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INITIAL BOUNDS FOR ANALYTIC FUNCTION CLASSES CHARACTERIZED BY CERTAIN SPECIAL FUNCTIONS AND BELL NUMBERS

Over the last few years, Geometric Function Theory (GFT) as one of the most prime branch of complex analysis has gained a considerable and an impressive attention from many researchers, largely because it deals with the study of the geometric properties of analytic functions and their numerous applications in various fields of mathematics such as in special functions, probability distributions, and fractional calculus. The investigations in this paper are on two new classes of analytic functions defined in the unit disk $\mathcal{E} = \{z \in \mathbb{C} : |z| < 1\}$ and denoted by $\chi\mathcal{S}_q(b,\mathcal{K})$ and $\chi\mathcal{T}_q(b,\mathcal{K})$. Function f in the classes satisfy the conditions $f(0) = f'(0) - 1 = 0$, hence can be of series type $f(z) = z + a_2z^2 + a_3z^3 + \cdots,\ z \in \mathcal{E}$. The definition of the two new classes of analytic functions embed some well-known special functions such as the Galuê-type Struve function, modified error function and a starlike function whose coefficients are Bell's numbers while some involving mathematical principles are the q-derivative, inequalities, convolution and subordination. The main results from these classes are however, the upper estimates of some initial bounds such as $|a_n|$ $(n = 2, 3, 4)$ and the Fekete-Szegö functional $|a_3 - \phi a_2^2|$ $(\phi \in \mathbb{C})$ of functions $f \in \chi\mathcal{S}_q(b,\mathcal{K})$ and $f \in \chi\mathcal{T}_q(b,\mathcal{K})$.
Key words: Analytic function, Schwarz function, Galuê-type Struve function, modified error function, Bell's numbers, coefficient estimate, Fekete-Szegö problem, subordination, convolution, q-derivative.

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Кейбір арнайы функциялармен және сандармен сипатталатын аналитикалық функциялар кластарының бастапқы шекаралары

Соңғы бірнеше жылда геометриялық функциялар теориясы (ГФТ) кешенді талдаудың ең басты саласының бірі ретінде көптеген зерттеушілер тарапынан айтарлықтай және әсерлі назарға ие болды, өйткені ол аналитикалық функциялардың геометриялық қасиеттерін зерттеумен айналысады және арнайы функциялар, ықтималдық үлестірімдер және бөлшек есептеу сияқты математиканың әртүрлі салаларындағы көптеген қолданбаларды зерттеумен айналысады. Бұл мақалада $\mathcal{E} = \{z \in \mathbb{C} : |z| < 1\}$ бірлік шеңберінде анықталған аналитикалық функциялардың екі жаңа класын қарастырады және $\chi\mathcal{S}_q(b,\mathcal{K})$ және $\chi\mathcal{T}_q(b,\mathcal{K})$ арқылы белгіленеді. f функциясы $f(0) = f'(0) - 1 = 0$ шарттарын қанағаттандырады, сондықтан $f(z) = z + a_2z^2 + a_3z^3 + \cdots,\ z \in \mathcal{E}$. қатардың типі түрінде болуы мүмкін. Аналитикалық функциялардың екі жаңа класының анықтамалары Галуэ типті Струве функциясын, өзгертілген қате функциясын, коэффициенттері Белл сандары, ал кейбір математикалық принциптері q-туынды болып табылатын жұлдыз функциясын, теңсіздіктерді, конвульсия және бағыну болып табылатын белгілі арнайы функцияларды қамтиды.

Дегенмен, бұл кластардың негізгі нәтижелері кейбір бастапқы шекаралардың жоғарғы бағалаулары болып табылады, яғни $|a_n|$ $(n = 2, 3, 4)$ және $f \in \chi\mathcal{S}_q(b, \mathcal{K})$ және $f \in \chi\mathcal{T}_q(b, \mathcal{K})$ функцияларының Фекете-Сеге функционалы $|a_3 - \phi a_2^2|$ $(\phi \in \mathbb{C})$.
Түйін сөздер: Аналитикалық функция, Шварц функциясы, Галуэ типті Струве функциясы, модификацияланған қате функциясы, Белл сандары, коэффициентті баға, Фекете-Сеге есебі, бағыну, ұйіртқі, q-туынды.

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Начальные границы классов аналитических функций, характеризующих определенных специальных функций и номеров

За последние несколько лет геометрическая теория функций (ГТФ) как одна из наиболее важных отраслей комплексного анализа привлекла значительное и впечатляющее внимание многих исследователей, главным образом потому, что она занимается изучением геометрических свойств аналитических функций и их многочисленные приложения в различных областях математики, таких как специальные функции, распределения вероятностей и дробное исчисление. В данной статье исследуются два новых класса аналитических функций, определенных в единичном круге $\mathcal{E} = \{z \in \mathbb{C} : |z| < 1\}$ и обозначается $\chi\mathcal{S}_q(b, \mathcal{K})$ и $\chi\mathcal{T}_q(b, \mathcal{K})$. Функция f в классах удовлетворяет следующим условиям $f(0) = f'(0) - 1 = 0$, следовательно, может иметь тип ряда $f(z) = z + a_2z^2 + a_3z^3 + \cdots,\ z \in \mathcal{E}$. В определения двух новых классов аналитических функций включены некоторые хорошо известные специальные функции, такие как функция Струве типа Галуэ, модифицированная функция ошибок и звездообразная функция, коэффициентами которой являются числа Белла, а некоторыми математическими принципами являются q-производная, неравенства, свертка и подчинение. Однако основными результатами этих классов являются верхние оценки некоторых начальных границ, таких как $|a_n|$ $(n = 2, 3, 4)$ и функционал Фекете-Сеге $|a_3 - \phi a_2^2|$ $(\phi \in \mathbb{C})$ функций $f \in \chi\mathcal{S}_q(b, \mathcal{K})$ и $f \in \chi\mathcal{T}_q(b, \mathcal{K})$.
Ключевые слова: Аналитическая функция, функция Шварца, функция Струве типа Галуэ, модифицированная функция ошибок, числа Белла, коэффициентная оценка, задача Фекете-Сегё, подчинение, свертка, q-производная.

1 Introduction and Definitions

Geometric Function Theory (GFT) is one of the most fascinating branches of complex analysis and it has gained a considerable attention from many researchers in pure mathematics. GFT deals with the study of the geometric properties of analytic functions with numerous applications in various fields of mathematics such as in the use of special functions, probability distributions, and fractional calculus.

Let $\mathcal{A}$ represent the set of normalized analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

with the conditions $f(0) = 0 = f'(0) - 1$ and $z \in \mathcal{E} := \{z \in \mathbb{C} : |z| < 1\}$. Also let $\mathcal{S}$, a subset of $\mathcal{A}$ be the set of analytic and univalent functions in $\mathcal{E}$.

The foundation of coefficient problems in the theory of univalent functions is traceable to Bieberbach conjecture or coefficient conjecture (see [7]) of 1916 where he conjuncted that

$|a_n| \leqslant n,\ \forall n \in \mathbb{N}$. In [7], Duren emphasized that *coefficient problem* is the determination of that part of the $(k-1)$-dimensional complex plane, occupied by the points $(a_2, a_3, a_4, \ldots, a_k)$ for function". In 1985, Branges [5] verified that the conjuncture was actually true and this affirmation subsequently elevated the theory to one of the growing areas of possible research. Some well-known subclasses of class $\mathcal{S}$ are therefore the classes of starlike, convex, close-to-convex, close-to-star and spirallike functions. In addition, the coefficient bounds, generalizations and the coefficient properties of several of the subclasses of class $\mathcal{S}$ have also been sought. In fact, the nature and properties of these subclasses which are largely based on the geometries of their domains are continuously been studied with no end at sight.

In this paper, represented by ∇ is the set of analytic functions of the form

$$w(z) = \sum_{n=1}^{\infty} w_n z^n \quad (z \in \mathcal{E}). \tag{2}$$

Set ∇ is known as the set of Schwarz functions and it is normalized by the conditions $w(0) = 0$, $|w(z)| < 1$ and $z \in \mathcal{E}$. Likewise, if $h_1, h_2 \in \mathcal{A}$, then $h_1 \prec h_2$ if $h_1(z) = h_2(w(z))$ for $z \in \mathcal{E}$. Should h_2 be univalent in $\mathcal{E}$, then $h_1(z) \prec h_2(z)$ if and only if $h_1(0) = h_2(0)$ and $h_1(\mathcal{E}) \subset h_2(\mathcal{E})$. The symbol '$\prec$' means subordination. Also, let

$$h_1 = z + \sum_{n=2}^{\infty} a_n z^n, h_1 = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A},\ \text{then } h_1 \star h_2 := z + \sum_{n=2}^{\infty} (a_n \times b_n) z^n$$

where the symbol '$\star$' means convolution or Hadamard product.

The sequence $\{\eta_n\}_0^\infty$ of numbers

$$1, 1, 2, 5, 15, 52, 203, 877, 4140, \ldots$$

was introduced by Bell [3, 4]. The Bell's numbers are generated as a result of observing the number of possible partitions of a set. In view of the Bell's number, Kumar et al. [15] established the function

$$\mathcal{K}(z) = e^{e^z - 1} = \sum_{n=0}^{\infty} \frac{\eta_n}{n!} z^n = 1 + z + z^2 + \frac{5}{6} z^3 + \frac{5}{8} z^4 + \cdots, \quad z \in \mathcal{E}. \tag{3}$$

and it was proved that function $\mathcal{K}(z)$ is starlike with respect to 1. This starlikeness property prompted the our interest to further investigate this function.

The Galuê-type Struve function (GTSF) was introduced in [16] and defined by

$$\alpha \mathcal{W}_{p,b,c,\xi}^{\lambda,\mu}(z) = z + \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma(\lambda n + \mu)\Gamma(\alpha n + \frac{p}{\xi} + \frac{b+2}{2})} \left(\frac{z}{2}\right)^{2n+p+1} \quad (z \in \mathcal{E}), \tag{4}$$

where $\alpha \in \mathbb{N}$, $z, p, b, c \in \mathbb{C}$, $\lambda > 0$, $\xi > 0$ and μ is an arbitrary parameter. It is evident that when $\lambda = \alpha = 1$, $\mu = \frac{3}{2}$ and $\xi = 1$ in (4), then we have the generalized Struve function (see [17]) defined by

$$\mathcal{H}_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma(n + \frac{3}{2})\Gamma(n + p + \frac{b+2}{2})} \left(\frac{z}{2}\right)^{2n+p+1} \quad (z \in \mathcal{E}), \tag{5}$$

where $z, p, b, c \in \mathbb{C}$. Using (4), consider the function

$$\mathcal{U}_{p,b,c,\xi}(z) = 2^p \sqrt{\pi}\, \Gamma\Big(\frac{p}{\xi} + \frac{b+2}{2}\Big) z^{\frac{-(p+1)}{2}} \alpha \mathcal{W}^{\lambda,\mu}_{p,b,c,\xi}(\sqrt{z}) \quad (z \in \mathcal{E}). \tag{6}$$

Using the Pochhammer (or Appell) symbol defined in terms of Euler's gamma function, Oyekan [19] presented the relation

$$(\gamma)_n = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)} = \gamma(\gamma+1)\ldots(\gamma+n-1)$$

so that from (6) we have

$$\mathcal{V}_{p,b,c,\xi}(z) = z\mathcal{U}_{p,b,c,\xi}(z) = z + \sum_{n=2}^{\infty} \Big(\frac{(\frac{-c}{4})^n}{(\mu)_{\lambda(n-1)}(\gamma)_{\alpha(n-1)}}\Big) z^n \quad (z \in \mathcal{E}). \tag{7}$$

Using the convolution principle, Oyekan [19] defined the function

$$\mathcal{L}^{\lambda,\mu,\xi}_{p,b,c}(z) = (f \star \mathcal{V}_{p,b,c,\xi})(z) = z + \sum_{n=2}^{\infty} \Big(\frac{(\frac{-c}{4})^n}{(\mu)_{\lambda(n-1)}(\gamma)_{\alpha(n-1)}}\Big) a_n z^n \quad (z \in \mathcal{E}), \tag{8}$$

where $p, b, c \in \mathbb{C}$, $\gamma = \frac{p}{\xi} + \frac{b+2}{2} \neq 0, -1, -2, \cdots$, $\alpha \in \mathbb{N}$, $\lambda, \xi > 0$ and μ is an arbitrary parameter. Function $\mathcal{V}$ in (7) is the normalized form of Galuê-type Struve function and is analytic in $\mathbb{C}$, while (8) is the simplified version.

A special function that occurs in probability, statistics, material science, and partial differential equation is the *error function*. The error function is use in quantum mechanics to eliminate the probability of observing a particle in a specified region. The error function

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^{n-1} z^{n+1}}{(2n+1)!} \quad (z \in \mathcal{E}) \tag{9}$$

was reported in [1] and for additional information see [6, 8]. In particular, Ramachandra et al. [25] made a slight modification to (9) and came up with the function

$$Erf(z) = z + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(2n-1)(n-1)!} z^n \quad (z \in \mathcal{E}). \tag{10}$$

where the function $Erf(z)$ was used to define a class of analytic functions and solved some coefficient problems.

Using the convolution concept, and in view of (8) and (10), we can deduce the function

$$\mathcal{G}(z) = (\mathcal{L}^{\lambda,\mu,\xi}_{p,b,c} \star Erf)(z) = z + \sum_{n=2}^{\infty} \frac{(-\frac{c}{4})^{n-1}}{(2n-1)(n-1)!(\mu)_{\lambda(n-1)}(\gamma)_{\alpha(n-1)}} a_n z^n. \tag{11}$$

The quantum derivative (q-derivative or Jackson's derivative) operator (see [9]) for function f in (1) is defined by

$$\left.\begin{aligned} &\mathcal{D}_q f(0) = f'(0) = 1 \quad (z=0) \quad \text{if it exists,} \\ &\mathcal{D}_q f(z) = \tfrac{f(z)-f(qz)}{z(1-q)} = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1} \quad (z \neq 0), \\ &\mathcal{D}_q^2 f(z) = \mathcal{D}_q(\mathcal{D}_q f(z)) = \sum_{n=2}^{\infty} [n]_q [n-1]_q a_n z^{n-2} \end{aligned}\right\} \tag{12}$$

such that the q-number n is defined by

$$[n]_q = \begin{cases} 0 & for \quad n = 0 \\ 1 & for \quad n = 1 \\ 1+q & for \quad n = 2 \\ \frac{1-q^n}{1-q} = \sum\limits_{n=0}^{n-1} q^n & for \quad n \in \mathbb{R} \end{cases} \tag{13}$$

and $\lim\limits_{q \uparrow 1}[n]_q = n$. The q-derivative is the q-analogue of the classical derivative of functions where it plays a significant role in defining many q-operators in various areas of q-analysis. For some historical details, properties, applications, and some results on some subclasses of analytic functions involving q-differentiation see [2, 9, 11–14].

Definition 1.1 *Let $q \in (0,1)$, $b \in \mathbb{C} \setminus \{0\}$, $\gamma \neq 0, -1, -2, \cdots$, $c \in \mathbb{C}$ and let $\mathcal{K}(z)$ be as defined in* (3). *The function $\mathcal{G}$ is said to belong to class $\chi\mathcal{S}_q(b, \mathcal{K})$, if*

$$1 + \frac{1}{b}\left(\frac{z\mathcal{D}_q\mathcal{G}(z)}{\mathcal{G}(z)} - 1\right) \prec \mathcal{K}(z) \tag{14}$$

and it is said to belong to the class $\chi\mathcal{T}_q(b, \mathcal{K})$, if

$$1 + \frac{1}{b}\left(\frac{z\mathcal{D}_q(\mathcal{D}_q\mathcal{G}(z))}{\mathcal{D}_q\mathcal{G}(z)}\right) \prec \mathcal{K}(z). \tag{15}$$

In this work we gave the estimates on the initial coefficients and on the Fekete-Szegö functionals for two classes of analytic functions.

2 Applicable Lemmas

Let $w \in \nabla$ in (2), then the following lemmas hold true.

Lemma 2.1 ([26]) *Let $w(z) \in \nabla$, then $|w_n| \leqslant 1$ $\forall n \in \mathbb{N}$. Equality occurs for functions $w(z) = e^{i\vartheta}z^n$ $(\vartheta \in [0, 2\pi))$.*

Lemma 2.2 ([10]) *Let $w \in \nabla$, then for $\phi \in \mathbb{C}$,*

$$|w_2 + \phi w_1^2| \leqslant \max\{1; |\phi|\}.$$

Equality holds for functions $w(z) = z$ or $w(z) = z^2$.

3 Main Results

Theorem 3.1 *Let $q \in (0,1)$, $b \in \mathbb{C} \setminus \{0\}$, $\gamma \neq 0, -1, -2, \cdots$, $c \in \mathbb{C}$ and let $\mathcal{K}(z)$ be as defined in* (3). *If $\mathcal{G}$ belongs to the class $\chi\mathcal{S}_q(b, \mathcal{K})$, then*

$$|a_2| \leqslant \frac{12(\mu)_\lambda(\gamma)_\alpha|b|}{cq},$$

$$|a_3| \leqslant \frac{40(\mu)_{2\lambda}(\gamma)_{2\alpha}|b|}{c^2q(1+q)} \max\left\{1, \left|\frac{q+b}{q}\right|\right\},$$

$$|a_4| \leqslant \frac{168(\mu)_{3\lambda}(\gamma)_{3\alpha}|b|}{c^3q(1+q+q^2)} \max\left\{1, \left|\sigma\Big[\frac{t}{\sigma} + \Big(\frac{q+b}{q}\Big) + \Big(1 + \frac{2}{\sigma}\Big)\Big]\right|\right\},$$

and for $\phi \in \mathbb{C}$,

$$|a_3 - \phi a_2^2| \leqslant \frac{40(\mu)_{2\lambda}(\gamma)_{2\alpha}|b|}{c^2 q(1+q)} \max\left\{1, \left|\frac{5(\mu)_{2\lambda}(\gamma)_{2\alpha}q^2 + 5(\mu)_{2\lambda}(\gamma)_{2\alpha}qb - 18\xi(\mu)_{2\lambda}^2(\gamma)_{2\alpha}^2 q(1+q)b}{5(\mu)_{2\lambda}(\gamma)_{2\alpha}q^2}\right|\right\}$$

where

$$\sigma = \frac{([2]_q - 1) + ([3]_q - 1)}{([2]_q - 1)([3]_q - 1)} b \quad \text{and} \quad t = \frac{5}{6} - \frac{b^2}{([2]_q - 1)^2}. \tag{16}$$

Proof. Suppose $\mathcal{G} \in \chi\mathcal{S}_q(b, \mathcal{K})$, then there exists a Schwarz function $w \in \nabla$ of the form (2) such that

$$1 + \frac{1}{b}\left(\frac{z\mathcal{D}_q\mathcal{G}(z)}{\mathcal{G}(z)} - 1\right) = \mathcal{K}(w(z)),$$

so that

$$[z\mathcal{D}_q\mathcal{G}(z) - \mathcal{G}(z)]\mathcal{G}^{-1}(z) = b[\mathcal{K}(w(z)) - 1]. \tag{17}$$

A careful expansion of (17) shows that

$$\begin{aligned}
[z\mathcal{D}_q\mathcal{G}(z) - \mathcal{G}(z)]\mathcal{G}^{-1}(z) &= ([2]_q - 1)\frac{c}{12(\mu)_\lambda(\gamma)_\alpha} a_2 z \\
&+ \left\{([3]_q - 1)\frac{c^2}{40(\mu)_{2\lambda}(\gamma)_{2\alpha}} a_3 - ([2]_q - 1)\frac{c^2}{144(\mu)_\lambda^2(\gamma)_\alpha^2} a_2^2\right\} z^2 \\
&- \left\{\Big(([2]_q - 1) + ([3]_q - 1)\Big)\frac{c^3}{480(\mu)_\lambda(\gamma)_\alpha(\mu)_{2\lambda}(\gamma)_{2\alpha}} a_2 a_3\right. \\
&\quad \left. - ([2]_q - 1)\frac{c^3}{1728(\mu)_\lambda^3(\gamma)_\alpha^3} a_2^3 - ([4]_q - 1)\frac{c^3}{168(\mu)_{3\lambda}(\gamma)_{3\alpha}} a_4\right\} z^3 \\
&+ \cdots
\end{aligned} \tag{18}$$

and

$$b[\mathcal{K}(w(z)) - 1] = bw_1 z + b(w_2 + w_1^2)z^2 + b(w_3 + 2w_1w_2 + \frac{5}{6}w_1^3)z^3 + \cdots. \tag{19}$$

Equating (18) and (19) gives

$$([2]_q - 1)\frac{c}{12(\mu)_\lambda(\gamma)_\alpha} a_2 = bw_1 \tag{20}$$

$$([3]_q - 1)\frac{c^2}{40(\mu)_{2\lambda}(\gamma)_{2\alpha}} a_3 - ([2]_q - 1)\frac{c^2}{144(\mu)_\lambda^2(\gamma)_\alpha^2} a_2^2 = b(w_2 + w_1^2). \tag{21}$$

$$\begin{aligned}
- \Big[([2]_q - 1) + ([3]_q - 1)\Big]\frac{c^3}{480(\mu)_\lambda(\gamma)_\alpha(\mu)_{2\lambda}(\gamma)_{2\alpha}} a_2 a_3 + ([2]_q - 1)\frac{c^3}{1728(\mu)_\lambda^3(\gamma)_\alpha^3} a_2^3 \\
+ ([4]_q - 1)\frac{c^3}{168(\mu)_{3\lambda}(\gamma)_{3\alpha}} a_4 = b(w_3 + 2w_1w_2 + \frac{5}{6}.w_1^3)
\end{aligned} \tag{22}$$

Now, from (20) we have

$$a_2 = \frac{12(\mu)_\lambda(\gamma)_\alpha b w_1}{c([2]_q - 1)} \tag{23}$$

so that by using (13) we get

$$|a_2| \leqslant \frac{12(\mu)_\lambda(\gamma)_\alpha |b||w_1|}{cq}$$

and the application of Lemma 2.1 achieves the result.

From (21) we have

$$a_3 = \frac{40(\mu)_{2\lambda}(\gamma)_{2\alpha} b}{c^2([3]_q - 1)}\left[w_2 + \left(\frac{[2]_q - 1 + b}{[2]_q - 1}\right)w_1^2\right] \tag{24}$$

so that using (13) gives

$$|a_3| \leqslant \frac{40(\mu)_{2\lambda}(\gamma)_{2\alpha}|b|}{c^2 q(1+q)}\left|w_2 + \left(\frac{[2]_q - 1 + b}{[2]_q - 1}\right)w_1^2\right|$$

and the application of Lemma 2.2 achieves the result.

From (22) we have

$$a_4 = \frac{168(\mu)_{3\lambda}(\gamma)_{3\alpha} b}{c^3([4]_q - 1)}\left[w_3 + \left(\frac{5}{6} - \frac{b^2}{([2]_q - 1))^2} + \frac{\left[([2]_q - 1) + ([3]_q - 1)\right]}{([2]_q - 1)([3]_q - 1)} b\left(\frac{[2]_q - 1 + b}{[2]_q - 1}\right)\right)w_1^3 \right.$$
$$\left. + \left(2 + \frac{\left[([2]_q - 1) + ([3]_q - 1)\right]}{([2]_q - 1)([3]_q - 1)}\right)w_1 w_2\right]$$

by using (16) we get

$$a_4 = \frac{168(\mu)_{3\lambda}(\gamma)_{3\alpha} b}{c^3([4]_q - 1)}\left[w_3 + \sigma\left(\frac{t}{\sigma} + \left(\frac{[2]_q - 1 + b}{[2]_q - 1}\right)\right)w_1^3 + \left(\frac{2}{\sigma} + 1\right)w_1 w_2\right]$$

by using (13) w get

$$|a_4| \leq \frac{168(\mu)_{3\lambda}(\gamma)_{3\alpha}|b|}{c^3 q(1+q+q^2)}\left|w_3 + \sigma\left(\frac{t}{\sigma} + \left(\frac{[2]_q - 1 + b}{[2]_q - 1}\right)\right)w_1^3 + \left(\frac{2}{\sigma} + 1\right)w_1 w_2\right|$$

and the application of Lemmas 2.1 and 2.2 achieves the result.

Lastly, from (23) and (24) and for $\phi \in \mathbb{C}$ we have

$$a_3 - \phi a_2^2 = \frac{40(\mu)_{2\lambda}(\gamma)_{2\alpha} b}{c^2([3]_q - 1)}\left[w_2 + \left(\frac{[2]_q - 1 + b}{([2]_q - 1)}\right)w_1^2\right] - \phi\frac{144(\mu)_\lambda^2(\gamma)_\alpha^2 b^2 w_1^2}{c^2([2]_q - 1)^2}$$
$$= \frac{40(\mu)_{2\lambda}(\gamma)_{2\alpha} b}{c^2([3]_q - 1)}\left[w_2 + \left(1 + \frac{b}{[2]_q - 1} - \frac{18\phi(\mu)_\lambda^2(\gamma)_\alpha^2 b([3]_q - 1)}{5([2]_q - 1)^2(\mu)_{2\lambda}(\gamma)_{2\alpha}}\right)w_1^2\right]$$

$$a_3-\phi a_2^2=\frac{40(\mu)_{2\lambda}(\gamma)_{2\alpha}b}{c^2([3]_q-1}\Bigg[w_2+ \left(\frac{5(\mu)_{2\lambda}(\gamma)_{2\alpha}([2]_q-1)^2+5b(\mu)_{2\lambda}(\gamma)_{2\alpha}([2]_q-1)-18\xi(\mu)_\lambda^2(\gamma)_\alpha^2 b([3]_q-1)}{5(\mu)_{2\lambda}(\gamma)_{2\alpha}([2]_q-1)^2}\right)w_1^2\Bigg]$$

by using (13) we get

$$a_3-\phi a_2^2=\frac{40(\mu)_{2\lambda}(\gamma)_{2\alpha}b}{c^2q(1+q)}\Bigg[w_2+ \left(\frac{5(\mu)_{2\lambda}(\gamma)_{2\alpha}([2]_q-1)^2+5b(\mu)_{2\lambda}(\gamma)_{2\alpha}([2]_q-1)-18\xi(\mu)_\lambda^2(\gamma)_\alpha^2 b([3]_q-1)}{5(\mu)_{2\lambda}(\gamma)_{2\alpha}([2]_q-1)^2}\right)w_1^2\Bigg]$$

and

$$|a_3-\phi a_2^2|\leqslant\frac{40(\mu)_{2\lambda}(\gamma)_{2\alpha}|b|}{c^2q(1+q)}\Bigg|w_2+ \left(\frac{5(\mu)_{2\lambda}(\gamma)_{2\alpha}([2]_q-1)^2+5b(\mu)_{2\lambda}(\gamma)_{2\alpha}([2]_q-1)-18\xi(\mu)_\lambda^2(\gamma)_\alpha^2 b([3]_q-1)}{5(\mu)_{2\lambda}(\gamma)_{2\alpha}([2]_q-1)^2}\right)w_1^2\Bigg|$$

and the application of Lemma 2.2 achieves the result.

Theorem 3.2 *Let $q\in(0,1)$, $b\in\mathbb{C}\setminus\{0\}$, $\gamma\neq 0,-1,-2,\cdots$, $c\in\mathbb{C}$ and let $\mathcal{K}(z)$ be as defined in* (3)*. If $\mathcal{G}$ belongs to the class $\chi\mathcal{T}_q(b,\mathcal{K})$, then*

$$|a_2|\leqslant\frac{12(\mu)_\lambda(\gamma)_\alpha|b|}{c(1+q)},$$
$$|a_3|\leqslant\frac{40(\mu)_{2\lambda}(\gamma)_{2\alpha}|b|}{c^2q(1+q)}\max\{1;|1+b|\},$$

and

$$|a_3-\xi a_2^2|\leqslant\frac{40(\mu)_{2\lambda}(\gamma)_{2\alpha}|b|}{c^2q(1+q)}\max\left\{1,\left|\frac{5(\mu)_{2\lambda}(\gamma)_{2\alpha}(1+q)^2(1+b)-18\xi(\mu)_\lambda^2(\gamma)_\alpha^2 q(1+q)b}{5(\mu)_{2\lambda}(\gamma)_{2\alpha}(1+q)^2}\right|\right\}.$$

Proof. Suppose $\mathcal{G}\in\chi\mathcal{T}_q(b,\mathcal{K})$, then there exists a Schwarz function $w\in\nabla$ of the form (2) such that

$$1+\frac{1}{b}\left(\frac{z\mathcal{D}_q\mathcal{G}(z)}{\mathcal{D}_q\mathcal{G}(z)}\right)=\mathcal{K}(w(z))$$

so that

$$(z\mathcal{D}_q\mathcal{G}(z))(\mathcal{D}_q\mathcal{G}(z))^{-1}=[\mathcal{K}(w(z))-1]b \tag{25}$$

from where we get

$$(z\mathcal{D}_q\mathcal{G}(z))(\mathcal{D}_q\mathcal{G}(z))^{-1}=z-[2]_q\frac{c}{12(\mu)_\lambda(\gamma)_\alpha}a_2z \tag{26}$$
$$+\left([3]_q\frac{c^2}{40(\mu)_{2\lambda}(\gamma)_{2\alpha}}a_3-[2]_q^2\frac{c^2}{144(\mu)_\lambda^2(\gamma)_\alpha^2}a_2^2\right)z^2+\cdots. \tag{27}$$

Equating (3) and (26) implies that

$$-[2]_q \frac{c}{12(\mu)_\lambda(\gamma)_\alpha} a_2 = bw_1 \tag{28}$$

and

$$[3]_q \frac{c^2}{40(\mu)_{2\lambda}(\gamma)_{2\alpha}} a_3 - [2]_q^2 \frac{c^2}{144(\mu)_\lambda^2(\gamma)_\alpha^2} a_2^2 = b(w_2 + w_1^2). \tag{29}$$

Now from (28) we get

$$a_2 = \frac{-12(\mu)_\lambda(\gamma)_\alpha bw_1}{c[2]_q}. \tag{30}$$

so that by using (13), taking modulus and applying Lemma 2.1 achieves the result.

From (29) we get

$$a_3 = \frac{40(\mu)_{2\lambda}(\gamma)_{2\alpha} b}{c^2[3]_q} \Big[w_2 + (1+b)w_1^2\Big], \tag{31}$$

and using (13) we get

$$a_3 = \frac{40(\mu)_{2\lambda}(\gamma)_{2\alpha} b}{c^2 q(1+q)} \Big[w_2 + (1+b)w_1^2\Big]$$

and

$$|a_3| \leqslant \frac{40(\mu)_{2\lambda}(\gamma)_{2\alpha} |b|}{c^2 q(1+q)} \Big|w_2 + (1+b)w_1^2\Big|$$

so that applying Lemma 2.2 achieves the result.

Now from (30) and (31) we get

$$\begin{aligned} a_3 - \varphi a_2^2 &= \frac{40(\mu)_{2\lambda}(\gamma)_{2\alpha} b}{c^2[3]_q} \Big[w_2 + (1+b)w_1^2\Big] - \xi \frac{144(\mu)_\lambda^2(\gamma)_\alpha^2 b^2 w_1^2}{c^2[2]_q^2} \\ &= \frac{40(\mu)_{2\lambda}(\gamma)_{2\alpha} b}{c^2[3]_q} \left[w_2 + \left(\frac{5(\mu)_{2\lambda}(\gamma)_{2\alpha}[2]_q^2(1+b) - 18\xi k_1^2 b[3]_q}{5(\mu)_{2\lambda}(\gamma)_{2\alpha}[2]_q^2}\right) w_1^2\right] \end{aligned}$$

and using (13) we get

$$a_3 - \varphi a_2^2 = \frac{40(\mu)_{2\lambda}(\gamma)_{2\alpha} b}{c^2 q(1+q)} \left[w_2 + \left(\frac{5(\mu)_{2\lambda}(\gamma)_{2\alpha}[2]_q^2(1+b) - 18\xi k_1^2 b[3]_q}{5(\mu)_{2\lambda}(\gamma)_{2\alpha}[2]_q^2}\right) w_1^2\right]$$

so that

$$|a_3 - \varphi a_2^2| \leqslant \frac{40(\mu)_{2\lambda}(\gamma)_{2\alpha} |b|}{c^2 q(1+q)} \left|w_2 + \left(\frac{5(\mu)_{2\lambda}(\gamma)_{2\alpha}[2]_q^2(1+b) - 18\xi k_1^2 b[3]_q}{5(\mu)_{2\lambda}(\gamma)_{2\alpha}[2]_q^2}\right) w_1^2\right|.$$

so that applying Lemma 2.2 achieves the result.

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