IRSTI 27.33.19 DOI: https://doi.org/10.26577/JMMCS2023v119i3a2

E.A. Bakirova[1,2*], N.B. Iskakova[1,3], Zh.M. Kadirbayeva[1,4]
[1]Institute of Mathematics and Mathematical Modeling, Kazakhstan, Almaty
[2]Kazakh National Women's Teacher Training University, Kazakhstan, Almaty
[3]Abai Kazakh National Pedagogical University, Kazakhstan, Almaty
[4]International Information Technology University, Kazakhstan, Almaty
*e-mail: e.bakirova@math.kz

NUMERICAL IMPLEMENTATION FOR SOLVING THE BOUNDARY VALUE PROBLEM FOR IMPULSIVE INTEGRO-DIFFERENTIAL EQUATIONS WITH PARAMETER

In this paper, a linear boundary value problem under impulse effects for the system of Fredholm integro-differential equations with a parameter is investigated. The purpose of this research is to provide a method for solving the studied problem numerically. The ideas of the Dzhumabaev parameterisation method, classical numerical methods of solving Cauchy problems and numerical integration techniques were used as a basis for achieving the goal. When applying the method of parameterisation by points of impulse effects, the interval on which the boundary value problem is considered is divided, additional parameters and new unknown functions are introduced. As a consequence, a problem with parameters equivalent to the original problem is obtained. According to the data of the matrices of the integral term of the equation, boundary conditions and impulse conditions, the SLAE with respect to the introduced parameters is compiled. And the unknown functions are found as solutions of the initial-special problem for the system of integro-differential equations. A numerical algorithm for finding a solution to the boundary value problem for impulse integro-differential equations with a parameter is constructed. Numerical methods for solving Cauchy problems for ODE and calculating definite integrals are used for numerical implementation of the constructed algorithm. Numerical calculations are verified on test problem.
Key words: boundary value problem, parametrization method, integro-differential equation with parameter, impulsive effect, numerical algorithm.

Э.А. Бакирова[1,2*], Н.Б. Искакова[1,3], Ж.М. Қадырбаева[1,4]
[1]Математика және математикалық моделдеу институты, Қазақстан, Алматы қ.
[2]Қазақ ұлттық қыздар педагогикалық университеті , Қазақстан, Алматы қ.
[3]Абай атындағы қазақ ұлттық педагогикалық университеті , Қазақстан, Алматы қ.
[4]Халықаралық ақпараттық технологиялар университеті, Қазақстан, Алматы қ.
*e-mail: e.bakirova@math.kz

Параметрі бар импульстік интегралдық - дифференциалдық теңдеулер үшін шеттік есепті шешудің сандық жүзеге асырылуы

Бұл жұмыста импульс әсерлі параметрі бар Фредгольм интегралдық-дифференциалдық теңдеулер жүйесі үшін сызықтық шеттік есеп зерттеледі. Зерттелетін есептің сандық шешімін табудың алгоритмін құру жұмыстың мақсаты болып табылады. Осы мақсатқа жетуге негіз болып Джумабаевтың параметрлеу әдісінің идеясы, Коши есептерін шешудің классикалық сандық әдістері және сандық интегралдау әдістері жатады. Параметрлеу әдісін қолдану кезінде шеттік есеп қарастырылатын аралықты бөлу импульс нүктелері арқылы жүзеге асырылады, қосымша параметрлер мен жаңа белгісіз функциялар енгізіледі. Нәтижесінде бастапқы есепке пара пар параметрлері бар есеп алынады. Теңдеудің интегралдық мүшесінің матрицалары, шеттік шарттары және импульстік әсер ету шарттары арқылы енгізілген параметрлерге қатысты сызықтық алгебралық теңдеулер жүйесі құрылады.

Ал белгісіз функциялар интегралдық-дифференциалдық теңдеулер жүйесі үшін бастапқы арнайы есептің шешімдері ретінде табылады. Импульс әсерлі параметрі бар Фредгольм интегралдық-дифференциалдық теңдеулер жүйесі үшін сызықтық шеттік есептің сандық шешімін табу алгоритмі ұсынылады. Құрылған алгоритмді сандық жүзеге асыру үшін жай дифференциалдық теңдеулер үшін Коши есептерін шешу және анықталған интегралдарды есептеу үшін сандық әдістер қолданылады. Сандық есептеулер тесттік есептермен тексеріледі.
Түйін сөздер: шеттік есеп, параметрлеу әдісі, параметрі бар интегралдық - дифференциалдық теңдеу, импульстік әсер, сандық алгоритм.

Э.А. Бакирова[1,2*], Н.Б. Искакова[1,3], Ж.М. Кадирбаева[1,4]
[1]Институт математики и математического моделирования, Казахстан, г. Алматы
[2]Казахский национальный женский педагогический университет, Казахстан, г. Алматы
[3]Казахский национальный педагогический университет имени Абая, Казахстан, г. Алматы
[4]Международный университет информационных технологий, Казахстан, г. Алматы
*e-mail: e.bakirova@math.kz

Численная реализация решения краевой задачи для импульсных интегро-дифференциальных уравнений с параметром

В работе при импульсных воздействиях для системы интегро-дифференциальных уравнений Фредгольма с параметром исследуется линейная краевая задача. Целью работы является построение алгоритма нахождения численного решения исследуемой задачи. В основу достижения цели легли идеи метода параметризации Джумабаева, классические численные методы решения задач Коши и приемы численного интегрирования. При применении метода параметризации точками импульсного воздействия производится разбиение интервала, на котором рассматривается краевая задача, вводятся дополнительные параметры и новые неизвестные функции. В следствии этого получается задача с параметрами, эквивалентная к исходной задаче. По данным матриц интегрального члена уравнения, краевых условий и условий импульсов составляется СЛАУ относительно введенных параметров. А неизвестные функции находятся как решения начально-специальной задачи для системы интегро-дифференциальных уравнений. Строится численный алгоритм поиска решения краевой задачи для импульсных интегро-дифференциальных уравнений с параметром. Для численной реализации построенного алгоритма привлекаются численные методы решения задач Коши для обыкновенных дифференциальных уравнений, и вычисления определенных интегралов. Численные расчеты проверяются на тестовой задаче.
Ключевые слова: краевая задача, метод параметризации, интегро-дифференциальное уравнение с параметром, импульсное воздействие, численный алгоритм.

1 Introduction and preliminaries

Active development of the theory of boundary value problems for impulsive integro-differential equations is associated with their use in various fields of applied mathematics, biomedicine, biophysics, chemistry and others. Methods of qualitative theory of differential equations, optimization theory and calculus of variations, etc. are used to investigate and solve such problems. Note that the construction of numerical algorithms for finding solutions of boundary value problems for impulsive integro-differential equations remains topical.

Impulsive differential equations and impulsive integro-differential equations are considered in [1]– [5] and references therein. In [6]– [10], problems for impulsive integro-differential equations have been studied by various methods.

On the interval $(0, T)$ we consider the boundary value problem for the Fredholm integro-differential equations with parameter

$$\frac{dz}{dt} = \mathcal{A}(t)z + \int_0^T \phi(t)\chi(s)z(s)ds + \mathcal{A}_0(t)\mu + f(t), \quad z \in R^n, \quad \mu \in R^l, \quad t \in (0,T), \quad (1)$$

$$B_0\mu + M_0z(0) + L_0z(T) = d_0, \quad d_0 \in R^{n+l}, \quad (2)$$

under impulse actions

$$M_iz(\theta_i - 0) - L_iz(\theta_i + 0) = d_i, \quad d_i \in R^n, \quad i = \overline{1,m}, \quad (3)$$

where the $(n\times n)-$matrices $\mathcal{A}(t)$, $\phi(t)$, $\chi(t)$ and the $(n\times l)-$matrix $\mathcal{A}_0(t)$ and the n-vector-function $f(t)$ are piecewise continuous on $[0,T]$; $t = \theta_j$, $(j = \overline{1,m})$ are impulse points; B_0, M_0, L_0, M_i and L_i, $(i = \overline{1,m})$ are constant matrices of dimensions $((n+l)\times l)$, $((n+l)\times n)$, $((n+l)\times n)$, $(n\times n)$ and $(n\times n)$ the respectively; $(n+l)-$ vector d_0 and $n-$vectors d_i, $(i = \overline{1,m})$ are constant, $0 = \theta_0 < \theta_1 < \ldots < \theta_{m+1} = T$.

A solution to the problem (1)-(3) is a pair $(z^*(t), \mu^*)$, where $\mu^* \in R^l$ and $z^*(t)$ is piecewise continuous differentiable on $(0,T)$ vector function, satisfying Equation (1), except for the points $t = \theta_j$, $j = \overline{1,m}$, boundary condition (2) for $\mu = \mu^*$ and impulsive input conditions (3).

Let's introduce notations:

(i) $\Delta_m(\theta)$ is a partition of the interval $[0,T) = \bigcup\limits_{r=1}^{m+1}[\theta_{r-1},\theta_r)$;

(ii) $PC([0,T],\theta_j,R^n)$ is space of piecewise continuous functions and $z(T) = \lim\limits_{t\to T-0} z(t)$;

(iii) $C([0,T],\Delta_m(\theta),R^{n(m+1)})$ is space of functions systems $z[t] = (z_1(t), z_2(t), ..., z_{m+1}(t))$ that are continuous on $[\theta_{r-1},\theta_r)$, $r = \overline{1,m+1}$, and have finite limits $\lim\limits_{t\to\theta_r-0} z_r(t)$ for all $r = \overline{1,m+1}$.

In the articles [11], [12] the boundary value problem with a parameter for Fredholm integro-differential equations (1), (2) was investigated. The criteria of unique solvability of the problem (1), (2) was established and a numerical algorithm for solving this problem was proposed. Necessary and sufficient conditions for the solvability of the problem (1)-(3) at $\mu = 0$ was obtained in [10].

2 Materials and methods

According to the idea of Dzhumabaev's parameterization method [13]- [15] in the problem (1)-(3) we partition $\Delta_m(\theta)$ by impulse points.

Denote by $z_r(t)$ a restriction of function $z(t)$ on r-th interval $[\theta_{r-1},\theta_r)$, i.e. $z_r(t) = z(t)$ for $t \in [\theta_{r-1},\theta_r)$, $r = \overline{1,m+1}$.

Then the function system $z[t] = (z_1(t), z_2(t), \ldots, z_{m+1}(t)) \in C([0,T],\Delta_m(\theta),R^{n(m+1)})$, and its elements $z_r(t)$, $r = \overline{1,m+1}$, satisfy the following problem

$$\frac{dz_r}{dt} = \mathcal{A}(t)z_r + \sum_{j=1}^{m+1}\int_{\theta_{j-1}}^{\theta_j} \phi(t)\chi(s)z_j(s)ds + \mathcal{A}_0(t)\mu + f(t), \quad t \in [\theta_{r-1},\theta_r), \ r = \overline{1,m+1}, \quad (4)$$

$$B_0\mu + M_0 z_1(0) + L_0 \lim_{t\to T-0} z_{m+1}(t) = d_0, \tag{5}$$

$$M_i \lim_{t\to\theta_i-0} z_i(t) - L_i z_i(\theta_i) = d_i, \quad i = \overline{1,m}. \tag{6}$$

A solution to the problem (4) – (6) is a pair $(z^*[t],\mu^*)$, where $\mu^* \in R^l$ and $z^*[t] = (z_1^*(t), z_2^*(t), \dots, z_{m+1}^*(t)) \in C([0,T], \Delta_m(\theta), R^{n(m+1)})$, whose components $z_r^*(t)$ are continuous differentiable on (θ_{r-1},θ_r), $r = \overline{1,m+1}$ and satisfy Equation (4), boundary condition (5) for $\mu = \mu^*$ and impulsive input conditions (6).

By introducing the parameters $\lambda_r = z_r(\theta_{r-1})$ for all $r = \overline{1,m+1}$, and $\lambda_{m+2} = \lim\limits_{t\to T-0} z_{m+1}(t)$ and substituting $v_r(t) = z_r(t) - \lambda_r$ at each r-interval $[\theta_{r-1},\theta_r)$ we obtain the following problem with parameters

$$\frac{dv_r}{dt} = \mathcal{A}(t)(v_r + \lambda_r) + \sum_{j=1}^{m+1} \int\limits_{\theta_{j-1}}^{\theta_j} \phi(t)\chi(s)(v_j(s)+\lambda_j)ds + \mathcal{A}_0(t)\mu + f(t), \quad t \in [\theta_{r-1},\theta_r), \tag{7}$$

$$v_r(\theta_{r-1}) = 0, \qquad r = \overline{1,m+1}, \tag{8}$$

$$B_0\mu + M_0\lambda_1 + L_0\lambda_{m+2} = d_0, \tag{9}$$

$$M_i(\lambda_i + \lim_{t\to\theta_i-0} v_i(t)) - L_i\lambda_{i+1} = d_i, \quad i = \overline{1,m}, \tag{10}$$

$$\lambda_{m+1} + \lim_{t\to T-0} v_{m+1}(t) = \lambda_{m+2}. \tag{11}$$

A solution to problem (7)–(11) is a pair $(\Lambda^*, v^*[t])$, with elements $\Lambda^* = (\lambda_1^*, \lambda_2^*, \dots, \lambda_{m+2}^*, \mu^*)$, $v^*[t] = \big(v_1^*(t), v_2^*(t), \dots, v_{m+1}^*(t)\big)$, where $v_r^*(t)$ are continuously differentiable on $[\theta_{r-1},\theta_r)$, $r = \overline{1,m+1}$, and satisfying the system (7), conditions (8)- (11) at the $\lambda_r = \lambda_r^*$, $r = \overline{1,m+1}$ and $\mu = \mu^*$.

The problem (1)-(3) and problem (7)-(11) are equivalent.

Let consider $\mathbb{X}(t)$ a fundamental matrix of the differential equation $\dfrac{dy}{dt} = \mathcal{A}(t)y(t)$ on $[0,T]$.

For fixed μ, λ_r, $r = \overline{1,m+1}$, the solution to the initial special problem (7), (8) can be represented in the equivalent integral form

$$v_r(t) = \mathbb{X}(t) \int\limits_{\theta_{r-1}}^{t} \mathbb{X}^{-1}(\tau)\Big[\mathcal{A}(\tau)\lambda_r + \sum_{j=1}^{m+1} \int\limits_{\theta_{j-1}}^{\theta_j} \phi(\tau)\chi(s)(v_j(s)+\lambda_j)ds + \mathcal{A}_0(\tau)\mu + f(\tau)\Big]d\tau, \quad t \in [\theta_{r-1},\theta_r), \quad r = \overline{1,m+1}. \tag{12}$$

Let denote $\beta = \sum\limits_{j=1}^{m+1} \int\limits_{\theta_{j-1}}^{\theta_j} \chi(s)v_j(s)ds$ and re-write the system (12) in the next form

$$v_r(t) = \mathbb{X}(t) \int\limits_{\theta_{r-1}}^{t} \mathbb{X}^{-1}(\tau)\Big[\mathcal{A}(\tau)\lambda_r + \phi(\tau)\beta + \sum_{j=1}^{m+1} \int\limits_{\theta_{j-1}}^{\theta_j} \phi(\tau)\chi(s)ds \cdot \lambda_j + \mathcal{A}_0(\tau)\mu + f(\tau)\Big]d\tau, \quad t \in [\theta_{r-1},\theta_r), \quad r = \overline{1,m+1}. \tag{13}$$

Multiplying $\chi(t)$ to both sides of (13), then integrating on the $[\theta_{r-1},\theta_r]$ and summarizing over r, we have the following system with respect to $\beta\in R^n$

$$\beta = G\big(\Delta_m(\theta)\big)\cdot\beta + \sum_{r=1}^{m+1} V_r\big(\Delta_m(\theta)\big)\lambda_r + W\big(\Delta_m(\theta)\big)\mu + g\big(f,\Delta_m(\theta)\big), \tag{14}$$

using the $(n\times n)-$matrices

$$G\big(\Delta_m(\theta)\big) = \sum_{r=1}^{m+1}\int_{\theta_{r-1}}^{\theta_r}\chi(\tau)\mathbb{X}(\tau)\int_{\theta_{r-1}}^{\tau}\mathbb{X}^{-1}(\tau_1)\phi(\tau_1)d\tau_1 d\tau,$$

$$V_r\big(\Delta_m(\theta)\big) = \int_{\theta_{r-1}}^{\theta_r}\chi(\tau)\mathbb{X}(\tau)\int_{\theta_{r-1}}^{\tau}\mathbb{X}^{-1}(\tau_1)\mathcal{A}(\tau_1)d\tau_1 d\tau +$$
$$+\sum_{j=1}^{m+1}\int_{\theta_{j-1}}^{\theta_j}\chi(\tau)\mathbb{X}(\tau)\int_{\theta_{j-1}}^{\tau}\mathbb{X}^{-1}(\tau_1)\phi(\tau_1)d\tau_1 d\tau\int_{\theta_{r-1}}^{\theta_r}\chi(\tau_2)d\tau_2,$$

the $(n\times l)-$matrix

$$W\big(\Delta_m(\theta)\big) = \sum_{r=1}^{m+1}\int_{\theta_{r-1}}^{\theta_r}\chi(\tau)\mathbb{X}(\tau)\int_{\theta_{r-1}}^{\tau}\mathbb{X}^{-1}(\tau_1)\mathcal{A}_0(\tau_1)d\tau_1 d\tau$$

and vectors of dimension n

$$g\big(f,\Delta_m(\theta)\big) = \sum_{r=1}^{m+1}\int_{\theta_{r-1}}^{\theta_r}\chi(\tau)\mathbb{X}(\tau)\int_{\theta_{r-1}}^{\tau}\mathbb{X}^{-1}(\tau_1)f(\tau_1)d\tau_1 d\tau.$$

Let $\big[I-G(\Delta_m(\theta))\big]^{-1} = \Theta\big(\Delta_m(\theta)\big)$, then (14) can be write in the following form

$$\beta = \sum_{j=1}^{m+1}\Theta\big(\Delta_m(\theta)\big)V_j\big(\Delta_m(\theta)\big)\lambda_j + \Theta\big(\Delta_m(\theta)\big)\Big[W\big(\Delta_m(\theta)\big)\mu + g\big(f,\Delta_m(\theta)\big)\Big]. \tag{15}$$

We obtain the representation of functions $v_r(t)$ in (13) by replacing β with the right-hand side of (15):

$$v_r(t)=\sum_{j=1}^{m+1}\mathbb{X}(t)\int\limits_{\theta_{r-1}}^{t}\mathbb{X}^{-1}(\tau)\phi(\tau)d\tau\Big[\Theta\big(\Delta_m(\theta)\big)V_j\big(\Delta_m(\theta)\big)+\int\limits_{\theta_{j-1}}^{\theta_j}\chi(\tau_1)d\tau_1\Big]\cdot\lambda_j+$$
$$+\mathbb{X}(t)\int\limits_{\theta_{r-1}}^{t}\mathbb{X}^{-1}(\tau)\mathcal{A}(\tau)d\tau\cdot\lambda_r+\mathbb{X}(t)\int\limits_{\theta_{r-1}}^{t}\mathbb{X}^{-1}(\tau)\Big[\phi(\tau)\Theta\big(\Delta_m(\theta)\big)W\big(\Delta_m(\theta)\big)+\mathcal{A}_0(\tau)\Big]d\tau\cdot\mu+$$
$$+\mathbb{X}(t)\int\limits_{\theta_{r-1}}^{t}\mathbb{X}^{-1}(\tau)\Big[\phi(\tau)\Theta\big(\Delta_m(\theta)\big)g\big(f,\Delta_m(\theta)\big)+f(\tau)\Big]d\tau,\ t\in[\theta_{r-1},\theta_r),\ r=\overline{1,m+1}. \tag{16}$$

Introduce notations

$$\widehat{D}_{r,j}\big(\Delta_m(\theta)\big)=\mathbb{X}(\theta_r)\int\limits_{\theta_{r-1}}^{\theta_r}\mathbb{X}^{-1}(\tau)\phi(\tau)d\tau\Big[\Theta\big(\Delta_m(\theta)\big)V_j\big(\Delta_m(\theta)\big)+$$
$$+\int\limits_{\theta_{j-1}}^{\theta_j}\chi(\tau_1)d\tau_1\Big],\ j\neq r,\ j=\overline{1,m+1}.$$

$$\widehat{D}_{r,r}\big(\Delta_m(\theta)\big)=\mathbb{X}(\theta_r)\int\limits_{\theta_{r-1}}^{\theta_r}\mathbb{X}^{-1}(\tau)\phi(\tau)d\tau\Big[\Theta\big(\Delta_m(\theta)\big)V_j\big(\Delta_m(\theta)\big)+\int\limits_{\theta_{j-1}}^{\theta_j}\chi(\tau_1)d\tau_1\Big]+$$
$$+\mathbb{X}(\theta_r)\int\limits_{\theta_{r-1}}^{\theta_r}\mathbb{X}^{-1}(\tau)\mathcal{A}(\tau)d\tau,$$

$$\widehat{W}_r\big(\Delta_m(\theta)\big)=\mathbb{X}(\theta_r)\int\limits_{\theta_{r-1}}^{\theta_r}\mathbb{X}^{-1}(\tau)\Big[\phi(\tau)\Theta\big(\Delta_m(\theta)\big)W\big(\Delta_m(\theta)\big)+\mathcal{A}_0(\tau)\Big]d\tau,$$

$$\widehat{F}_r\big(\Delta_m(\theta)\big)=\mathbb{X}(\theta_r)\int\limits_{\theta_{r-1}}^{\theta_r}\mathbb{X}^{-1}(\tau)\Big[\phi(\tau)\Theta\big(\Delta_m(\theta)\big)g\big(f,\Delta_m(\theta)\big)+f(\tau)\Big]d\tau,\quad r=\overline{1,m+1}.$$

Then from (16) we have

$$\lim_{t\to\theta_r-0}v_r(t)=\sum_{j=1}^{m+1}\widehat{D}_{r,j}\big(\Delta_m(\theta)\big)\cdot\lambda_j+\widehat{W}_r\big(\Delta_m(\theta)\big)\cdot\mu+\widehat{F}_r\big(\Delta_m(\theta)\big). \tag{17}$$

The following system with respect to parameters μ, λ_r, $r=\overline{1,m+2}$ is obtained when the right side of (17) is substituted into the conditions (9) - (11):

$$B_0\mu + M_0\lambda_1 + L_0\lambda_{m+2} = d_0, \tag{18}$$

$$M_i\big(\lambda_i + \sum_{j=1}^{m+1}\widehat{D}_{i,j}\big(\Delta_m(\theta)\big)\cdot\lambda_j + \widehat{W}_i\big(\Delta_m(\theta)\big)\mu\big) - L_i\lambda_{i+1} = d_i - M_i\widehat{F}_i, \quad i=\overline{1,m}, \tag{19}$$

$$\lambda_{m+1} + \sum_{j=1}^{m+1}\widehat{D}_{m+1,\,j}\big(\Delta_m(\theta)\big)\cdot\lambda_j + \widehat{W}_{m+1}\big(\Delta_m(\theta)\big)\mu - \lambda_{m+2} = -\widehat{F}_{m+1}\big(\Delta_m(\theta)\big). \tag{20}$$

Left hand side of the (18)-(20) can be denoted as $Q_*\big(\Delta_m(\theta)\big)$ and written as

$$Q_*\big(\Delta_m(\theta)\big)\Lambda = -F_*\big(\Delta_m(\theta)\big), \quad \Lambda = (\lambda_1,...,\lambda_{m+2},\mu)\in R^{n(m+2)+l}, \tag{21}$$

where

$$F_*\big(\Delta_m(\theta)\big) = \Big(-d_0, -d_1 + M_1\widehat{F}_1\big(\Delta_m(\theta)\big), ... \\ ..., -d_m + M_m\widehat{F}_m\big(\Delta_m(\theta)\big), \widehat{F}_{m+1}\big(\Delta_m(\theta)\big)\Big) \in R^{n(m+2)+l}.$$

3 The Main results

Cauchy problems for ODE on subintervals

$$\frac{ds}{dt} = \mathcal{A}(t)s + \mathbb{P}(t), \quad s(\theta_{r-1}) = 0, \quad t\in[\theta_{r-1},\theta_r], \tag{22}$$

are a significant part of proposed algorithm. In this case, $\mathbb{P}(t)$ is either a $(n\times n)$-matrix or a $n-$vector, both of which are continuous on $[\theta_{r-1},\theta_r]$. So, a square matrix or a vector of size n is the solution to problem (22). The Cauchy problem's (22) solution is denoted by the $\widehat{b}_r(\mathbb{P},t)$. Undoubtedly,

$$\widehat{b}_r(\mathbb{P},t) = \mathbb{X}(t)\int_{\theta_{r-1}}^{t}\mathbb{X}^{-1}(\tau)\mathbb{P}(\tau)d\tau, \quad t\in[\theta_{r-1},\theta_r].$$

We suggest the following algorithm to solve the problem (1)- (3).

Step 1. With step $h_r=(\theta_r-\theta_{r-1})/N_r$, $r=\overline{1,m+1}$, each r-th interval $[\theta_{r-1},\theta_r]$ is divided into N_r parts.

Step 2.By solving auxiliary Cauchy problems (22) with $\mathbb{P}(t)\in\{\phi(t),\mathcal{A},\mathcal{A}_0,f(t)\}$ find the functions $\widehat{b}_r^{h_r}(\phi,t)$, $\widehat{b}_r^{h_r}(\mathcal{A},t)$, $\widehat{b}_r^{h_r}(\mathcal{A}_0,t)$, $\widehat{b}_r^{h_r}(f,t)$, $t\in[\theta_{r-1},\theta_r]$, $r=\overline{1,m+1}$.

Step 3. The following expressions are calculated using the Simpson's method:

$$\chi_r^{h_r} = \int_{\theta_{r-1}}^{\theta_r}\chi(\tau)d\tau, \quad \chi_r^{h_r}(\phi) = \int_{\theta_{r-1}}^{\theta_r}\chi(\tau)\widehat{b}_r(\phi,\tau)d\tau, \quad \chi_r^{h_r}(\mathcal{A}) = \int_{\theta_{r-1}}^{\theta_r}\chi(\tau)\widehat{b}_r(\mathcal{A},\tau)d\tau,$$

$$\chi_r^{h_r}(\mathcal{A}_0) = \int\limits_{\theta_{r-1}}^{\theta_r} \chi(\tau)\widehat{b}_r(\mathcal{A}_0,\tau)d\tau, \qquad \chi_r^{h_r}(f) = \int\limits_{\theta_{r-1}}^{\theta_r} \chi(\tau)\widehat{b}_r(f,\tau)d\tau, \qquad r=\overline{1,m+1}.$$

Step 4. Then construct the follows:

$$G^{h_r}(\Delta_m(\theta)) = \sum_{r=1}^{m+1}\chi_r^{h_r}(\phi), \quad W^{h_r}(\Delta_m(\theta)) = \sum_{r=1}^{m+1}\chi_r^{h_r}(\mathcal{A}_0), \quad g^{h_r}(f,\Delta_m(\theta)) = \sum_{r=1}^{m+1}\chi_r^{h_r}(f),$$

$$V_r^{h_r}(\Delta_m(\theta)) = \chi_r^{h_r}(\mathcal{A}) + \sum_{j=1}^{m+1}\chi_j^{h_r}(\phi)\cdot\chi_r^{h_r}, \qquad r=\overline{1,m+1}.$$

Step 5. We have the following approximate system with respect to Λ :

$$Q_*^{h_r}\big(\Delta_m(\theta)\big)\Lambda = -F_*^{h_r}\big(\Delta_m(\theta)\big), \quad \Lambda\in R^{n(m+2)+l}, \tag{23}$$

Solving the system (23), we determine $\Lambda^{h_r} = (\lambda_1^{h_r},...,\lambda_{m+2}^{h_r},\mu^{h_r})$. Take note that the values of the solution to the problem (1)-(3) are represented by the components $\lambda_1^{h_r}$, $\lambda_2^{h_r}$,...,$\lambda_{m+1}^{h_r}$: $z^{h_r}(\theta_{r-1}) = \lambda_r^{h_r}$, $r=\overline{1,m+1}$ and $\lim\limits_{t\to T-0} z^{h_r}(t) = \lambda_{m+2}^{h_r}$.

Step 6. When the matrix $I - G^{h_r}(\Delta_m(\theta))$ is invertible, we find:

$$\Theta^{h_r}(\Delta_m(\theta)) = [I - G^{h_r}(\Delta_m(\theta))]^{-1},$$
$$\beta^{h_r} = \sum_{j=1}^{m+1}\Theta^{h_r}(\Delta_m(\theta))V_j^{h_r}(\Delta_m(\theta))\lambda_j^{h_r} + \Theta^{h_r}(\Delta_m(\theta))\Big[W^{h_r}(\Delta_m(\theta))\mu^{h_r} + g^{h_r}(f,\Delta_m(\theta))\Big].$$

Step 7. Solving the following problems

$$\frac{dz}{dt} = \mathcal{A}(t)z + \mathcal{F}^{h_r}(t), \quad z(\theta_{r-1}) = \lambda_r^{h_r}, \quad t\in[\theta_{r-1},\theta_r), \quad r=\overline{1,m+1},$$

where

$$\mathcal{F}^{h_r}(t) = \phi(t)\Big[\beta^{h_r} + \sum_{r=1}^{m+1}\chi_r^{h_r}\cdot\lambda_r^{h_r}\Big] + \mathcal{A}_0(t)\mu^{h_r} + f(t)$$

and find the values of the $z^{h_r}(t)$ at the remaining subintervals' points.

As a result, the proposed algorithm gives us the numerical solution to the problem with parameter (1)-(3).

To illustrate the suggested method for numerically solving problems (1)-(3) based on a modification of the Dzhumabaev parametrization method, consider the example below.

4 Example

We have the following problem with parameter

$$\frac{dz}{dt} = \mathcal{A}(t)z + \int\limits_0^T \phi(t)\chi(s)z(s)ds + \mathcal{A}_0(t)\mu + f(t), \quad z\in R^2, \quad \mu\in R^1, \quad t\in(0,1), \tag{24}$$

$$B_0\mu + M_0 z(0) + L_0 z(T) = d_0, \quad d_0 \in R^3, \tag{25}$$

$$M_1 z\Big(\frac{1}{4} - 0\Big) - L_1 z\Big(\frac{1}{4} + 0\Big) = d_1, \quad d_1 \in R^2, \tag{26}$$

Here $\theta_0 = 0$, $\theta_1 = \frac{1}{4}$, $\theta_2 = T = 1$,

$$\mathcal{A}(t) = \begin{pmatrix} 4t & 5 \\ -3 & 3t^2 \end{pmatrix}, \quad \mathcal{A}_0(t) = \begin{pmatrix} t^2+2 \\ t-4 \end{pmatrix}, \quad \phi(t) = \begin{pmatrix} 2 & 6t \\ t^3 & -3 \end{pmatrix}, \quad \chi(t) = \begin{pmatrix} 6 & 24 \\ -64t & 0 \end{pmatrix},$$

$$B_0 = \begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix}, \quad M_0 = \begin{pmatrix} 3 & 7 \\ 0 & -5 \\ 3 & -6 \end{pmatrix}, \quad L_0 = \begin{pmatrix} -3 & 8 \\ 0 & 2 \\ 3 & 11 \end{pmatrix}, \quad d_0 = \begin{pmatrix} 200 \\ 124 \\ 482 \end{pmatrix},$$

$$M_1 = \begin{pmatrix} 2 & -5 \\ 0 & 4 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 3 & 0 \\ -1 & -4 \end{pmatrix}, \quad d_1 = \begin{pmatrix} -7 \\ 38 \end{pmatrix},$$

$$f(t) = \begin{pmatrix} -64t^3 - 13t^2 + 1358t - 741 \\ -389t^3 + 57t^2 - 13t - 624 \end{pmatrix}, \text{ at } t \in \big[0, \tfrac{1}{4}\big),$$

$$f(t) = \begin{pmatrix} -205t^2 + 1384t - 793 \\ -96t^4 - 347t^3 - 27t^2 + 75t - 629 \end{pmatrix}, \text{ at } t \in \big[\tfrac{1}{4}, 1\big].$$

To solve (24)–(26), we employ a numerical implementation of the our method. The accuracy of the solution is dependent on how well the Cauchy problem is solved on subintervals. By partitioning the $\big[0, \frac{1}{4}\big]$, $\big[\frac{1}{4}, 1\big]$ with step $h = 0.025$, we can provide the outcomes of the numerical implementation of the method.

Exact solution of problem (24)–(26) is the following:

$$z^*(t) = \begin{pmatrix} 16t^2 - 1 \\ 8t - 3 \end{pmatrix} \text{ at } t \in \big[0, \tfrac{1}{4}\big),$$

$$z^*(t) = \begin{pmatrix} 8t + 2 \\ 32t^2 - 6t + 9 \end{pmatrix} \text{ at } t \in \big[\tfrac{1}{4}, 1\big],$$

$$\mu^* = 13.$$

The following Table 1 shows the variations between the exact and approximate solutions to problems (24)–(26):

Таблица 1: Absolute Errors

k	t	$\lvert z^*_{(1)}(t) - \widetilde{z}_{(1)}(t)\rvert$	$\lvert z^*_{(2)}(t) - \widetilde{z}_{(2)}(t)\rvert$	k	t	$\lvert z^*_{(1)}(t) - \widetilde{z}_{(1)}(t)\rvert$	$\lvert z^*_{(2)}(t) - \widetilde{z}_{(2)}(t)\rvert$
0	0	2.101e-7	2.258e-7	20	0.5	4.920e-8	2.342e-7
1	0.025	2.437e-7	1.792e-7	21	0.525	2.340e-8	2.251e-7
2	0.05	2.713e-7	1.306e-7	22	0.55	1.300e-9	2.149e-7
3	0.075	2.928e-7	8.050e-8	23	0.575	2.510e-8	2.036e-7
4	0.1	3.081e-7	2.920e-8	24	0.6	4.770e-8	1.913e-7
5	0.125	3.172e-7	2.270e-8	25	0.625	6.900e-8	1.780e-7
6	0.15	3.199e-7	7.480e-8	26	0.65	8.900e-8	1.637e-7
7	0.175	3.162e-7	1.268e-7	27	0.675	1.074e-7	1.484e-7
8	0.2	3.061e-7	1.782e-7	28	0.7	1.243e-7	1.322e-7
9	0.225	2.894e-7	2.286e-7	29	0.725	1.392e-7	1.150e-7
10	0.25	3.461e-7	2.565e-7	30	0.75	1.521e-7	9.690e-8
11	0.275	3.140e-7	2.606e-7	31	0.775	1.627e-7	7.780e-8
12	0.3	2.823e-7	2.632e-7	32	0.8	1.707e-7	5.790e-8
13	0.325	2.510e-7	2.643e-7	33	0.825	1.758e-7	3.710e-8
14	0.35	2.202e-7	2.639e-7	34	0.85	1.776e-7	1.540e-8
15	0.375	1.900e-7	2.622e-7	35	0.875	1.757e-7	0.700e-8
16	0.4	1.604e-7	2.591e-7	36	0.9	1.697e-7	3.010e-8
17	0.425	1.315e-7	2.547e-7	37	0.925	1.590e-8	5.370e-8
18	0.45	1.033e-7	2.491e-7	38	0.95	1.429e-7	7.790e-8
19	0.475	7.580e-8	2.422e-7	39	0.975	1.207e-7	1.023e-7
20	0.5	4.920e-7	2.342e-7	40	1	9.180e-8	1.267e-7
$\lvert\mu^* - \widetilde{\mu}\rvert = 4.608e-7$							

5 Conclusion

We suggest a method for solving the problem with parameter (1)-(3) numerically.The nonlinear boundary value problem for the system of impulsive Fredholm integro-differential equations may be solved using the suggested approach.

6 Acknowledgement

This research is funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan. (Grants No. AP19675193, BR20281002)

References

[1] Halanay. A., Wexler. D., "Qualitative Theory of Impulse Systems", (ditura Academiei Republici Socialiste Romania, Bucuresti, 1968)

[2] Lakshmikantham V., Bainov D.D., Simeonov P.S., "Theory of Impulsive Differential Equations", (World Scientific, Singapore, (1989)

[3] Samoilenko A.M., Perestyuk N.A., "Impulsive Differential Equations", (World World Scientific, Singapore (1995)

[4] Nieto J.J., Rodrigues-Lopez R., "New comparison results for impulsive integro-differential equations and applications", *Journal of Mathematical Analysis and Applications,* 328, (2007): 1343–1368. https://doi.org/10.1016/j.jmaa.2006.06.029

[5] Sousa J., Oliveira D., Oliveira E., "On the existence and stability for noninstantaneous impulsive fractional integrodifferential equation", *Mathematical Methods in the Applied Sciences,* 42(4), (2018): 1249-1261. https://doi.org/10.1002/mma.5430

[6] Akhmetov M.U., Zafer A., Sejilova R.D., "The control of boundary value problems for quasilinear impulsive integro-differential equations", *Nonlinear Analysis,* 48(2), (2002): 271-286.

[7] He Z., He X., "Monotone iterative technique for impulsive integrodifferential equations with periodic boundary conditions", *Computers and Mathematics with Applications,* 48, (2004): 73-84. https://doi.org/10.1016/j.camwa.2004.01.005

[8] Luo Z., Nieto J.J., "New results for the periodic boundary value problem for impulsive integro-differential equations", *Nonlinear Analysis: Theory and Methods and Applications,* 70, (2009): 2248-260. https://doi.org/10.1016/j.na.2008.03.004

[9] Wang X., Zhang J., "Impulsive anti-periodic boundary value problem for first order integro-differential equations", *Journal of Computational and Applied Mathematics* , 234, (2010): 3261-3267. https://doi.org/10.1016/j.cam.2010.04.024

[10] Dzhumabaev D.S. "Solvability of a linear boundary value problem for a fredholm integro-differential equation with impulsive inputs", *Differential Equations,* 51(9), (2015): 1180-1196. https://doi.org/10.1134/S0012266115090086

[11] Assanova A.T., Bakirova E.A., Kadirbayeva Zh.M. and Uteshova R.E., "A computational method for solving a problem with parameter for linear systems of integro-differential equations", *Comp. Appl. Math.,* 39, (2020): 248. https://doi.org/10.1007/s40314-020-01298-1

[12] Bakirova E.A., Assanova A.T., Kadirbayeva Zh.M., "A problem with parameter for the integro-differential equations", *Mathematical Modelling and Analysis,* 26, (2021): 34–54. https://doi.org/10.3846/mma.2021.11977

[13] Dzhumabaev D.S., "Criteria for the unique solvability of a linear boundary-value problem for an ordinary differential equation", *USSR Comput. Math. Math. Phys.,* 1, (1989): 34–46. https://doi.org/10.1016/0041-5553(89)90038-4

[14] Dzhumabaev D.S. "On one approach to solve the linear boundary value problems for fredholm integro-differential equations", *Journal of Computational and Applied Mathematics,* 294(2), (2016): 342-357. https://doi.org/10.1016/j.cam.2015.08.023

[15] Dzhumabaev D.S. "New general solutions to linear Fredholm integro-differential equations and their applications on solving the boundary value problems", *Journal of Computational and Applied Mathematics,* 327, (2018): 79–108. https://doi.org/10.1016/j.cam.2017.06.010