1-бөлім **Раздел 1** **Section 1**

Математика **Математика** **Mathematics**

IRSTI 27.39.21 DOI: https://doi.org/10.26577/JMMCS2024-122-02-b1

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Analytical Solution of Initial Value Problem for Ordinary Differential Equation with Singular Perturbation and Piecewise Constant Argument

The study of differential equations with piecewise constant arguments has been treated widely in the literature. This type of equation, in which techniques of differential and difference equations are combined, models, among others, some biological phenomena , the stabilization of hybrid control systems with feedback discrete controller or damped oscillators.The first studies in this field have been given in 1984, after this, some papers related with stability, oscillation properties and existence of periodic outcomes have been treated by several authors.The manuscript is crafted as follows: Section 2 outlines the primary methodologies adopted throughout the inquiry.Section 3 is dedicated to obtaining the exclusive outcome to the issue. We formulate a series of difference equations overseeing the vector $\begin{pmatrix} y(\theta_i) \\ y'(\theta_i) \end{pmatrix}$, $i = \overline{1,p}$ which portray the constituents of the outcome. This generalized approach allows for a broader understanding of how to tackle such differential equations across various scenarios. These equations now form a recognized branch of the field of differential equations, and they are frequently used in biological and economic models. Undoubtedly, their applications will continue to increase in the future.
Key words: initial functions, initial value problem, EPCA.

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Сингулярлы ауытқуы және бөлікті-тұрақты аргументті бар қарапайым дифференциалдық теңдеу үшін бастапқы мән есебінің аналитикалық шешімі

Мақалада жалпыланған түрдегі бөлікті тұрақты аргументті кіші параметрлі жай дифференциалдық теңдеу үшін бастапқы мән есебі қарастырылған. Бөлікті тұрақты аргументті кіші параметрлі біртекті емес дифференциалдық теңдеуге сәйкес біртекті бөлікті тұрақты аргументті сингулярлы ауытқыған дифференциалдық теңдеудің іргелі шешімдер жүйесі құрылды. Шешімнің құрамындағы $\begin{pmatrix} y(\theta_i) \\ y'(\theta_i) \end{pmatrix}$, $i = \overline{1,p}$ векторын анықтайтын айырымдық теңдеулер жүйесі алынды. Осы құрылған айырымдық теңдеулер жүйесінің шешімі анықталды. Редукция тәсілін қолданып, қойылған бөлікті тұрақты аргументті кіші параметрлі бастапқы мән есебінің шешімінің аналитикалық формуласы алынды. Шешімнің аналитикалық формуласы туралы теорема қорытылып шығарылды. Қарастырылған жалпыланған түрдегі бөлікті тұрақты сингулярлы ауытқыған бастапқы мән есебінің дербес жағдайларда сызықты гармоникалық осциллятор болып табылады.
Түйін сөздер: бөлікті-константалы аргумент, гармоникалық осциллятор, бастапқы функциялар, бастапқы мән есебі.

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Аналитическое решение задачи начального значения для обыкновенного дифференциального уравнения с сингулярным возмущением и кусочно-постоянным аргументом

В статье исследуется задача начального значения для обыкновенного дифференциального уравнения с возмущением малого параметра и кусочно-постоянным аргументом в обобщенном виде. В соответствии с этим уравнением мы разрабатываем систему фундаментальных решений для однородного сингулярно возмущенного дифференциального уравнения, которое зависит от кусочно-постоянного аргумента. Выведем систему разностных уравнений, описывающую вектор $\begin{pmatrix} y(\theta_i) \\ y'(\theta_i) \end{pmatrix}$, $i = \overline{1,p}$ компонентов решения. Установлено решение полученной системы разностных уравнений. Используя редукционный подход, мы получили аналитическую формулу для решения задачи начального значения для обыкновенного дифференциального уравнения с кусочно- постоянным аргументом в обобщенном виде, включающую малый параметр. Была выведена и доказана теорема, устанавливающая аналитическую формулу для решения. Конкретный пример задачи начального значения в рамках сингулярно возмущенного обыкновенного дифференциального уравнения, зависящего от кусочно- постоянного аргумента в обобщенной форме с малым параметром, соответствует задаче линейного гармонического осциллятора.

Ключевые слова: гармонический осциллятор, начальные функций, задача начального значения.

1 Introduction

We consider the Cauchy problem for a linear differential equation with a singular perturbed piecewise constant argument:

$$\varepsilon y''(t) + A(t)y'(t) + B(t)y(t) + C(t)y\left(\beta(t)\right) = F(t) \tag{1}$$

$$y(0,\varepsilon) = d_0,\ y'(0,\varepsilon) = d_1, \tag{2}$$

where $\varepsilon > 0$ is a small parameter, d_0, d_1 are given constants.

If $t \in [\theta_i, \theta_{i+1})$, $i = \overline{1,p}$, $0 < \theta_1 < \theta_2 < \ldots < \theta_p < T$ a piecewise constant function is defined as $\beta(t) = \theta_i$.

Let the following conditions hold true:

(C1) The roles $A(t), B(t), C(t)$ and $F(t)$ are continuously differentiable in the span $0 \leq t \leq T$.

(C2) $A(t) > 0$, $0 \leq t \leq T$.

Differential equations with piecewise constant argument (EPCA) were proposed for investigations in $[1, 2]$ by founders of the theory, K. Cook, S. Busenberg, J. Wiener, and S. Shah. They are named as differential EPCA. In the last three decades, many interesting results have been obtained, and applications have been realized in this theory. Existence and uniqueness of solutions, oscillations and stability, integral manifolds and periodic solutions, and many other questions of the theory have been intensively discussed. The founders proposed that the method of investigation of these equations is based on a reduction to discrete systems. That is, only values of solutions at moments, which are integers or multiples of integers, were discussed. Moreover, systems must be linear with respect to the values of solutions, if the argument is not deviated. It reduces the theoretical depth of the investigations as well as the number of real-world problems, which can be modeled by using these equations.Through the application of reduction techniques, we derive an analytical formula for solving the IVP for this ODE with a piecewise constant argument and small parameter.

2 Supplementary Resources

This section presents the fundamental system of solutions and discusses the initial functions, along with demonstrating their crucial properties relevant to our research.

2.1 A fundamental system of solutions

According to equation (1), the homogeneous differential equation has the form

$$\varepsilon y''(t) + A(t)y'(t) + B(t)y(t) = 0 \tag{3}$$

Lemma 1 *Under that stipulations (C1) and (C2) hold, the system of fundamental solutions for the homogeneous equation* (3) *can be explicitly expressed in the form denoted by*

$$\begin{gathered} y_1^{(j)}(t,\varepsilon) = y_{10}^{(j)}(t) + O(\varepsilon),\, j = 0,1 \\ y_2^{(j)}(t,\varepsilon) = \frac{1}{\varepsilon^j} e^{-\frac{1}{\varepsilon}\int_{\theta_i}^t A(x)dx} \left((-A(t))^j \cdot y_{20}(t) + O(\varepsilon)\right),\, j = 0,1 \end{gathered} \tag{4}$$

in the interval $\theta_i \le t < \theta_{i+1},\ \ i = \overline{0,p}$, *where* $y_{10}(t) = \exp\left(-\int_{\theta_i}^t \frac{B(x)}{A(x)}dx\right)$ *and* $y_{20}(t) = \frac{A(\theta_i)}{A(t)},\ \ i = \overline{0,p},\ \ \theta_0 = 0,\ \ \theta_{p+1} = T.$

Proof. First of all, we will consider the $[0,\theta_1)$ interval. The system of fundamental solutions of the homogeneous equation (3) is searched as follows:

$$\begin{gathered} y_1(t,\varepsilon) = y_{10}(t) + \varepsilon y_{11}(t) + \varepsilon^2 y_{12}(t) + \ldots \\ y_2(t,\varepsilon) = e^{-\frac{1}{\varepsilon}\int_0^t A(x)dx} \left(y_{20}(t) + \varepsilon y_{21}(t) + \varepsilon^2 y_{22}(t) + \ldots\right), \end{gathered} \tag{5}$$

where $y_{ik}(t),\ i = 1,2,\ \ k = 0,1,2,\ldots$ are unknown coefficients.

To determine these coefficients, we take the first and second derivatives of the functions

(5):

$$
\begin{aligned}
y_1'(t,\varepsilon) &= y_{10}'(t) + \varepsilon y_{11}'(t) + \varepsilon^2 y_{12}'(t) + \dots \\
y_2'(t,\varepsilon) &= \tfrac{1}{\varepsilon} e^{-\frac{1}{\varepsilon}\int_0^t A(x)dx}\Big(-A(t)y_{20}(t) + \varepsilon\big(y_{20}'(t) \\
&\quad -A(t)y_{21}(t)\big) + O\left(\varepsilon^2\right)\Big), \\
y_1''(t,\varepsilon) &= y_{10}''(t) + \varepsilon y_{11}''(t) + \varepsilon^2 y_{12}''(t) + \dots \\
y_2''(t,\varepsilon) &= \tfrac{1}{\varepsilon^2} e^{-\frac{1}{\varepsilon}\int_0^t A(x)dx}\Big(A^2(t)y_{20}(t) + \varepsilon\big(A^2(t)y_{21}(t) \\
&\quad -2A(t)y_{20}'(t) - A'(t)y_{20}(t)\big) + O\left(\varepsilon^2\right)\Big).
\end{aligned}
\tag{6}
$$

Substituting formulas (5), (6) into equation (3), equating the coefficients in front of the small parameter ε to the same degree, we get problems defining unknown coefficients $y_{10}(t)$, $y_{20}(t)$:

$$
A(t)y_{10}'(t) + B(t)y_{10}(t) = 0,\ y_{10}(0) = 1. \tag{7}
$$

The solution to the IVP (7) is determined as $y_{10}(t) = e^{-\int_0^t \frac{B(x)}{A(x)}dx}$.

$$
\begin{cases}
A(t)y_{20}'(t) + A'(t)y_{20}(t) = 0, \\
y_{20}(0) = 1.
\end{cases}
\tag{8}
$$

The solution to the IVP (8) is determined as $y_{20}(t) = \frac{A(0)}{A(t)}$.

Continuing this process, we will prove in the interval $\theta_i \le t < \theta_{i+1}$, $i = \overline{1,p}$. Lemma 1 is proved.

2.2 The initial functions

Definition 1

$$
\varepsilon K_l''(t,s,\varepsilon) + A(t)K_l'(t,s,\varepsilon) + B(t)K_l(t,s,\varepsilon) = 0,\ l = 1,2, \tag{9}
$$

$$
K_l^{(j)}(s,s,\varepsilon) = \delta_{l-1,j},\ l = 1,2,\ j = 0,1 \tag{10}
$$

the functions $K_l(t,s,\varepsilon), ; l = 1,2,$ defined for $\theta_i \le s \le t \le \theta_{i+1}$, $i = 0,\dots,p$, represent solutions to the problem described by equations (9) *and* (10), *and are referred to as initial functions. And*

$$
K_l^{(q)}(t,s,\varepsilon) = \frac{W_{lt}^{(q)}(t,s,\varepsilon)}{W(s,\varepsilon)},\quad l = 1,2,\ q = 0,1. \tag{11}
$$

Here, $\delta_{l-1,j}$ represents the Kronecker delta symbol. $W_l(t,s,\varepsilon)$ is a second-order determinant in which the $l-$th row of the Wronskian $W(s,\varepsilon)$ is replaced by the set of fundamental solutions represented by (4).

Lemma 2 *Assuming stipulations (C1) and (C2) hold. Then the initial functions* $K_l^{(q)}(t,s,\varepsilon)$, $l=1,2,$ $q=0,1$ *have the following asymptotic behavior when* $\varepsilon\to 0$ *on* $\theta_i\le t<\theta_{i+1}$, $i=\overline{0,p}$

$$K_1^{(q)}(t,s,\varepsilon)=\frac{y_{10}^{(q)}(t)}{y_{10}(s)}+\varepsilon^{1-q}\frac{(-A(t))^q y_{20}(t)y_{10}'(s)}{A(s)y_{10}(s)y_{20}(s)}e^{-\frac{1}{\varepsilon}\int_s^t A(x)dx}$$
$$+O\left(\varepsilon+\varepsilon^{2-q}e^{-\frac{1}{\varepsilon}\int_s^t A(x)dx}\right),$$
$$K_2^{(q)}(t,s,\varepsilon)=\varepsilon\Big[\frac{y_{10}^{(q)}(t)}{A(s)y_{10}(s)}-\frac{(-A(t))^q y_{20}(t)}{\varepsilon^q\cdot A(s)y_{20}(s)}e^{-\frac{1}{\varepsilon}\int_s^t A(x)dx}$$
$$+O\left(\varepsilon+\frac{1}{\varepsilon^{q-1}}e^{-\frac{1}{\varepsilon}\int_s^t A(x)dx}\right)\Big],q=0,1. \tag{12}$$

Proof. First of all,we consider the interval $[0,\theta_1)$. By leveraging the fundamental solutions (4), we derive the asymptotic behavior of $W(s,\varepsilon)$**:**

$$W(s,\varepsilon)=\begin{vmatrix} y_{10}(s)+O(\varepsilon)e^{-\frac{1}{\varepsilon}\int_0^s A(x)dx}\left(y_{20}(s)+O(\varepsilon)\right) \\ y_{10}'(s)+O(\varepsilon)\frac{1}{\varepsilon}e^{-\frac{1}{\varepsilon}\int_0^s A(x)dx}\left(-A(s)y_{20}(s)+O(\varepsilon)\right)\end{vmatrix}$$
$$=\frac{1}{\varepsilon}e^{-\frac{1}{\varepsilon}\int_0^s A(x)dx}\left(-A(s)y_{10}(s)y_{20}(s)+O(\varepsilon)\right)$$
$$-e^{-\frac{1}{\varepsilon}\int_0^s A(x)dx}\left(y_{10}'(s)y_{20}(s)+O(\varepsilon)\right)$$
$$=\frac{1}{\varepsilon}e^{-\frac{1}{\varepsilon}\int_0^s A(x)dx}\left(-A(s)y_{10}(s)y_{20}(s)+O(\varepsilon)\right)\neq 0. \tag{13}$$

We calculate the asymptotics of the determinant $W_1^{(q)}(t,s,\varepsilon)$, $q=0,1$

$$W_1^{(q)}(t,s,\varepsilon)=\begin{vmatrix} y_{10}^{(q)}(t)+O(\varepsilon)\frac{1}{\varepsilon^q}e^{-\frac{1}{\varepsilon}\int_0^t A(x)dx}\left((-A(t))^q y_{20}(t)+O(\varepsilon)\right) \\ y_{10}'(s)+O(\varepsilon)\frac{1}{\varepsilon}e^{-\frac{1}{\varepsilon}\int_0^s A(x)dx}\left(-A(s)y_{20}(s)+O(\varepsilon)\right)\end{vmatrix}$$
$$=\frac{1}{\varepsilon}e^{-\frac{1}{\varepsilon}\int_0^s A(x)dx}\left(-A(s)y_{10}^{(q)}(t)y_{20}(s)+O(\varepsilon)\right)$$
$$-\frac{1}{\varepsilon^q}e^{-\frac{1}{\varepsilon}\int_0^t A(x)dx}\left((-A(t))^q y_{10}'(s)y_{20}(t)+O(\varepsilon)\right). \tag{14}$$

We calculate the asymptotics of the determinant $W_2^{(q)}(t,s,\varepsilon)$, $q=0,1$

$$W_2^{(q)}(t,s,\varepsilon)=\begin{vmatrix} y_{10}(s)+O(\varepsilon)e^{-\frac{1}{\varepsilon}\int_0^s A(x)dx}\left(y_{20}(s)+O(\varepsilon)\right) \\ y_{10}^{(q)}(t)+O(\varepsilon)\frac{1}{\varepsilon^q}e^{-\frac{1}{\varepsilon}\int_0^t A(x)dx}\left((-A(t))^q y_{20}(t)+O(\varepsilon)\right)\end{vmatrix}$$
$$=\frac{1}{\varepsilon^q}e^{-\frac{1}{\varepsilon}\int_0^t A(x)dx}\left((-A(t))^q y_{10}(s)y_{20}(t)+O(\varepsilon)\right)$$
$$-e^{-\frac{1}{\varepsilon}\int_0^s A(x)dx}\left(y_{10}^{(q)}(t)y_{20}(s)+O(\varepsilon)\right). \tag{15}$$

Substituting the formulas (13)-(15) into (11) we obtain (12).

Similarly, we also prove for the interval $\theta_i \le t < \theta_{i+1}, \quad i = \overline{1,p}$. By (4) we show the asymptotic behavior of $W(s,\varepsilon)$:

$$W(s,\varepsilon) = \begin{vmatrix} y_{10}(s) + O(\varepsilon) e^{-\frac{1}{\varepsilon}\int_{\theta_i}^{s} A(x)dx} \left(y_{20}(s) + O(\varepsilon)\right) \\ y'_{10}(s) + O(\varepsilon)\frac{1}{\varepsilon} e^{-\frac{1}{\varepsilon}\int_{\theta_i}^{s} A(x)dx} \left(-A(s) y_{20}(s) + O(\varepsilon)\right) \end{vmatrix}$$

$$= \frac{1}{\varepsilon} e^{-\frac{1}{\varepsilon}\int_{\theta_i}^{s} A(x)dx} \left(-A(s) y_{10}(s) y_{20}(s) + O(\varepsilon)\right) \tag{16}$$

$$- e^{-\frac{1}{\varepsilon}\int_{\theta_i}^{s} A(x)dx} \left(y'_{10}(s) y_{20}(s) + O(\varepsilon)\right)$$

$$= \frac{1}{\varepsilon} e^{-\frac{1}{\varepsilon}\int_{\theta_i}^{s} A(x)dx} \left(-A(s) y_{10}(s) y_{20}(s) + O(\varepsilon)\right) \neq 0.$$

We calculate the asymptotics of the determinant $W_1^{(q)}(t,s,\varepsilon)$,$q = 0,1$:

$$W_1^{(q)}(t,s,\varepsilon) = \begin{vmatrix} y_{10}^{(q)}(t) + O(\varepsilon)\frac{1}{\varepsilon^q} e^{-\frac{1}{\varepsilon}\int_{\theta_i}^{t} A(x)dx} \left((-A(t))^q y_{20}(t) + O(\varepsilon)\right) \\ y'_{10}(s) + O(\varepsilon)\frac{1}{\varepsilon} e^{-\frac{1}{\varepsilon}\int_{\theta_i}^{s} A(x)dx} \left(-A(s) y_{20}(s) + O(\varepsilon)\right) \end{vmatrix}$$

$$= \frac{1}{\varepsilon} e^{-\frac{1}{\varepsilon}\int_{\theta_i}^{s} A(x)dx} \left(-A(s) y_{10}^{(q)}(t) y_{20}(s) + O(\varepsilon)\right) \tag{17}$$

$$- \frac{1}{\varepsilon^q} e^{-\frac{1}{\varepsilon}\int_{\theta_i}^{t} A(x)dx} \left((-A(t))^q y'_{10}(s) y_{20}(t) + O(\varepsilon)\right).$$

We calculate the asymptotics of the determinant $W_2^{(q)}(t,s,\varepsilon)$,$q = 0,1$:

$$W_2^{(q)}(t,s,\varepsilon) = \begin{vmatrix} y_{10}(s) + O(\varepsilon) e^{-\frac{1}{\varepsilon}\int_{\theta_i}^{s} A(x)dx} \left(y_{20}(s) + O(\varepsilon)\right) \\ y_{10}^{(q)}(t) + O(\varepsilon)\frac{1}{\varepsilon^q} e^{-\frac{1}{\varepsilon}\int_{\theta_i}^{t} A(x)dx} \left((-A(t))^q y_{20}(t) + O(\varepsilon)\right) \end{vmatrix}$$

$$= \frac{1}{\varepsilon^q} e^{-\frac{1}{\varepsilon}\int_{\theta_i}^{t} A(x)dx} \left((-A(t))^q y_{10}(s) y_{20}(t) + O(\varepsilon)\right) \tag{18}$$

$$- e^{-\frac{1}{\varepsilon}\int_{\theta_i}^{s} A(x)dx} \left(y_{10}^{(q)}(t) y_{20}(s) + O(\varepsilon)\right).$$

Putting the formulas(16)-(18) into (11) we obtain (12).Lemma 2 is proven.

3 Main results and methods

3.1 Derivation of analytical solutions

Problem (1)–(2) has the following form for $t \in [0, \theta_1)$:

$$\begin{cases} \varepsilon y''(t) + A(t)y'(t) + B(t)y(t) = F(t) - C(t)d_0, \\ y(0,\varepsilon) = d_0, \\ y'(0,\varepsilon) = d_1 \end{cases} \tag{19}$$

The solution to problem (19) is provided by the following analytical expression:

$$\begin{aligned} y(t,\varepsilon) &= d_0K_1(t,0,\varepsilon) + d_1K_2(t,0,\varepsilon) + \tfrac{1}{\varepsilon}\textstyle\int_0^t K_2(t,s,\varepsilon)\left(F(s) - C(s)d_0\right)ds, \\ y'(t,\varepsilon) &= d_0K_1'(t,0,\varepsilon) + d_1K_2'(t,0,\varepsilon) + \tfrac{1}{\varepsilon}\textstyle\int_0^t K_2'(t,s,\varepsilon)\left(F(s) - C(s)d_0\right)ds, \end{aligned} \tag{20}$$

where $K_l^{(q)}(t,s,\varepsilon), \quad l = 1,2, \quad q = 0,1$ are initial functions.

The Cauchy problem (1)-(2) has the following form for $t \in [\theta_i, \theta_{i+1}), \quad i = \overline{1,p}$:

$$\varepsilon y''(t) + A(t)y'(t) + B(t)y(t) = F(t) - C(t)y(\theta_i), \tag{21}$$

$$\begin{aligned} y(t,\varepsilon)|_{t=\theta_i} &= y(\theta_i), \\ y'(t,\varepsilon)|_{t=\theta_i} &= y'(\theta_i). \end{aligned} \tag{22}$$

Theorem 1 *Assuming stipulations (C1) and (C2) are gratified, the Cauchy problem defined by equations* (21) *and* (22) *possesses a unique solution over the interval* $t \in [\theta_i, \theta_{i+1}), \quad i = \overline{1,p}$ *, which is able to be formulated in the shape:*

$$\hat{y}(t,\varepsilon) = Q(t,\theta_i,\varepsilon)\hat{y}(\theta_i) + \hat{U}(t,\theta_i,\varepsilon), \ t \in [\theta_i, \ \theta_{i+1}), \ i = \overline{0,p}, \tag{23}$$

where

$$\hat{y}(t,\varepsilon) = \begin{pmatrix} y(t,\varepsilon) \\ y'(t,\varepsilon) \end{pmatrix},$$

$$Q(t,\theta_i,\varepsilon) = \begin{pmatrix} Q_1(t,\theta_i,\varepsilon) & Q_2(t,\theta_i,\varepsilon) \\ Q_1'(t,\theta_i,\varepsilon) & Q_2'(t,\theta_i,\varepsilon) \end{pmatrix}, \tag{24}$$

$$\hat{U}(t,\theta_i,\varepsilon) = \begin{pmatrix} U(t,\theta_i,\varepsilon) \\ U'(t,\theta_i,\varepsilon) \end{pmatrix},$$

where the roles $Q_1^{(q)}(t,\theta_i,\varepsilon), \ \ Q_2^{(q)}(t,\theta_i,\varepsilon), \ \ U^{(q)}(t,\theta_i,\varepsilon), \ q = 0,1$ *and* $\hat{y}(\theta_i)$ *are defined by*

$$Q_1^{(q)}(t,\theta_i,\varepsilon)=K_1^{(q)}(t,\theta_i,\varepsilon)-\tfrac{1}{\varepsilon}\int_{\theta_i}^{t}K_2^{(q)}(t,s,\varepsilon)C(s)ds,\quad q=0,1,$$

$$Q_2^{(q)}(t,\theta_i,\varepsilon)=K_2^{(q)}(t,\theta_i,\varepsilon),\quad q=0,1, \tag{25}$$

$$U^{(q)}(t,\theta_i,\varepsilon)=\tfrac{1}{\varepsilon}\int_{\theta_i}^{t}K_2^{(q)}(t,s,\varepsilon)F(s)ds,\quad q=0,1,$$

$$\begin{aligned}\hat{y}(\theta_i)=&\prod_{j=0}^{i-1}Q\left(\theta_{j+1},\theta_j,\varepsilon\right)\hat{y}(0,\varepsilon)+\sum_{l=1}^{i-1}\prod_{j=l}^{i-1}Q\left(\theta_{j+1},\theta_j,\varepsilon\right)\hat{U}\left(\theta_l,\theta_{l-1},\varepsilon\right)\\&+\hat{U}\left(\theta_i,\theta_{i-1},\varepsilon\right).\end{aligned} \tag{26}$$

Proof. To determine outcome of the issue (1)–(2) for $t\in[\theta_i,\theta_{i+1})$, $i=\overline{1,p}$, we change the variables as $s=t-\theta_i$, $t=\theta_i \quad\Rightarrow\quad s=0$ and as an result we obtain the problem

$$\left[\begin{array}{l}\varepsilon\frac{d^2y}{ds^2}+A(s)\frac{dy}{ds}+B(s)y(s)=F(s)-C(s)y(\theta_i),\\ \qquad y(s,\varepsilon)\big|_{s=0}=y(\theta_i),\quad \frac{dy}{ds}\big|_{s=0}=y'\ (\theta_i).\end{array}\right. \tag{27}$$

As the problem type in (27) closely resembles the IVP described in (19), the solution to IVP (27) for all $s\in[0,\theta_1)$ is structured as follows:

$$\begin{aligned}y(s,\varepsilon)&=y(\theta_i)K_1(s,\theta_i,\varepsilon)+y'(\theta_i)K_2(s,\theta_i,\varepsilon)\\&\quad+\frac{1}{\varepsilon}\int_0^s K_2(s,p,\varepsilon)\left(F(p)-C(p)y(\theta_i)\right)dp,\\y'(s,\varepsilon)&=y(\theta_i)K_1'(s,\theta_i,\varepsilon)+y'(\theta_i)K_2'(s,\theta_i,\varepsilon)\\&\quad+\frac{1}{\varepsilon}\int_0^s K_2'(s,p,\varepsilon)\left(F(p)-C(p)y(\theta_i)\right)dp.\end{aligned} \tag{28}$$

By changing variable from s to t in (28) we derive outcome to the issue (21)–(22) for $t\in[\theta_i,\theta_{i+1})$, $i=\overline{1,p}$ in the following form

$$\begin{aligned}y(t,\varepsilon)=&\left(K_1(t,\theta_i,\varepsilon)-\frac{1}{\varepsilon}\int_{\theta_i}^{t}K_2(t,s,\varepsilon)C(s)ds\right)y(\theta_i)\\&+K_2(t,\theta_i,\varepsilon)y'(\theta_i)+\frac{1}{\varepsilon}\int_{\theta_i}^{t}K_2(t,s,\varepsilon)F(s)ds,\end{aligned} \tag{29}$$

$$\begin{aligned}y'(t,\varepsilon)=&\left(K_1'(t,\theta_i,\varepsilon)-\frac{1}{\varepsilon}\int_{\theta_i}^{t}K_2'(t,s,\varepsilon)C(s)ds\right)y(\theta_i)\\&+K_2'(t,\theta_i,\varepsilon)y'(\theta_i)+\frac{1}{\varepsilon}\int_{\theta_i}^{t}K_2'(t,s,\varepsilon)F(s)ds.\end{aligned} \tag{30}$$

In light of (24),(25) and using (29),(30) we obtain (23).

To find the unknown vector $\begin{pmatrix} y(\theta_i) \\ y'(\theta_i) \end{pmatrix}$, $i=\overline{1,p}$ we put $t=\theta_{i+1}$ into (29) and (30), then we obtain the following difference system of equations for $\begin{pmatrix} y(\theta_i) \\ y'(\theta_i) \end{pmatrix}$, $i=\overline{1,p}$:

$$\begin{aligned} y(\theta_{i+1},\varepsilon) &= \left(K_1(\theta_{i+1},\theta_i,\varepsilon) - \frac{1}{\varepsilon}\int_{\theta_i}^{\theta_{i+1}} K_2(\theta_{i+1},s,\varepsilon)C(s)ds\right) y(\theta_i) \\ &\quad + K_2(\theta_{i+1},\theta_i,\varepsilon)y'(\theta_i) + \frac{1}{\varepsilon}\int_{\theta_i}^{\theta_{i+1}} K_2(\theta_{i+1},s,\varepsilon)F(s)ds, \\ y'(\theta_{i+1},\varepsilon) &= \left(K_1'(\theta_{i+1},\theta_i,\varepsilon) - \frac{1}{\varepsilon}\int_{\theta_i}^{\theta_{i+1}} K_2'(\theta_{i+1},s,\varepsilon)C(s)ds\right) y(\theta_i) \\ &\quad + K_2'(\theta_{i+1},\theta_i,\varepsilon)y'(\theta_i) + \frac{1}{\varepsilon}\int_{\theta_i}^{\theta_{i+1}} K_2'(\theta_{i+1},s,\varepsilon)F(s)ds. \end{aligned} \tag{31}$$

By using the formulas (24),(25) we reduce the system (31) into the following vector form:

$$\hat{y}(\theta_{i+1},\varepsilon) = N\left(\theta_{i+1},\theta_i,\varepsilon\right)\hat{y}(\theta_i) + \hat{P}\left(\theta_{i+1},\theta_i,\varepsilon\right). \tag{32}$$

In view of (32) we obtain

$$\hat{y}(\theta_i,\varepsilon) = N\left(\theta_i,\theta_{i-1},\varepsilon\right)\hat{y}(\theta_{i-1}) + \hat{P}\left(\theta_i,\theta_{i-1},\varepsilon\right). \tag{33}$$

Substituting (33) into (32), we get

$$\begin{aligned} \hat{y}(\theta_{i+1},\varepsilon) &= N\left(\theta_{i+1},\theta_i,\varepsilon\right)\cdot N\left(\theta_i,\theta_{i-1},\varepsilon\right)\hat{y}(\theta_{i-1}) \\ &\quad + N\left(\theta_{i+1},\theta_i,\varepsilon\right)\hat{P}\left(\theta_i,\theta_{i-1},\varepsilon\right) + \hat{P}\left(\theta_{i+1},\theta_i,\varepsilon\right). \end{aligned} \tag{34}$$

Using (32) we have

$$\hat{y}(\theta_{i-1},\varepsilon) = N\left(\theta_{i-1},\theta_{i-2},\varepsilon\right)\hat{y}(\theta_{i-2}) + \hat{P}\left(\theta_{i-1},\theta_{i-2},\varepsilon\right). \tag{35}$$

Putting (35) into (34)we obtain

$$\begin{aligned} \hat{y}(\theta_{i+1},\varepsilon) &= N\left(\theta_{i+1},\theta_i,\varepsilon\right)N\left(\theta_i,\theta_{i-1},\varepsilon\right)N\left(\theta_{i-1},\theta_{i-2},\varepsilon\right)\hat{y}(\theta_{i-2}) \\ &\quad + N\left(\theta_{i+1},\theta_i,\varepsilon\right)N\left(\theta_i,\theta_{i-1},\varepsilon\right)\hat{P}\left(\theta_{i-1},\theta_{i-2},\varepsilon\right) \\ &\quad + N\left(\theta_{i+1},\theta_i,\varepsilon\right)\hat{P}\left(\theta_i,\theta_{i-1},\varepsilon\right) + \hat{P}\left(\theta_{i+1},\theta_i,\varepsilon\right). \end{aligned} \tag{36}$$

Repeating this process till $i+1$ we get

$$\begin{aligned} \hat{y}(\theta_{i+1},\varepsilon) &= \prod_{j=0}^{i} N\left(\theta_{j+1},\theta_j,\varepsilon\right)\hat{y}(0,\varepsilon) \\ &\quad + \sum_{l=1}^{i}\prod_{j=l}^{i} N\left(\theta_{j+1},\theta_j,\varepsilon\right)\hat{P}\left(\theta_l,\theta_{l-1},\varepsilon\right) + \hat{P}\left(\theta_{i+1},\theta_i,\varepsilon\right). \end{aligned} \tag{37}$$

By (37) we obtain (26). Theorem 1 is proven.

Theorem 2 *Assuming conditions (C1) and (C2) hold true, the Cauchy problem defined by equations* (1) *and* (2) *possesses a unique solution over the interval* $[0, T]$*, which can be expressed as follows:*

$$\hat{y}(t,\varepsilon) = \begin{cases} Q(t,0,\varepsilon)\hat{y}(0) + \hat{U}(t,0,\varepsilon), & t \in [0,\theta_1)\,, \\ Q(t,\theta_1,\varepsilon)\hat{y}(\theta_1) + \hat{U}(t,\theta_1,\varepsilon), & t \in [\theta_1,\theta_2)\,, \\ \ldots \\ Q(t,\theta_p,\varepsilon)\hat{y}(\theta_p) + \hat{U}(t,\theta_p,\varepsilon), & t \in [\theta_p,T]\,. \end{cases} \tag{38}$$

Here, $\hat{y}(t,\varepsilon)$, $\hat{U}(t,\theta_i,\varepsilon)$, $i = 0,\ldots,p$ *are vector functions, and* $Q(t,\theta_i,\varepsilon),;i = 0,\ldots,p$ *is a* 2×2 *matrix with elements defined by equations* (24) *and* (25), $\hat{y}(0) = \begin{pmatrix} d_0 \\ d_1 \end{pmatrix}$ *and* $\hat{y}(\theta_i)$, $i = \overline{1,p}$ *is vector functions with the elements which defined by* (26).

4 Conclusion

In this study, we explored the IVP associated with a singularly perturbed ODE that depends on a piecewise constant argument in a generalized form with a small parameter. Employing a reduction approach, we derived an analytical solution formula for this IVP.

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