, because we also should consider the discontinuity points of the function. For this reason, we need a new concept known as Poisson couple, that is, a couple of a sequence of discontinuity points and convergence sequence that has the kappa property. Moreover, we meet the challenges for the stability by considering functions on diagonals in the space of arguments. Examples of Poisson stable functions are given to illustrate the theoretical results. The method and results can be effectively used in the study of different types of functional differential equations, impulsive differential equations and differential equations generalized piecewise constant argument, as well as their application.
Key words: $B-$topology, discontinuous Poisson stable function, compartmental functions, Poisson sequence, Poisson couple.

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Үзілісті көпкомпонентті периодты Пуассон бойынша орнықты функциялар

Рекуррентік функциялардың ішінде Пуассон бойынша орнықты функциялар ең күрделі болып табылады. Үзілісті функциялар үшін орнықтылыққа қатысты нәтижелер өте аз. Бұл зерттеудің басты бағыты үзілісті көппомпонентті периодты Пуассон бойынша орнықты функциялар болып табылады. Функциялардың үзіліс нүктелері ретінде арнайы уақыттық тізбек, яғни Пуассон тізбегі қарастырылады. Пуассон бойынша орнықты және периодты екі құраушысы бар үзілісті функциялар алғаш рет зерттелуде. Пуассон бойынша орнықтылық пен периодттылықты біріктіру үшін үзіліссіз функциялар жағдайында арнайы каппа қасиеті бар жинақтылық тізбегі қолданылған болатын [1, 2]. Үзілісті функциялар үшін бұл қасиет жеткіліксіз, өйткені функцияның үзіліс нүктелерін де ескеруіміз керек. Осы себепті бізге каппа қасиетке ие жинақтылық тізбегі мен үзіліс нүктелірің тізбектерінен құралған Пуассон жұбы деп аталатын жаңа ұғым қажет. Сонымен қатар, аргументтер кеңістігіндегі диагональдардағы функцияларды қарастыру арқылы орнықтылық мәселелерін шешеміз. Теориялық нәтижелерді көрсету үшін Пуассон бойынша орнықты функциялардың мысалдары келтірілген. Әдіс пен нәтижелерді импульстік дифференциалдық теңдеулер, жалпыланған бөлікті тұрақты аргументті дифференциалдық теңдеулер және функционалдық дифференциалдық теңдеулердің әртүрлі түрлерін, сондай-ақ олардың қолданысын зерттеуде тиімді пайдалануға болады.
Түйін сөздер: $B-$топология, Пуассон бойынша орнықты функция, көпкомпонентті функциялар, Пуассон тізбегі, Пуассон жұбы.

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Разрывные многокомпанентные периодические устойчивые по Пуассону функции

Среди рекуррентных функций наиболее сложными являются устойчивые по Пуассону функции. Для разрывных функций существует очень мало результатов, где рассматривается указанная устойчивость. В центре внимания настоящего исследования находятся разрывные многокомпонентные периодические устойчивые по Пуассону функции. В качестве точек разрыва этих функций рассматриваются специальные последовательности Пуассона. Впервые исследуются разрывные функции с двумя компонентами, периодическими и устойчивыми по Пуассону. Чтобы объединить периодичность и устойчивость по Пуассону для непрерывных функций была использована последовательность сходимости со специальным каппа-свойством [1, 2]. Для разрывных функций этого свойства недостаточно, так как мы должны учитывать точки разрыва. По этой причине нам нужно новое понятие, известное как пара Пуассона, состоящая из последовательности точек разрыва и последовательности сходимости, обладающей каппа-свойством. Мы решаем проблемы устойчивости по Пуассону, рассматривая функции на диагоналях в пространстве аргументов. Для иллюстрации теоретических результатов приведены примеры устойчивых по Пуассону функций. Метод и результаты могут быть эффективно использованы при изучении различных типов импульсных дифференциальных уравнений, дифференциальных уравнений с обобщенным кусочно-постоянным аргументом и функционально-дифференциальных уравнений, а также для их применения.

Ключевые слова: $B-$топология, разрывная устойчивая по Пуассону функция, многокомпонентные функции, последовательность Пуассона, пара Пуассона.

1 Introduction

The creation of a mathematical model for any phenomenon is accomplished by discrete, continuous or discontinuous functions. Therefore, theory of functions should be followed by a number of extension methods and numerical representations. Beginning from simple algebraic operations, they can be also Fourier series and even results of operators' theory. The construction and numerical analysis of the discontinuous Poisson stable functions are given in the paper. A new way to determine discontinuous Poisson stable functions is proposed. Discontinuous functions with two compartments, periodic and Poisson stable are in the focus of research. To establish the correspondence between periodicity and Poisson stability, the special *kappa property* is utilized [1, 2]. For functions that depend on the two variables, the method of diagonals of arguments is used [3–5]. Poisson stability is investigated based on the $B-$topology [6].

Due to the widespread of chaos theory, the properties of periodic or Poisson stable functions are insufficient to fully describe the behavior of nonlinear dynamical systems. In this regard, a class of new recurrent functions with the separation property was introduced [7]. Unpredictable functions cause the Poincare chaos. The study of unpredictable solutions of differential equations has led to the possibility of studying chaos based on the laws of the qualitative theory of differential equations. In recent years, the existence and stability of solutions of discrete, linear and quasilinear differential equations with unpredictable solutions has been studied and significant results have been obtained [8]. Thus, for the study of sophisticated processes, compartmental functions are needed. For the first time, a theoretical connection was established between the recurrent functions in [1]. The functions, formed by a combination of Poisson stable, quasi-periodic and periodic functions, are called compartmental functions, and their properties have been studied in articles [1, 2]. The effectiveness of the compartmental method to the unpredictability was given by analysis of contributions of unpredictability and periodicity [2]. It was just the first step towards the application of the study, since the next ones will be related to the control of chaos that will be used for the parameters of the compartments individually. In this paper, we want to extend the class with discontinuous compartmental functions. That is, we consider discontinuous compartmental periodic Poisson stable functions.

2 Poisson sequences

This section contains the basic definitions of special time sequences. Correspondingly, we investigate the properties of these sequences that will be utilized to study discontinuous compartmental functions.

Definition 1 *[9] A bounded sequence $\mu_i, i \in \mathbb{Z}$, in $\mathbb{R}$ is said to be Poisson stable, provided that there exists a sequence $l_m \to \infty$, $m \in \mathbb{N}$, of positive integers, which satisfies $\mu_{i+l_m} \to \mu_i$ as $m \to \infty$ on bounded intervals of integers.*

Fix two sequences of real numbers $t_m, \theta_i, m \in \mathbb{N}, i \in \mathbb{Z}$, strictly increasing with respect to the index and unlimited in both directions, which we call as *Poisson sequences* throughout this paper. Moreover, it is expected that there exists a number $\underline{\theta} > 0$ such that $\underline{\theta} < \theta_{i+1} - \theta_i$ for all $i \in \mathbb{Z}$.

Definition 2 *[6] A sequence τ_i, $i \in \mathbb{Z}$, is called with $(w,p)-$property, provided that there exist an integer p and a real number $w > 0$, which satisfy $\tau_{i+p} - \tau_i = w$ for all $i \in \mathbb{Z}$.*

Definition 3 *A couple (t_m, θ_i) of the sequences t_m, θ_i, $m \in \mathbb{N}, i \in \mathbb{Z}$, is called Poisson couple, provided that there exists a sequence l_m, $m \in \mathbb{N}$, of integers, which diverges to infinity, such that*

$$\theta_{i+l_m} - t_m - \theta_i \to 0 \text{ as } m \to \infty \tag{1}$$

uniformly on each bounded interval of integers i.

To investigate compartmental discontinuous functions, we need sufficient conditions that connect periodicity with Poisson stability.

Lemma 1 *Suppose that the couple (t_m, θ_i) of sequences t_m, θ_i, $m \in \mathbb{N}, i \in \mathbb{Z}$, satisfies the following conditions*

(i) $t_m = mw$, where $m \in \mathbb{N}$, $w \in \mathbb{R}$;

(ii) θ_i has the $(w,p)-$ property.

Then the couple (t_m, θ_i) is Poisson couple.

Proof. Because $(w, p)-$ property and it holds for $\theta_{i+p} = \theta_i + w$, $i \in \mathbb{Z}$. By taking $l_m = mp$ for $m \in \mathbb{N}$, one can get $\theta_{i+mp} = \theta_i + mw$, $i \in \mathbb{Z}$.

We proceed to proof that

$$\theta_{i+l_m} - t_m - \theta_i = \theta_{i+mp} - mw - \theta_i = \theta_i + mw - mw - \theta_i = 0.$$

Since the sequence consist of zeros, it follows that

$$\lim_{m\to\infty} (\theta_{i+l_m} - t_m - \theta_i) = 0,$$

as $m \to \infty$, on each bounded intervals of integers i. So, the condition (1) is satisfied. □

According to the Lemma A1 [1], for a number $w > 0$ and an arbitrary sequence of positive real numbers $t_m, m = 1, 2, ...$, one can find a subsequence $t_{m_l}, l = 1, 2, ...$, and a number τ_w, $0 \le \tau_w < w$, which satisfies $t_{m_l} \to \tau_w (mod\ w)$ as $l \to \infty$.

Next, we will consider the application of this assertion in the proof of the Poisson stability. The number τ_w is called as the *Poisson shift* with respect to the w. The set of Poisson shifts T_w consists of an infinite number of terms, that is, it is not empty. Denote infimum of the set T_w as κ_w, and call it as *Poisson number* with respect to w or shortly the *Poisson number*. We say that a sequence t_m admits the *kappa property* with respect to the number w, if the Poisson number is equal to 0.

We will use the following assertion [1].

Lemma 2 *[1]* $\kappa_w \in T_w$.

Lemma 3 *If the couple* (t_m, θ_i) *consist the sequence* $t_m, m \in \mathbb{N}$, *which has the kappa property with respect to* w, *and the sequence* $\theta_i, i \in \mathbb{Z}$, *with the* $(w, p)-$ *property, then it is Poisson couple.*

Proof. Taking into account that the sequence t_m has the kappa property and using the $(w, p)-$ property of the sequence θ_i, we get that

$$\lim_{m\to\infty} |\theta_{i+l_m} - t_m - \theta_i| = 0$$

as $m \to \infty$, on each bounded intervals of integers i. □

Next, consider an example of a sequence that has the kappa property.

Example 1 *Consider the sequence* t_m *is defined as following form* $t_m = \frac{(m-1)w}{n}$, $m \in \mathbb{N}$ *with fixed numbers* $n \in \mathbb{N}$ *and* $w > 0$. *If* $n = 1$, *then from* $t_m \equiv 0(mod\ w)$, *one can see that there exists a unique Poisson shift* $\tau_w = 0$, *so the set* $T_w = \{0\}$ *consists of one element. If* $n = 2$, *then we get* $t_m \equiv \frac{w}{2}(mod\ w)$ *for odd numbers* m *and* $t_m \equiv 0(mod\ w)$ *for even numbers* m. *Hence, the set of Poisson shifts consists of two elements, that is* $T_w = \{0, \frac{w}{2}\}$. *In a similar way, one can show that for any* $n = 1, 2, \ldots$, *the sequence* t_m *has a non empty set of Poisson shifts, that is* $T_w = \{0, \frac{w}{n}, \frac{2w}{n}, \ldots, \frac{(n-1)w}{n}\}$. *So, one can see that the Poisson number is* $\kappa_w = \inf T_w = 0$, *that is,* t_m *satisfies the kappa property with respect to* w.

3 Compartmental Poisson stable functions

A discontinuous function $u(t)\colon \mathbb{R} \to \mathbb{R}$, is called *conditionally uniform continuous*, if for any number $\epsilon > 0$ there exists a number $\sigma > 0$ such that $|u(t_1) - u(t_2)| < \epsilon$ whenever the points t_1 and t_2 belong to the same continuity interval and $|t_1 - t_2| < \sigma$ [10].

Denote by $\mathcal{G}$ the space of piecewise continuous functions $u(t)\colon \mathbb{R} \to \mathbb{R}$ with countable set of discontinuity moments of the first kind. The functions are left-continuous and conditionally uniform continuous. The sets of discontinuity moments are unbounded from both sides and do not have finite accumulation endpoints. Moreover, the set of discontinuity points is strictly ordered and enumerated with integers.

Definition 4 *An element $f(t)$ of $\mathcal{G}$ is called discontinuous periodic, if there exist an integer p and a real number $w > 0$, such that the set of its moments of discontinuities θ_i, $i \in \mathbb{Z}$, satisfy $(w,p)-$property and $f(t+w) = f(t)$ for $t \in \mathbb{R}$.*

The functions $\phi(t)$ and $\psi(t)$ from $\mathcal{G}$, are called $\epsilon-$*equivalent* on a bounded open interval J, if the moments of discontinuities of $\phi(t)$ and $\psi(t)$ in J can be numerated with multiplicity one, θ_i^ϕ and θ_i^ψ, $i = 1,2,...,l$, such that $|\theta_i^\phi - \theta_i^\psi| < \epsilon$ for each $i = 1,2,\ldots,l$, and $|\phi(t) - \psi(t)| < \epsilon$, for all $t \in J$, except possibly those between θ_i^ϕ and θ_i^ψ, $i = 1,2,\ldots,l$. If ϕ, ψ are $\epsilon-$equivalent on J, then we say that the functions are in $\epsilon-$neighborhoods of each other on J. The topology defined on the $\epsilon-$ neighborhoods basis is said to be *B-topology* [6].

Definition 5 *An element $g(t)$ of $\mathcal{G}$ is called discontinuous Poisson stable, if there exist a sequence $t_m \to \infty$ of real numbers such that (t_m, θ_i) is a Poisson couple and $g(t+t_m) \to g(t)$ as $m \to \infty$ on each bounded intervals of real numbers in B-topology.*

Definition 6 *A function $h(t) = f(t) + g(t)$, where $f(t)$ and $g(t)$ are members of $\mathcal{G}$ with common set of discontinuities, is called discontinuous modulo periodic Poisson stable function, if $f(t)$ is a discontinuous periodic function and $g(t)$ is a discontinuous Poisson stable function.*

Definition 7 *A product $f(t)g(t)$, where $f(t)$ and $g(t)$ are members of $\mathcal{G}$ with common set of discontinuities, is called discontinuous factor periodic Poisson stable function, if $f(t)$ is a discontinuous periodic function and $g(t)$ is a discontinuous Poisson stable function.*

Definition 8 *A function $h(t) : \mathbb{R} \to \mathbb{R}$ is called discontinuous compartmental periodic Poisson stable, if $h(t) = Q(t,t)$, where $Q(u,v)$ is a discontinuous function with common set of discontinuities for u and v, discontinuous periodic in u uniformly with respect to v, and discontinuous Poisson stable in v uniformly with respect to u. That is, there exists number w such that $Q(u+w,v) = Q(u,v)$ and there exists a sequence $t_m \to \infty$, which satisfies $Q(u, v+t_m) \to Q(u,v)$ as $m \to \infty$ uniformly on each bounded intervals of v in B-topology.*

Remark 1 *Since the compartments in Definitions 6-8 have a common set of discontinuities, it satisfies the $(w,p)-$property.*

4 Main results

In this part of research we investigate properties of discontinuous compartmental periodic Poisson stable functions.

Theorem 1 *A discontinuous periodic function is Poisson stable.*

Proof. Consider discontinuous periodic function $f(t)$ with discontinuity moments $\theta_i,\ i \in \mathbb{Z}$. Together with the sequence θ_i, we fix a sequence t_m, $m \in \mathbb{Z}$, such that (t_m, θ_i), is a Poisson couple in the sense of Definition 3. For a fixed $m \in \mathbb{Z}$, consider the function $f(t+t_m)$, if $\theta'_i \le t < \theta'_{i+1}$, where $\theta'_i = \theta_{i+l_m} - t_m$, $i \in \mathbb{Z}$.

According to Lemma 1, the set of discontinuity moments θ'_{i+1}, $i \in \mathbb{Z}$ of $f(t+t_m)$ are coincide with the set $\theta_i,\ i \in \mathbb{Z}$. We take $t_m = mw$ for $n \in \mathbb{N}$, it is easy to prove that the functions $f(t+t_m) - f(t)$ are identically zeros.

So, the Poisson stability of $f(t)$ is proved. □

Theorem 2 *Suppose that a bounded and piecewise continuous function $Q(u, v)$ is $w-$periodic in u. Then $h(t) = Q(t, t)$ is discontinuous Poisson stable, with common set of discontinuities $\theta_i,\ i \in \mathbb{Z}$, for u and v, which admits the $(w, p)-$ property such that*

(a) for each $\epsilon > 0$ there exists a number $\sigma > 0$ which satisfies $|Q(t_1, t) - Q(t_2, t)| < \epsilon$, where the points t, $t+t_m$ are taken from the same continuity interval and $|t_1 - t_2| < \sigma$, $t \in \mathbb{R}$;

there exists a sequence t_m, $t_m \to \infty$ as $m \to \infty$ and satisfies following conditions

(b) t_m satisfies the kappa property with respect to the period w;

(c) $|Q(t, t+t_m) - Q(t, t)| \to 0$ as $m \to \infty$ on each bounded interval $I \subset \mathbb{R}$ of t in B-topology.

Proof. Because t_m satisfies kappa property, there exists a subsequence t_{m_l}, $t_{m_l} \to \tau_w (mod\ w)$ as $l \to \infty$. Suppose that $t_m \to 0 (mod\ w)$ as $m \to \infty$. Let us fix $\epsilon > 0$ and an interval $I = [b, c] \subset \mathbb{R}$. Consequently, for arbitrarily fixed number $\epsilon > 0$, the bounded interval I, and by condition (a), one can find sufficiently large m such that

$$|Q(t+t_m, t+t_m) - Q(t, t+t_m)| < \epsilon/2 \tag{2}$$

for all $t \in \mathbb{R}$, where the points t, $t+t_m$ are taken from the same continuity interval.

Let θ_i and θ'_i, be the discontinuity moments of the functions $Q(t, t)$, $Q(t, t+t_m)$ in I respectively, where $\theta'_i = \theta_{i+l_m} - t_m$. Assume that $\theta_i \le \theta_{i+l_m} - t_m$ and consider discontinuity moments θ_i, $i = k+1, k+2, \dots, k+r-1$, of the interval $[b, c]$ such that

$$\theta_k \le b < \theta_{k+1} < \theta_{k+2} < \cdots < \theta_{k+r-1} < c \le \theta_{k+r}.$$

Let us fix i, $i = k, k+1, \dots, k+r$, and for fixed i, it follows that $Q(t, t)$, $t \in [\theta_i, \theta_{i+1})$ and $Q(t, t+t_m)$, $t \in [\theta'_i, \theta'_{i+1})$. Thus, for sufficiently large m we have non-empty interval $(\theta'_i, \theta_{i+1})$. According to (1) and by condition (c) the following inequalities are valid:

$$|\theta'_i - \theta_i| < \epsilon, \tag{3}$$

for all $i = k, k+1, \ldots, k+r$, and

$$|Q(t, t+t_m) - Q(t,t)| < \epsilon/2 \tag{4}$$

for $t \in (\theta_i', \theta_{i+1})$.

Applying (2) and (4), we get that

$$|h(t+t_m) - h(t)| = |Q(t+t_m, t+t_m) - Q(t,t)| \leq$$
$$|Q(t+t_m, t+t_m) - Q(t, t+t_m)| + |Q(t, t+t_m) - Q(t,t)| \leq \epsilon/2 + \epsilon/2 = \epsilon$$

for all $t \in (\theta_i', \theta_{i+1})$, $i = k, k+1, \ldots, k+r$. Consequently, $h(t+t_m) \to h(t)$ uniformly on each arbitrary bounded time interval in B-topology. □

Theorem 3 *Assume that $h(t) = f(t)+g(t)$ is a discontinuous modulo periodic Poisson stable function. The discontinuity moments θ_i, $i \in \mathbb{Z}$ is common for $f(t)$ and $g(t)$ and satisfies the $(w,p)-$ property, the sequence t_m satisfies the kappa property with respect to the number w. Then $h(t)$ is discontinuous Poisson stable.*

Theorem 4 *Assume that $h_1(t) = f(t)g(t)$ is a discontinuous factor periodic Poisson stable function. The discontinuity moments θ_i, $i \in \mathbb{Z}$ is common for $f(t)$ and $g(t)$ and satisfies the $(w,p)-$ property, the sequence t_m satisfies the kappa property with respect to the number w. Then the function $h_1(t)$ is discontinuous Poisson stable.*

The technique of proofs of the Theorems 3, 4 is similar to that for Theorems 1, 2, and the B-topology is used.

In the following theorems we consider specific discontinuous compartmental periodic Poisson stable functions.

Fix a Poisson stable sequence $\lambda_i, i \in \mathbb{Z}$, such that there exist a sequence l_m which diverges to infinity, $|\lambda_{i+l_m} - \lambda_i| \to 0$ as $m \to \infty$ for each i in bounded intervals of integers.

Theorem 5 *[2] Let $\xi(t) : (0, d] \to \mathbb{R}$, where $d > 0$, be a bounded function. Then function $\zeta(t) = \lambda_i \xi(t - id)$, $t \in (id, (i+1)d]$, $i \in \mathbb{Z}$, is discontinuous Poisson stable.*

Proof. Fix a number $i \in \mathbb{Z}$ and an interval (γ, δ) so that $(\gamma, \delta) \subset ((i-1)d, (i+s+1)d]$ for $s \in \mathbb{N}$. For $t_m = l_m d, n = 1, 2, \ldots$ from interval $t \in (jd, (j+1)d], i-1 \leq j \leq i+s$, we have that $t + l_m d \in ((j+l_m)d, (j+l_m+1)d]$ and $\xi(t - (j+l_m)d) = \xi(t - jd)$.

Let us denote $N = \sup\limits_{t \in (0,d]} |\xi(t)|$. For an arbitrary $\epsilon > 0$ and sufficiently large m, it is true that $|\lambda_{j+l_m} - \lambda_j| < \frac{\epsilon}{N}$, $i-1 \leq j \leq i+s$. We fixed integer number l in $i-1 \leq l \leq i+s$. If $t \in (ld, (l+1)d]$, then $\zeta(t) = \xi(t-ld) = \lambda_l$ and $\zeta(t+t_m) = \zeta(t+l_m d) = \xi(t-(l+l_m)d) = \lambda_{l+l_m}$. This is why, for $t \in (ld, (l+1)d]$, $l \in [i-1, i+s]$, we have that

$$|\zeta(t+t_m) - \zeta(t)| = |\zeta(t+l_m d) - \zeta(t)| =$$
$$|\lambda_{l+l_m}\xi(t-(l+l_m)d) - \lambda_l \xi(t-lw)| \leq |\lambda_{l+l_m} - \lambda_l||\xi(t-ld)| \leq$$
$$|\lambda_{l+l_m} - \lambda_l| N < \epsilon.$$

It follows that $l \in [i-1, i+s]$, and, in consequence, $|\zeta(t+t_m) - \zeta(t)| < \epsilon$, $t \in (\gamma, \delta)$. Thus, it have been proved that $\zeta(t)$ is a Poisson stable function.

Theorem 6 *The function* $\eta(t) = \lambda_i$, $\theta_i < t \leq \theta_{i+1}$, $i = 0, 1, 2, \ldots$, *is a discontinuous Poisson stable with the sequence of convergence* $t_m, m = 1, 2, \ldots$, *if sequences* t_m *and* θ_i *make a Poisson couple* (t_m, θ_i).

Proof. Consider the function $\eta(t + t_m)$ for fixed $m \in \mathbb{N}$. If $\eta(t) = \lambda_i$, $\theta_i < t \leq \theta_{i+1}$, then it is possible to show that $\eta(t + t_m) = \lambda_{i+l_m}$ for $\theta'_i \leq t < \theta'_{i+1}$, where $\theta'_i = \theta_{i+l_m} - t_m$, $i \in \mathbb{Z}$.

Fix an interval $[b_1, c_1]$ where $c_1 > b_1$, and an arbitrary $\epsilon > 0$ so that $2\epsilon < \underline{\theta}$ on this interval. Suppose that $\theta_i \leq \theta_{i+l_m} - t_m$ and consider discontinuity moments θ_i, $i = k+1, k+2, \ldots, k+r_1-1$, in the interval $[b_1, c_1]$ such that

$$\theta_k \leq b_1 < \theta_{k+1} < \theta_{k+2} < \cdots < \theta_{k+r_1-1} < c_1 \leq \theta_{k+r_1}.$$

We will prove that for sufficiently large m the inequalities $|\theta'_i - \theta_i| < \epsilon$ for all $i = k, k+1, \ldots, k+r_1$, and $|\eta(t+t_m) - \eta(t)| < \epsilon$ for each $t \in [b_1, c_1]$ are valid, except those between θ_i and θ'_i for each i.

Fix i, $i = k, k+1, \ldots, k+r_1$. We have that $\eta(t) = \lambda_i$, for $t \in (\theta_i, \theta_{i+1}]$ and $\eta(t+t_m) = \lambda_{i+l_m}$, $t \in (\theta'_i, \theta'_{i+1}]$. Hence, for sufficiently large m the interval $(\theta'_i, \theta_{i+1})$ is non-empty. From (1) it implies that $|\theta'_i - \theta_i| < \epsilon$ is valid. Moreover, for sufficiently large m,

$$|\eta(t + t_m) - \eta(t)| = |\lambda_{i+l_m} - \lambda_i| < \epsilon$$

for $t \in (\theta'_i, \theta_{i+1})$. Thus, $\eta(t)$ is discontinuous Poisson stable function.

5 Examples and Numerical Simulations

Next, we will construct examples of discontinuous Poisson stable functions.

In [1, 7], a Poisson stable sequence was constructed as a solution to the logistic map

$$\chi_{j+1} = \nu \chi_j (1 - \chi_j). \tag{5}$$

Moreover, it is shown that for each ν from the interval $[3 + (2/3)^{1/2}, 4]$, there exists a Poisson stable solution $z_j, j \in \mathbb{Z}$, of (5) so that the sequence belongs to $[0, 1]$. That is, there exist a sequence l_m, $l_m \to \infty$ as $m \to \infty$ such that $|z_{j+l_m} - z_j| \to 0$ as $m \to \infty$ for each j in bounded intervals of integers.

Example 2 *Consider the function* $\zeta_1(t) = z_i \xi(t - id)$, $t \in (id, (i+1)d]$, $i = 0, 1, 2, \ldots$, *with* $\xi(t - id) = 1$, *and* $d = 5$, *such that* z_i *is a Poisson stable solution of (5). Prove that the function* $\zeta_1(t)$ *is discontinuous Poisson stable in accordance with Theorem 5.*

Fix an interval (γ_1, δ_1) *and number* i *such that* $(\gamma_1, \delta_1) \subset (5(i-1), 5(i+s+1)]$ *for* $s \in \mathbb{N}$. *Assume that* γ_1 *and* δ_1 *are integers. For* $t_m = 5l_m, m = 1, 2, \ldots$ *and* $t \in (5j, 5(j+1)], i - 1 \leq j \leq i + s$, *we have that* $t + 5l_m \in (5(j + l_m), 5(j + l_m + 1)]$.

For an arbitrary number $\epsilon > 0$ *and sufficiently large* m, *the inequality* $|z_{j+l_m} - z_j| < \epsilon$, $i - 1 \leq i \leq i + s$ *is fulfilled. For* $t \in (5l, 5(l+1)]$, $l \in [i-1, i+s]$, *we have that*

$$|\zeta_1(t + t_m) - \zeta_1(t)| = |\zeta_1(t + 5l_m) - \zeta_1(t)| \leq |z_{l+l_m} - z_l| < \epsilon.$$

From the above it follows that for $l \in [i-1, i+s]$, *and so* $|\zeta_1(t+t_m) - \zeta_1(t)| \to 0$, *as* $n \to \infty$ *uniformly on the interval* $t \in (\gamma_1, \delta_1)$. *So, the Poisson stability of* $\zeta_1(t)$ *have been proved (Figure 1).*

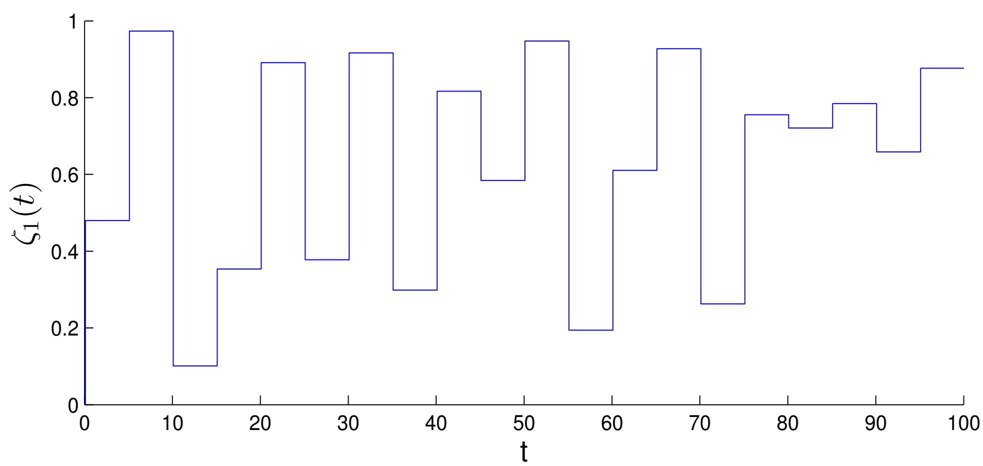

Figure 1: The graph of function $\zeta_1(t)$, $t \in (5i, 5(i+1)]$, $i = 0, 1, 2, \ldots$. The vertical lines connecting the pieces of the graph are drawn for better visibility.

Example 3 *Consider a function $\eta_1(t) = z_i$, $\theta_i < t \le \theta_{i+1}$, where $\theta_i = \frac{1}{12}(6i + (-1)^i - 1)$, $i \in \mathbb{Z}$, $i = 0, 1, 2, \ldots$, is unbounded sequence. Let us demonstrate that $\eta_1(t)$ is a discontinuous Poisson stable.*

Fix a sequence t_m, $m \in \mathbb{Z}$, such that (t_m, θ_i) is a Poisson couple in the sense of Definition 3. One can verify that the sequence θ_i satisfies $(1,2)-$property. It is known that, if $\eta_1(t) = z_i$, $t \in (\theta_i, \theta_{i+1}]$, $i \in \mathbb{Z}$, then $\eta_1(t + t_m) = z_{i+l_m}$ for $\theta'_i \le t < \theta'_{i+1}$, where $\theta'_i = \theta_{i+l_m} - t_m$, $i \in \mathbb{Z}$.

Fix an interval $[b_2, c_2]$, where $c_2 > b_2$, and an arbitrary number $\epsilon > 0$ such that $2\epsilon < \underline{\theta}$ on this interval. Assume that $\theta_i \le \theta_{i+l_m} - t_m$ and consider discontinuity points θ_i, $i = l+1, l+2, \ldots, l+r_2-1$, of the interval $[b_2, c_2]$.

Let us fix i, $i = l, l+1, \ldots, l+r_2$, and for fixed i we have that $\eta_1(t) = z_i$, for $t \in (\theta_i, \theta_{i+1}]$ and $\eta_1(t + t_m) = z_{i+l_m}$, $t \in (\theta'_i, \theta'_{i+1}]$. From (1) $|\theta'_i - \theta_i| < \epsilon$ is valid. Moreover it implies that for sufficiently large n,

$$|\eta_1(t + t_m) - \eta_1(t)| = |z_{i+l_m} - z_i| < \epsilon$$

for $t \in (\theta'_i, \theta_{i+1})$. Thus, $\eta_1(t)$ is discontinuous Poisson stable (Figure 2).

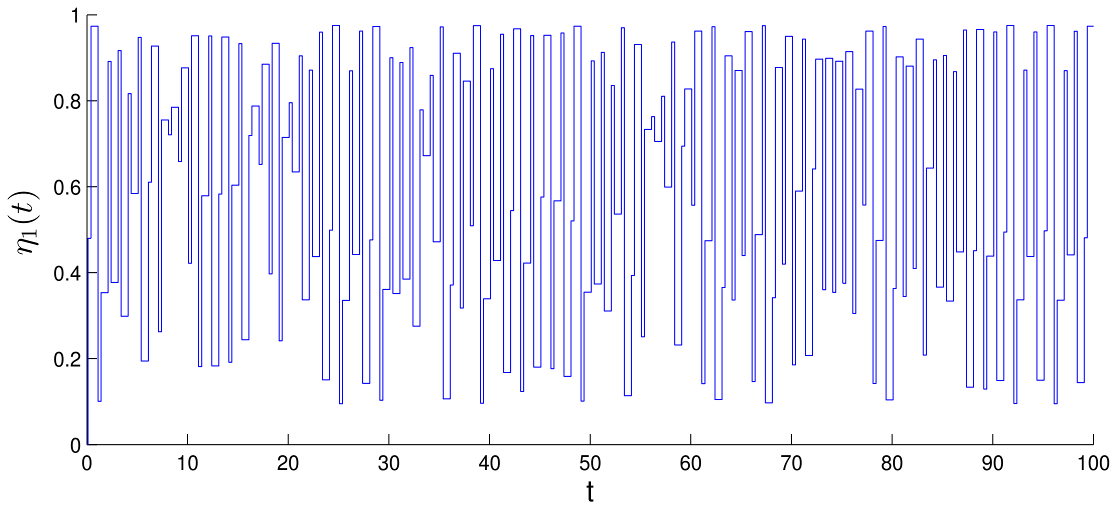

Figure 2: The graph of function $\eta_1(t) = z_i,\ t \in (\theta_i, \theta_{i+1}]$, with $\theta_i = \frac{1}{12}(6i + (-1)^i - 1)$, $i = 0, 1, 2, \ldots$. The vertical lines connecting the pieces of the graph are drawn for better visibility.

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