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ON ONE-DIMENSIONAL HELMHOLTZ EQUATION

The study of time-periodic solutions of the multidimensional wave equation on the entire 3D space is an important field of research in applied mathematics. It is known that this study leads to the Sommerfeld radiation condition at infinity. The radiation condition states that for a solution to a one-dimensional wave equation, such as the Helmholtz equation or the wave equation, to represent an outgoing wave at infinity. The Helmholtz equation in 1D, which models the propagation of electromagnetic waves in systems effectively reduced to one dimension, is equivalent to the time-independent Schrodinger equation. The one-dimensional Helmholtz potential is widely used in various areas of physics and engineering, such as electromagnetics, acoustics, and quantum mechanics. The Sommerfeld problem in the one-dimensional case requires special investigation, and the radiation conditions in the one-dimensional case differ from those in the multidimensional case. These differences are related to the peculiarities of the fundamental solutions. In this paper, we constructed the fundamental solution of the one-dimensional Helmholtz equation. Then, we found the boundary conditions for the one-dimensional Helmholtz potential. Finally, the equivalent conditions with Sommerfeld radiation conditions were found for the one-dimensional Helmholtz equation.
Key words: Fundamental solution, one-dimensional Sommerfeld problem, one-dimensional Helmholtz equation.

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Бір өлшемді Гельмгольц теңдеуі туралы

Бүкіл үш өлшемді кеңістікте көпөлшемді толқын теңдеудің уақыт бойынша периодты шешімдерін зерттеу қолданбалы математикадағы маңызды зерттеу саласы болып табылады. Бұл зерттеу Зоммерфельдтің шексіздікте сәулелену есебіне келетіні белгілі. Сәулелену шарты Гельмгольц теңдеуі немесе толқын теңдеуі сияқты бір өлшемді толқын теңдеуін шешу үшін бастапқы толқынды шексіздіктегі бейнелеуін сипаттайды.Бір өлшемге тиімді келтірілген жүйелердегі электромагниттік толқындардың таралуын модельдейтін бір өлшемді Гельмгольц теңдеуі уақытқа тәуелсіз Шредингер теңдеуімен эквивалентті. Гельмгольц потенциалы толқын өткізгіштердегі электромагниттік толқындар, акустикалық толқын өткізгіштердегі дыбыс толқындары және кванттық механика сияқты толқындардың таралу есептеріндегі негізгі ұғым болып табылады. Бір өлшемді жағдайда Зоммерфельд есебі арнайы зерттеуді қажет етеді, ал бір өлшемді жағдайда сәулелену шарты көп өлшемді жағдайдан өзгешеленеді. Бұл айырмашылықтар іргелі шешімдердің өзгешеліктерімен байланысты. Бұл жұмыста біз бір өлшемді Гельмгольц теңдеуінің іргелі шешімін құрдық. Содан кейін, бір өлшемді Гельмгольц потенциалының шекаралық шарттарын таптық. Сонымен қатар, бір өлшемді Гельмгольц теңдеуі үшін Зоммерфельд сәулелену шартына эквивалент шарт табылды.
Түйін сөздер: Іргелі шешім, бір өлшемді Зоммерфельд есебі, бір өлшемді Гельмгольц теңдеуі.

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Об одномерном уравнении Гельмгольца

Изучение периодических во времени решений многомерного волнового уравнения во всем трехмерном пространстве является важной областью исследований в прикладной математике. Известно, что это исследование приводит к условию излучения Зоммерфельда на бесконечности. Условие излучения гласит, что для решения одномерного волнового уравнения, такого как уравнение Гельмгольца или волновое уравнение, необходимо представлять исходящую волну на бесконечности. Уравнение Гельмгольца в одномерном пространстве, которое моделирует распространение электромагнитных волн в системах, эффективно сведенных к одному измерению, эквивалентно не зависящему от времени уравнению Шредингера. Потенциал Гельмгольца является фундаментальным понятием в задачах распространения волн, таких как электромагнитные волны в волноводах, звуковые волны в акустических волноводах и квантовая механика. Задача Зоммерфельда в одномерном случае требует специального исследования, а условия излучения в одномерном случае отличаются от условий в многомерном случае. Эти различия связаны с особенностями фундаментальных решений. В этой работе, мы построили фундаментальное решение одномерного уравнения Гельмгольца. Затем, мы нашли граничные условия для одномерного потенциала Гельмгольца. Также, для одномерного уравнения Гельмгольца найдена условия, эквивалентные условиям излучения Зоммерфельда .
Ключевые слова: Фундаментальное решение, одномерная задача Зоммерфельда, одномерное уравнение Гельмгольца.

1 Introduction

The idea of constructing boundary value problems for differential equations by transferring boundary conditions started from the work [1], where the boundary value problem for an ordinary second-order differential equation was studied. The method of boundary condition transfer has also been used in [2, 3] for solving one-dimensional problems. Further developments of this technique for systems of ordinary differential equations are presented in [4] and [5]. In [6], the operator method of transferring boundary conditions from infinity was first used for the Helmholtz equation in a half-cylinder with ordinary conditions on the boundary. In the case of a finite right-hand side, it is shown that the condition of boundedness of the solution at infinity is equivalent to an operator boundary condition on a certain cross-section of the waveguide. Pseudodifferential boundary conditions ware equivalent to radiation conditions found on a sphere in [7]. In [8], boundary conditions for solving the Helmholtz equation inside a bounded domain with an artificially smooth boundary are proposed. That is, the author adds some assumptions to the sufficient smoothness of the boundary and proves the uniform convergence of solutions of the interior problem to the solution of the problem posed in an unbounded domain with Sommerfeld radiation conditions at infinity as the size of the domain increases without bound.

In [9], a new formulation of non-local type boundary conditions for the Helmholtz equation, which are equivalent to the Sommerfeld radiation conditions, is proposed. These boundary conditions have the property that stationary waves coming from the domain Ω to $\partial\Omega$ pass through $\partial\Omega$ without reflection, i.e., they act as transparent boundary conditions. In [10], the problem of reducing the Sommerfeld problem to a boundary problem in a finite domain $\mathbb{R}^n$, $n \geq 3$ is solved in the general case. The Sommerfeld problem in the one-dimensional case requires special investigation, and the radiation conditions in the one-dimensional case differ from those in the multidimensional case. These differences are related to the peculiarities of the fundamental solutions.

2 Fundamental solution to one-dimensional Helmholtz equation

In this section, we will find the fundamental solution of the one-dimensional Helmholtz equation.

Let us consider the inhomogeneous Helmholtz equation in 1D

$$\frac{d^2}{dx^2}u(x) + \lambda^2 u(x) = f(x), \tag{1}$$

where f is a given function with compact support representing a bounded source of energy in one dimension, and $\lambda \in \mathbb{C}$ is a constant, called the wavenumber.

Lemma 1 *A function*

$$\varepsilon_1(x-\xi) = \frac{\sin \lambda|x-\xi|}{2\lambda}$$

is a fundamental solution of the one-dimensional Helmholtz equation, such that

$$\frac{d^2}{dx^2}\varepsilon_1(x-\xi) + \lambda^2\varepsilon_1(x-\xi) = \delta(x-\xi),$$

where $\delta(x-\xi)$ is the Dirac delta function, defined as: $\langle\delta(x-\xi), \varphi\rangle = \varphi(\xi)$ for any test function φ.

Proof. First, let us find the second derivative of $\frac{\sin \lambda|x-\xi|}{2\lambda}$. Apply the chain rule to the outermost function, we obtain

$$\begin{aligned}
\frac{d^2}{dx^2}\frac{\sin \lambda|x-\xi|}{2\lambda} &= \frac{d}{dx}\left[\frac{d}{dx}\frac{\sin \lambda|x-\xi|}{2\lambda}\right] \\
&= \frac{d}{dx}\left[\lambda \cdot \frac{\cos \lambda|x-\xi|}{2\lambda} \cdot \frac{d}{dx}|x-\xi|\right] \\
&= \frac{1}{2}\frac{d}{dx}\cos \lambda|x-\xi| \cdot \frac{d}{dx}|x-\xi| + \frac{1}{2}\cos \lambda|x-\xi| \cdot \frac{d^2}{dx^2}|x-\xi| \\
&= -\frac{\lambda \sin \lambda\,|x-\xi|}{2}\left(\frac{d}{dx}|x-\xi|\right)^2 + \frac{1}{2}\cos \lambda|x-\xi| \cdot \frac{d^2}{dx^2}|x-\xi|.
\end{aligned} \tag{2}$$

The second derivative of $|x-\xi|$ is not defined at $x=\xi$, as the function has a sharp corner at that point. However, we can still define its generalized derivative using the theory of distributions.

To find the generalized derivative of $\frac{d^2}{dx^2}|x-\xi|$, we can start by finding its first derivative using the *sign* function: $\frac{d}{dx}|x-\xi| = \operatorname{sign}(x-\xi)$ where $\operatorname{sign}(x-\xi)$ is the *sign* function, defined as:

$$\operatorname{sign}(x-\xi) = \begin{cases} 1 & \text{if } x > \xi, \\ -1 & \text{if } x < \xi. \end{cases}$$

The *sign* function is not differentiable at $x=\xi$, so we need to define its generalized derivative using the distributional derivative formula:

$$\langle\frac{d}{dx}\operatorname{sign}(x-\xi), \varphi\rangle = -\langle\operatorname{sign}(x-\xi), \varphi'(x)\rangle$$

where φ is a smooth test function. Using this formula, we can find the generalized derivative of $\frac{d^2}{dx^2}|x-\xi|$ as:

$$\frac{d}{dx}\operatorname{sign}(x-\xi) = 2\delta(x-\xi),$$

for any test function φ. Then, from (2) it follows

$$\begin{aligned}\frac{d^2}{dx^2}\frac{\sin\lambda\,|x-\xi|}{2\lambda} &= -\frac{\lambda\sin\lambda\,|x-\xi|}{2}(\operatorname{sgn}(x-\xi))^2 + \cos\lambda\,|x-\xi|\,\delta(x-\xi)\\ &= -\frac{\lambda\sin\lambda\,|x-\xi|}{2} + \delta\,(x-\xi)\,.\end{aligned}$$

Here $\cos\lambda\,|x-\xi|\,\delta\,(x-\xi) = \delta(x-\xi)$. Indeed, both $\cos\lambda\,|x-\xi|\,\delta\,(x-\xi)$ and $\delta(x-\xi)$ are zero for all $(x-\xi)$ except $x=\xi$, and infinity at $x=\xi$.

Therefore,

$$\begin{aligned}\frac{d^2}{dx^2}\frac{\sin\lambda|x-\xi|}{2\lambda} + \lambda^2\cdot\frac{\sin\lambda|x-\xi|}{2\lambda} &= -\frac{\lambda\sin\lambda\,|x-\xi|}{2} + \delta(x-\xi) + \frac{\lambda\sin\lambda|x-\xi|}{2}\\ &= \delta(x-\xi),\end{aligned}$$

which completes the proof.

Corollary 1 *A function*

$$\varepsilon_1^*(x-\xi) = \frac{\cos\lambda|x-\xi|}{2\lambda}$$

satisfies the following equation

$$\frac{d^2}{dx^2}\varepsilon_1^*(x-\xi) + \lambda^2\varepsilon_1^*(x-\xi) = 0.$$

Proof. As in the proof of Lemma 1, we can see that

$$\begin{aligned}\frac{d^2}{dx^2}\frac{\cos\lambda\,|x-\xi|}{2\lambda} &= -\frac{\lambda\cos\lambda\,|x-\xi|}{2}(\operatorname{sgn}(x-\xi))^2 - \sin\lambda\,|x-\xi|\,\delta(x-\xi)\\ &= -\frac{\lambda\cos\lambda\,|x-\xi|}{2} - \sin\lambda\,|x-\xi|\,\delta(x-\xi).\end{aligned}$$

Since $|x-\xi|\delta(x-\xi) = |x-\xi|\delta(|x-\xi|) = 0$ for all $|x-\xi|$, it is easily seen that

$$\sin\lambda\,|x-\xi|\,\delta(x-\xi) = \frac{\sin\lambda\,|x-\xi|}{|x-\xi|}\cdot|x-\xi|\delta(x-\xi) = 0,$$

and the corollary follows.

Furthermore, we will refer to the integral that follows as a one-dimensional Helmholtz potential

$$u(x) = \int_0^1 \varepsilon_1(x-\xi,\lambda)f(\xi)d\xi, \tag{3}$$

where f is a given function with compact support.

According to Lemma 1, the one-dimensional Helmholtz potential (3) satisfies the following one-dimensional inhomogeneous Helmholtz equation (1).

3 Boundary conditions for the one-dimensional Helmholtz potential

Now, we will find the potential boundary condition for the one-dimensional Helmholtz equation.

Let

$$u(x) = \int_0^1 \varepsilon_1(x-\xi,\lambda) f(\xi) d\xi = \int_0^1 \frac{\sin\lambda|x-\xi|}{2\lambda} f(\xi) d\xi, \tag{4}$$

where $f(\xi) = \left(\frac{d^2}{d\xi^2} + \lambda^2\right) u(\xi)$.

Replace $f(\xi)$ with $\left(\frac{d^2}{d\xi^2} + \lambda^2\right) u(\xi)$ in (4), we have

$$\begin{aligned} u(x) &= \int_0^1 \frac{\sin\lambda|x-\xi|}{2\lambda} \left(\frac{d^2}{d\xi^2} + \lambda^2\right) u(\xi) d\xi \\ &= \int_0^1 \frac{\sin\lambda|x-\xi|}{2\lambda} \left(\frac{d^2}{d\xi^2} u(\xi)\right) d\xi + \lambda^2 \int_0^1 \frac{\sin\lambda|x-\xi|}{2\lambda} u(\xi) d\xi. \end{aligned} \tag{5}$$

Applying integration by parts twice to the first integral in equation (5), we get

$$\begin{aligned} u(x) &= \frac{\sin\lambda|x-\xi|}{2\lambda} \frac{d}{d\xi} u(\xi)\Big|_0^1 - u(\xi) \frac{d}{d\xi} \frac{\sin\lambda|x-\xi|}{2\lambda}\Big|_0^1 \\ &+ \int_0^1 \frac{d^2}{d\xi^2} \left(\frac{\sin\lambda|x-\xi|}{2\lambda}\right) u(\xi) d\xi + \lambda^2 \int_0^1 \frac{\sin\lambda|x-\xi|}{2\lambda} u(\xi) d\xi. \end{aligned} \tag{6}$$

Since

$$\frac{d^2}{d\xi^2} \frac{\sin\lambda|x-\xi|}{2\lambda} + \lambda^2 \frac{\sin\lambda|x-\xi|}{2\lambda} = \delta(x-\xi),$$

and $\int_0^1 \delta(x-\xi) u(\xi) d\xi = u(x)$, we can rewrite the equality (6) as follows

$$u(x) = \frac{\sin\lambda|x-\xi|}{2\lambda} \frac{d}{d\xi} u(\xi)\Big|_0^1 - u(\xi) \frac{d}{d\xi} \frac{\sin\lambda|x-\xi|}{2\lambda}\Big|_0^1 + u(x). \tag{7}$$

Hence

$$\frac{\sin\lambda|x-\xi|}{2\lambda} \frac{d}{d\xi} u(\xi)\Big|_0^1 - u(\xi) \frac{d}{d\xi} \frac{\sin\lambda|x-\xi|}{2\lambda}\Big|_0^1 = 0. \tag{8}$$

Since

$$\frac{d}{d\xi} \frac{\sin\lambda\,|x-\xi|}{2\lambda} = \frac{\cos\lambda|x-\xi| \cdot \operatorname{sgn}(x-\xi)}{2}, \tag{9}$$

and $\sin\lambda(1-x) = \sin\lambda\cos\lambda x - \sin\lambda x\cos\lambda$, it follows that

$$\begin{aligned}\frac{\sin\lambda\,|x-\xi|}{2\lambda}u'(\xi)\Big|_0^1 &= u'(1)\frac{\sin\lambda|1-x|}{2\lambda} - \frac{\sin\lambda x}{2\lambda}u'(0)\\ &= -\frac{\sin\lambda x}{2\lambda}\left(\cos\lambda u'(1)+u'(0)\right)+\frac{\sin\lambda}{2\lambda}\cos\lambda x u'(1).\end{aligned}\tag{10}$$

In the same manner we can see that

$$\begin{aligned}-u(\xi)\frac{d}{d\xi}\frac{\sin\lambda\,|x-\xi|}{2\lambda}\Big|_0^1 &= -u(\xi)\frac{\cos\lambda|x-\xi|}{2}\operatorname{sgn}(x-\xi)\Big|_0^1\\ &= u(1)\frac{\cos\lambda(1-x)}{2}+\frac{u(0)\cos\lambda x}{2}\\ &= \frac{u(1)}{2}\left[\cos\lambda\cos\lambda x+\sin\lambda\sin\lambda x\right]+\frac{u(0)}{2}\cos\lambda x\\ &= \frac{\cos\lambda x}{2}\cdot\left[u(1)\cos\lambda+u(0)\right]+\frac{u(1)}{2}\sin\lambda\sin\lambda x.\end{aligned}\tag{11}$$

Combining (10) with (11) we can rewrite (8) as

$$\begin{aligned}0 = &-\frac{\sin\lambda x}{2\lambda}\left[u'(0)+\cos\lambda u'(1)-u(1)\cdot\lambda\sin\lambda\right]\\ &-\cos\lambda x\left[-\frac{\sin\lambda}{2\lambda}u'(1)-\frac{u(1)\cos\lambda+u(0)}{2}\right].\end{aligned}\tag{12}$$

As $\sin\lambda x$ and $\cos\lambda x$ are linearly independent in $L_2(0,1)$, from (12) we can conclude that

$$u'(0)+\cos\lambda u'(1)-u(1)\lambda\sin\lambda = 0,\tag{13}$$

and

$$\frac{\sin\lambda}{\lambda}u'(1)+u(1)\cos\lambda+u(0)=0.\tag{14}$$

Thus, it follows that:

Theorem 1 *The one-dimensional Helmholtz potential*

$$u(x)=\int_0^1\varepsilon_1(x-\xi,\lambda)f(\xi)d\xi\tag{15}$$

for any $f\in L_2(0,1)$ *satisfies the conditions* (13) *and* (14).

Example. For $\lambda=0$, the one-dimensional Helmholtz potential coincides with the following one-dimensional Newton potential

$$u(x)=\frac{1}{2}\int_0^1|x-\xi|f(\xi)d\xi,\tag{16}$$

which satisfies the following conditions [11]

$$u'(0)+u'(1)=0,\tag{17}$$

$$u'(1)=u(1)+u(0).\tag{18}$$

4 Equivalent conditions with one-dimensional Sommerfeld radiation condition

For $\lambda \in \mathbb{R}$, by lemma 1 and corollary 2 we may actually assume that

$$\varepsilon_1(x-\xi,\lambda) = \frac{e^{i\lambda|x-\xi|}}{2i\lambda} \tag{19}$$

is a fundamental solution of the one-dimensional Helmholtz equation of the form

$$u''(x) - \lambda^2 u(x) = f(x). \tag{20}$$

A regular solution to the homogeneous equation $u''(x) + \lambda^2 u(x) = 0$ in an interval (a, b) can be rewritten as

$$u(x) = \int_a^b \varepsilon_1(x-\xi,\lambda)\left(\frac{d^2}{d\xi^2}u(\xi) + \lambda^2 u(\xi)\right) d\xi, \tag{21}$$

where $\varepsilon_1(x-\xi,\lambda) = \frac{e^{i\lambda|x-\xi|}}{2i\lambda}$, and $\lambda \in \mathbb{R}$. By (21), using integration by parts we obtain

$$\begin{aligned} u(x) &= \int_a^b \varepsilon_1(x-\xi,\lambda)\left(\frac{d^2}{d\xi^2}u(\xi) + \lambda^2 u(\xi)\right) d\xi \\ &= \varepsilon_1(x-\xi,\lambda)\frac{d}{d\xi}u(\xi)\Big|_a^b - u(\xi)\frac{d}{d\xi}\varepsilon_1(x-\xi,\lambda)\Big|_a^b \\ &+ \int_a^b u(\xi)\left(\frac{d^2}{d\xi^2} + \lambda^2\right)\varepsilon_1(x-\xi,\lambda)d\xi, \end{aligned}$$

such as $\left(\frac{d^2}{d\xi^2} + \lambda^2\right)\varepsilon_1(x-\xi,\lambda) = \delta(x-\xi)$, and $\int\limits_a^b u(\xi)\delta(x-\xi)d\xi = u(x)$ it follows that

$$0 = \varepsilon_1(x-\xi,\lambda)\frac{d}{d\xi}u(\xi)\Big|_a^b - u(\xi)\frac{d}{d\xi}\varepsilon_1(x-\xi,\lambda)\Big|_a^b$$

replacing $\varepsilon_1(x-\xi,\lambda)$ by $\frac{e^{i\lambda|x-\xi|}}{2i\lambda}$, we can see that

$$\begin{aligned} 0 &= \frac{e^{i\lambda|x-\xi|}}{2i\lambda}\frac{d}{d\xi}u(\xi)\Big|_a^b - u(\xi)\frac{d}{d\xi}\frac{e^{i\lambda|x-\xi|}}{2i\lambda}\Big|_a^b \\ &= \frac{e^{i\lambda|x-\xi|}}{2i\lambda}\frac{d}{d\xi}u(\xi)\Big|_a^b - u(\xi)i\lambda\cdot\operatorname{sgn}(x-\xi)\frac{e^{i\lambda|x-\xi|}}{2i\lambda}\Big|_a^b \\ &= \frac{e^{i\lambda|x-b|}}{2i\lambda}u'(b) - \frac{e^{i\lambda|x-a|}}{2i\lambda}u'(a) - i\lambda u(b)\frac{e^{i\lambda|x-b|}}{2i\lambda} - i\lambda u(a)\frac{e^{i\lambda|x-a|}}{2i\lambda} \\ &= \frac{e^{i\lambda|x-b|}}{2i\lambda}\left(u'(b) - i\lambda u(b)\right) - \frac{e^{i\lambda|x-a|}}{2i\lambda}(u'(a) + i\lambda u(a)), \end{aligned} \tag{22}$$

Since λ is real number, the functions $e^{i\lambda|x-b|}$ and $e^{i\lambda|x-a|}$ are bounded and linearly independent for any x on $\mathbb{R}$, even at infinity. Then we may assume that $|x| \gg |b|$, and $|x| \gg |a|$. Therefore, by (22) implies that

$$u'(b) - i\lambda u(b) = 0, \tag{23}$$

$$u'(a) + i\lambda u(a) = 0. \tag{24}$$

On the other hand, in the one-dimensional case, the solution of the Helmholtz equation that satisfies the Sommerfeld radiation condition at infinity [12]:

$$\lim_{x\to\pm\infty} \left(\frac{du}{dx} \mp i\lambda u\right) = 0,$$

is given by the following form

$$u(x) = \int\limits_{-\infty}^{+\infty} \frac{e^{i\lambda|x-\xi|}}{2i\lambda} f(\xi) d\xi, \tag{25}$$

where the function $f(\xi)$ with compact support in $[a, b]$.
Note that we have actually proved that:

Theorem 2 *Let $f(x)$ be a function with compact support in $[a, b]$, then there exists a unique solution to the Helmholtz equation that satisfies* (23)-(24). *Moreover, the Sommerfeld radiation condition and conditions* (23)-(24) *are equivalent.*

5 Conclusion

In conclusion, this paper presented a comprehensive analysis of the one-dimensional Helmholtz equation. The fundamental solution was successfully constructed. The determination of boundary conditions for the one-dimensional Helmholtz potential was a significant contribution, offering a framework for understanding and solving problems within this specific domain. Furthermore, the paper extended the understanding of the equation by establishing the equivalent conditions with Sommerfeld radiation conditions, enhancing the applicability and theoretical foundations of the one-dimensional Helmholtz equation. Overall, these findings significantly advance our understanding of this fundamental equation and pave the way for future research in related areas.

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