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1-бөлім **Раздел 1** **Section 1**

Математика **Математика** **Mathematics**

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THE COEFFICIENT INVERSE PROBLEM FOR A PSEUDOPARABOLIC EQUATION OF THE THIRD ORDER

In this paper, we consider the coefficient inverse problem for a third-order pseudoparabolic equation, which represents mathematical model for the movement of moisture and salts in soils. Such non-classical equations are also called Sobolev-type equations. At present, the study of direct and inverse problems for a pseudoparabolic equation is readily developing due to the needs of modeling and controlling processes in hydrodynamics, mechanics, thermal physics and continuum mechanics. At the same time, the investigation of coefficient inverse problems is also important, since they are used in solving problems of planning the development of oil fields, in particular, in determining the filtration parameters of fields, in creating new types of measuring equipment, in solving environmental monitoring problems, etc. Thus both trend directions such as pseudoparabolic equations and coefficient inverse problems are relevant due to the abundance of various applications where such non-classical objects arise. In this work, the Galerkin method is used to prove the existence of the solution for the inverse coefficient problem and obtained sufficient conditions for the blow up of its solution in a finite time in a bounded domain. Moreover, authors developed the algorithm for the numerical solution of the given problem by using the finite difference method. In addition, computational experiments were carried out illustrating the theoretical calculations obtained in the work.
Key words: pseudoparabolic equation, coefficient inverse problem, blow up solution, numerical solution, numerical experiments.

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Үшінші ретті псевдопараболалық теңдеу үшін коэффициентті кері есебі

Бұл жұмыста топырақтағы ылғал мен тұздардың қозғалысының математикалық моделі болып табылатын үшінші ретті псевдопараболалық теңдеу үшін коэффициент кері есебін қарастырамыз. Мұндай классикалық емес теңдеулер Соболев типті теңдеулер деп те аталады. Қазіргі уақытта гидродинамика, механика, жылу физикасы, тұтас орта механикасында процестерді модельдеу және басқару қажеттілігіне байланысты псевдопараболалық теңдеу үшін тура және кері есептерді зерттеу белсенді түрде дамып келеді. Сонымен қатар, коэффициент кері есептерін зерттеу маңызды, өйткені олар мұнай кен орындарын игеруді жоспарлау мәселелерін шешуде, атап айтқанда, кен орындарының сүзу параметрлерін анықтауда, өлшеу құралдарының жаңа түрлерін жасауда, қоршаған ортаны бақылау мәселелері және т.б. есептерді шешуде қолданылады.

Осылайша, псевдопараболалық теңдеу мен коэффициентті кері есептер екі бағыты, классикалық емес объектілер пайда болатын әртүрлі қолдануларының көптігіне байланысты өзекті болып табылады. Бұл жұмыста Галеркин әдісімен коэффициентті кері есебінің шешімі бар екені дәлелденді және шектелген облыста шешімнің ақырлы уақытта қирауы үшін жеткілікті шарттары алынды. Сонымен қатар, осы есептің сандық шешімінің алгоритмі құрастырылып, жұмыста алынған теориялық есептеулерді көрсету үшін сандық тәжірибелер жүргізілді.
Түйін сөздер: псевдопараболалық теңдеу, коэффициентті кері есеп, шешімнің қирауы, сандық шешім, сандық тәжірибе.

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Коэффициентная обратная задача для псевдопараболического уравнения третьего порядка

В данной работе рассматривается коэффициентная обратная задача для псевдопараболического уравнения третьего порядка, которое представляет собой математическую модель движения влаги и солей в почвах. Такие неклассические уравнения еще называют уравнениями соболевского типа. В настоящее время исследование прямых и обратных задач для псевдопараболического уравнения активно развивается в связи с необходимостью моделирования и управления процессами в гидродинамике, механике, теплофизике и механике сплошных сред. В то же время важное значение имеет исследование коэффициентных обратных задач, поскольку они используются при решении задач планирования разработки нефтяных месторождений, в частности, при определении фильтрационных параметров месторождений, при создании новых типов измерительной техники, при решении задач экологического мониторинга и т. д. Таким образом, оба трендовых направления псевдопараболические уравнения и коэффициентные обратные задачи актуальны в силу обилия разнообразных приложений, в которых возникают такие неклассические объекты. В данной работе методом Галеркина доказано существование решения обратной коэффициентной задачи и получены достаточные условия разрушения решения за конечное время в ограниченной области. Кроме того, построен алгоритм численного решения данной задачи и проведены численные эксперименты, иллюстрирующие полученные в работе теоретические выкладки.

Ключевые слова: псевдопараболическое уравнение, коэффициентная обратная задача, разрушение решения, численное решение, численный эксперимент.

1 Introduction

The work is devoted to the solvability of coefficient inverse problem for a pseudoparabolic equation, also called Sobolev-type equation. Furthermore, in this work, we conducted computational experiments, and developed the numerical solution of the given coefficient inverse problem. A problem is said to be a coefficient inverse problem where should be determined the coefficient (coefficients) of the equation with its solution. Interest in them is caused primarily by their important applied values. Coefficient inverse problems find applications in planning the development of oil fields (determining the filtration parameters of fields), in creating new types of measuring equipment, in solving environmental monitoring problems, etc.

On the other hand, pseudoparabolic equations describe various physical processes. It is well known that many issues on fluid filtration in fractured-porous media [1–2], heat transfer in a heterogeneous medium [3], moisture transfer in soils [4–5] lead to the study of initial, initial-boundary and inverse problems for third-order pseudoparabolic equations.

Several direct and inverse problems for pseudoparabolic equations are studied in the works of G.I. Barenblatt [2], Korpusov M.O., A.B. Alshin, Pletner Yu.D. [6], D.Colton [7-8], R.C.Rao [9-10], W. Rundell [11-12], R.Showalter [13-14], T.W.Ting [15], Karch [16], Yaman [17], Khompysh [18] and others.

Most of the papers cited above consider questions on the solvability of local boundary value problems for pseudoparabolic equations. In particular, D.Colton [8] was the first to construct the Riemann function for a special one-dimensional pseudoparabolic equation and proved the existence and uniqueness of the regular solution for the characteristic Goursat problem and the first initial boundary value problem. Further, M.Kh. Shkhanukov [19-20], V. A. Vodakhova [21] studied the solvability of boundary value problems for pseudoparabolic equations of the third order with non-local conditions of the Samarskii-Bitsadze type and with integral conditions. In the monograph of Sveshnikov A.G., Korpusov M.O., Yu.D. Pletner [6] are investigated the solvability and blow-up solutions for a whole class of model equations of pseudoparabolic type with non-linear elliptic operators at the time derivative, as well as the existence and non-existence of a time-global solution.

To date, inverse problems for equations of parabolic and hyperbolic types are studied quite well. Coefficient inverse problems for non-classical equations seem to be much less considered; in particular, for Sobolev-type equations. Similar problems, but for equations of parabolic type, are considered in the works of E.G.Savateev [22–24]. Let us remark that one dimensional inverse problems for pseudoparabolic equations of the third order are investigated in the works of E.R. Atamanova, M.Sh. Mamayusupova [25], B.S. Ablabekov [26], A.I. Kozhanov [27], [28], S.N. Antontsev [29], S.E. Aitzhanov [30] and others.

Methods for the numerical solution of the coefficient inverse problem are developed by authors such as AL. Bukhgeim, A.B. Bakushinsky, A.V. Goncharsky, A.A. Samarsky, O.M. Alifanov, P.N. Vabishchevich, A.Yu. .M.Ibragimov, A.D.Iskanderov and others. Here it is also important to note the works of M.Kh. Beshtokov [31], [32], where author considers numerical methods of solving local and non-local boundary value problems for one dimensional and multidimensional generalized Sobolev-type equations in Cartesian, cylindrical and spherical coordinates.

2 Statement of the problem

We consider in the cylinder $Q_T = \{(x,t): \ x \in \Omega, \ 0 < t < T\}, \ \Omega \subset R^n, \ n \geq 1$ the inverse problem of determining a pair of functions $\{u(x,t), \ f(t) + g(x)\}$ for the pseudoparabolic equation

$$u_t - \Delta u_t - \Delta u = (f(t) + g(x))u(x,t), \tag{1}$$

with initial condition

$$u(x,0) = u_0(x), \ \ x \in \Omega, \tag{2}$$

boundary condition

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0, \ \ t \in [0,T], \tag{3}$$

and overdetermination conditions

$$u_t(x,0) = u_1(x), \;\; x \in \Omega, \tag{4}$$

$$\int_\Omega u(x,t)dx = e(t), \;\; t \in [0,T]. \tag{5}$$

Let the conditions be satisfied for the known data of the inverse problem (1)-(5):

$$\begin{array}{l} e(t) \in W_2^1(0,T), \;\; e(t) \neq 0, \;\; \forall t \in [0,T], \\ u_0(x), \, u_1(x) \in W_2^2(\Omega), \;\; u_0(x) \neq 0, \;\; \forall x \in \Omega. \end{array} \tag{6}$$

Our main result is the following.

Theorem. Let the conditions (6) be satisfied, then the inverse problem (1)-(5) has a unique solution $u(x,t) \in L_\infty\left(0,T; \overset{0}{W_2^1}(\Omega)\right) \cap L_\infty\left(0,T; W_2^2(\Omega)\right), \; u_t(x,t) \in L_2\left(0,T; W_2^2(\Omega)\right), \; f(t)+g(x) \in L_2(Q_T)$.

Remark. If it is additionally known that $f(0) \neq 0$, then $f(t)$ and $g(x)$ are defined uniquely.

3 Materials and methods

Let us put into equation (1) the $t = 0$, considering (2) and (4), we get

$$u_t(x,0) - \Delta u_t(x,0) - \Delta u(x,0) = (f(0)+g(x))u(x,0),$$

$$u_1(x) - \Delta u_1(x) - \Delta u_0(x) = (f(0)+g(x))u_0(x),$$

$$f(0)+g(x) = (u_0(x))^{-1}\left[u_1(x) - \Delta u_1(x) - \Delta u_0(x)\right]. \tag{7}$$

Denote by $h(x) \equiv (u_0(x))^{-1}\left[u_1(x) - \Delta u_1(x) - \Delta u_0(x)\right]$.

Multiplying the relation (7) by the function $u(x,t)$ and integrating over the domain Ω, by virtue of the notation we obtain

$$f(0)e(t) + g(x)e(t) = h(x)e(t).$$

$$f(0)\int_\Omega udx + \int_\Omega g(x)udx = \int_\Omega h(x)udx.$$

$$f(0)e(t) + \int_\Omega g(x)udx = \int_\Omega h(x)udx. \tag{8}$$

Now we integrate the equation (1) over the domain Ω, taking into account the condition (5) and Green's formula, we get

$$\int_\Omega u_t dx - \int_\Omega \Delta u_t dx - \int_\Omega \Delta u dx = f(t)\int_\Omega udx + \int_\Omega g(x)udx,$$

$$e'(t) = f(t)e(t) + \int_\Omega g(x)udx. \tag{9}$$

We multiply the relation (7) by the function $e(t)$ and add to the ratio (9), also considering the equality (8), we get

$$f(t)e(t) + \int_\Omega g(x)udx + f(0)e(t) + g(x)e(t) = \\ = e'(t) + h(x)e(t)$$

$$f(t) + g(x) = e^{-1}(t)\left(e'(t) + h(x)e(t) - \int_\Omega h(x)u(x,t)dx\right). \tag{10}$$

Thus we have reduced the inverse problem (1)-(5) to a direct problem for the integro-differential equation. In the cylinder $Q_T = \{(x,t):\ x \in \Omega,\ 0 < t < T\},\ \Omega \subset R^n,\ n \geq 1$, let us consider the initial-boundary value problem of determining the function $u(x,t)$ for the integro-differential equation

$$u_t - \Delta u_t - \Delta u = h_1(x,t)u - e^{-1}(t)u(x,t)\int_\Omega h(x)u(x,t)dx, \tag{11}$$

with initial condition

$$u(x,0) = u_0(x),\ \ x \in \Omega, \tag{12}$$

and boundary condition

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0,\ \ t \in [0,T], \tag{13}$$

where $h_1(x,t) = e^{-1}(t)[e'(t) + h(x)e(t)]$.

Now we prove the existence using the Galerkin method. We are looking for an approximate solution in the form

$$u_N(x,t) = \sum_{j=1}^{N} C_{Nj}(t)\Psi_j(x), \tag{14}$$

where $\Psi_j(x)$ orthonormal basis from the space $\overset{0}{W}{}_2^2(\Omega)$, they are found from the Sturm-Liouville problem

$$\Delta\Psi_j(x) + \lambda_j\Psi_j(x) = 0, \quad \left.\frac{\partial\Psi_j}{\partial n}\right|_{\partial\Omega} = 0,$$

where λ_j eigenvalues and $\Psi_j(x)$are eigengunctions. $C_{Nj}(t)$ are unknown functions, they are defined from the following Cauchy problem (systems of differential equations with initial conditions)

$$\int_\Omega (u_{Nt} - \Delta u_{Nt} - \Delta u_N)\Psi_j(x)dx = h_1(x,t)\int_\Omega u_N\Psi_j(x)dx- \\ -e^{-1}(t)\int_\Omega u_N\Psi_j(x)dx\int_\Omega h(x)u_N dx, \tag{15}$$

$$u_N(x,0)=\sum_{j=1}^{N}C_{Nj}(0)\Psi_j(x)=u_{0N}=\sum_{j=1}^{N}u_{0j}\Psi_j(x).$$

We multiply the ratio (15) by $C_{Nj}(t)$ and sum over $j=1,...,N$, then we get

$$\int_\Omega (u_t-\Delta u_t-\Delta u)udx=\int_\Omega h_1(x,t)u^2dx-e^{-1}(t)\int_\Omega h(x)udx\int_\Omega u^2dx.$$

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}\left(\|u_N\|_{2,\Omega}^2+\|\nabla u_N\|_{2,\Omega}^2\right)+\|\nabla u_N\|_{2,\Omega}^2=\\&=\textstyle\int_\Omega h_1(x,t)u_N^2dx+e^{-1}(t)\int_\Omega h(x)u_Ndx\,\|u_N\|_{2,\Omega}^2.\end{aligned}\tag{16}$$

Let us estimate the right side of identity (16) using the Hölder and Young inequalities:

$$\left|\int_\Omega h_1(x,t)u_N^2dx\right|\le h_{01}\|u_N\|_{2,\Omega}^2\le\frac{1}{4}\|u_N\|_{2,\Omega}^4+h_{01}^2.$$

$$\left|e^{-1}(t)\int_\Omega h(x)u_Ndx\right|\|u_N\|_{2,\Omega}^2\le e_0\|h\|_{2,\Omega}\|u_N\|_{2,\Omega}^3\le\frac{1}{4}\|u_N\|_{2,\Omega}^4+\frac{27}{4}e_0^4\|h\|_{2,\Omega}^4.$$

$$\frac{1}{2}\frac{d}{dt}\left(\|u_N\|_{2,\Omega}^2+\|\nabla u_N\|_{2,\Omega}^2\right)+\|\nabla u_N\|_{2,\Omega}^2\le\frac{1}{2}\|u_N\|_{2,\Omega}^4+h_{01}^2+\frac{27}{4}e_0^4\|h\|_{2,\Omega}^4.$$

From Poincaré's inequality $\|\nabla u\|_{2,\Omega}^2\ge\lambda_1\|u\|_{2,\Omega}^2$ we can get the next inequality

$$\|\nabla u\|_{2,\Omega}^2=\frac{1}{1+\lambda_1}\|\nabla u\|_{2,\Omega}^2+\frac{\lambda_1}{1+\lambda_1}\|\nabla u\|_{2,\Omega}^2\ge\frac{\lambda_1}{1+\lambda_1}\left(\|u\|_{2,\Omega}^2+\|\nabla u\|_{2,\Omega}^2\right).$$

Substituting the obtained inequalities into the identity (16), we find

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|_{2,\Omega}^2+\|\nabla u\|_{2,\Omega}^2\right)+\frac{\lambda_1}{1+\lambda_1}\left(\|u\|_{2,\Omega}^2+\|\nabla u\|_{2,\Omega}^2\right)\le\frac{1}{2}\|u\|_{2,\Omega}^4+h_{01}^2+\frac{27}{4}e_0^4\|h\|_{2,\Omega}^4.$$

$$\frac{d}{dt}\left(\|u_N\|_{2,\Omega}^2+\|\nabla u_N\|_{2,\Omega}^2\right)+C_0\left(\|u_N\|_{2,\Omega}^2+\|\nabla u_N\|_{2,\Omega}^2\right)\le\left(\|u_N\|_{2,\Omega}^2+\|\nabla u_N\|_{2,\Omega}^2\right)^2+C_1,$$

where $C_0=\frac{2\lambda_1}{1+\lambda_1},\ C_1=2h_{01}^2+\frac{27}{2}e_0^4\|h\|_{2,\Omega}^4$.

We denote by $y(t)=\|u_N\|_{2,\Omega}^2+\|\nabla u_N\|_{2,\Omega}^2$. Then the last inequality can be written as

$$\frac{dy(t)}{dt}+C_0y(t)\le y^2(t)+C_1.$$

$$\frac{d}{dt}\left(e^{C_0t}y(t)\right)\le y^2(t)e^{C_0t}+C_1e^{C_0t}.$$

$$e^{C_0t}y(t)\le y(0)+\int_0^t e^{C_0\tau}y^2(\tau)d\tau+\frac{C_1}{C_0}\left(e^{C_0t}-1\right).$$

$$y(t) \le y(0)e^{-C_0 t} + \int_0^t e^{C_0(\tau - t)} y^2(\tau) d\tau + \frac{C_1}{C_0}\left(1 - e^{-C_0 t}\right).$$

$$y(t) \le y(0)e^{-C_0 t} + \frac{1}{4}\int_0^t e^{2C_0(\tau - t)} d\tau + \int_0^t y^4(\tau) d\tau + \frac{C_1}{C_0}.$$

$$y(t) \le y(0) + \frac{C_1}{C_0} + \frac{1}{8C_0} + \int_0^t y^4(\tau) d\tau.$$

Applying Bihari's lemma, we obtain the required estimate

$$y(t) \le \frac{y(0) + \frac{C_1}{C_0} + \frac{1}{8C_0}}{\left[1 - 3\left(y(0) + \frac{C_1}{C_0} + \frac{1}{8C_0}\right)^3 t\right]^{\frac{1}{3}}}.$$

Thus we have obtained an a priori estimate, so the next inequality is fairly true

$$ess\sup_{t\in[0,T]} \left(\|u_N\|_{2,\Omega}^2 + \|\nabla u_N\|_{2,\Omega}^2\right) \le C_2, \;\; \forall T < T_0, \tag{17}$$

where the constant C_2 does not depend on N.

Now we multiply the ratio (15) by $\lambda_j C'_{Nj}(t)$ and sum over $j = 1, ..., N$, and get

$$-\int_\Omega (u_{Nt} - \Delta u_{Nt} - \Delta u_N)\Delta u_{Nt} dx = -\int_\Omega h_1(x,t) u_N \Delta u_{Nt} dx +$$
$$+ e^{-1}(t) \int_\Omega h(x) u_N dx \int_\Omega u_N \Delta u_{Nt} dx.$$

$$\begin{aligned} &\|\nabla u_{Nt}\|_{2,\Omega}^2 + \|\Delta u_{Nt}\|_{2,\Omega}^2 + \tfrac{1}{2}\tfrac{d}{dt}\|\Delta u_N\|_{2,\Omega}^2 = \\ &= -\int_\Omega h_1(x,t) u_N \Delta u_{Nt} dx + e^{-1}(t)\int_\Omega h(x) u_N dx \int_\Omega u_N \Delta u_{Nt} dx. \end{aligned} \tag{18}$$

Similarly, we estimate the right side using the Hölder and Young inequalities, also the estimate (17), as a result, we obtain the following estimate

$$ess\sup_{t\in[0,T]} \|\Delta u_N\|_{2,\Omega}^2 + \int_0^T \|\nabla u_{Nt}\|_{2,\Omega}^2 dt + \int_0^T \|\Delta u_{Nt}\|_{2,\Omega}^2 dt \le C_3. \tag{19}$$

Suppose that the problem (11)-(13) has two solutions $u_1(x,t)$ и $u_2(x,t)$. The difference of two solutions $u(x,t) = u_1(x,t) - u_2(x,t)$ satisfies the next equation

$$\begin{aligned} &u_t - \Delta u_t - \Delta u = h_1(x,t)u - e^{-1}(t) u_1(x,t) \int_\Omega h(x) u(x,t) dx - \\ &- e^{-1}(t) u(x,t) \int_\Omega h(x) u_2(x,t) dx, \end{aligned} \tag{20}$$

with the initial condition

$$u(x,0) = 0, \;\; x \in \Omega, \tag{21}$$

and the boundary condition

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0, \ \ t \in [0,T], \tag{22}$$

Multiplying the equation (20) by the function $u(x,t)$ and integrating over Ω, we get

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}\left(\|u\|^2_{2,\Omega} + \|\nabla u\|^2_{2,\Omega}\right) + \|\nabla u\|^2_{2,\Omega} = \\ &= \textstyle\int_\Omega h_1(x,t)u^2dx + e^{-1}(t)\int_\Omega u_1 u dx \int_\Omega h(x)udx + e^{-1}(t)\int_\Omega h(x)u_1dx\,\|u\|^2_{2,\Omega}.\end{aligned} \tag{23}$$

Let us estimate the right side of identity (16) using the Hölder and Young inequalities:

$$\left|\int_\Omega h_1(x,t)u^2dx\right| \le h_{01}\,\|u\|^2_{2,\Omega}.$$

$$\left|e^{-1}(t)\int_\Omega u_1 u dx \int_\Omega h(x)udx\right| \le e_0\,\|h\|_{2,\Omega}\,\|u_1\|_{2,\Omega}\,\|u\|^2_{2,\Omega}.$$

$$\left|e^{-1}(t)\int_\Omega h(x)u_1dx\right|\|u\|^2_{2,\Omega} \le e_0\,\|h\|_{2,\Omega}\,\|u_1\|_{2,\Omega}\,\|u\|^2_{2,\Omega}.$$

Substituting the obtained estimates into the relation (23), we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|^2_{2,\Omega} + \|\nabla u\|^2_{2,\Omega}\right) + \|\nabla u\|^2_{2,\Omega} \le \left(h_{01} + 2e_0\,\|h\|_{2,\Omega}\,\|u_1\|_{2,\Omega}\right)\|u\|^2_{2,\Omega},$$

or

$$\frac{d}{dt}\left(\|u\|^2_{2,\Omega} + \|\nabla u\|^2_{2,\Omega}\right) \le 2\left(h_{01} + 2e_0\,\|h\|_{2,\Omega}\,\underset{t\in[0,T]}{ess\sup}\,\|u_1\|_{2,\Omega}\right)\left(\|u\|^2_{2,\Omega} + \|\nabla u\|^2_{2,\Omega}\right).$$

After integrating with respect to τ from 0 to T and applying Gronwall's lemma, considering the condition (21), we get $\|u\|^2_{2,\Omega} + \|\nabla u\|^2_{2,\Omega} \le 0, \ \ \forall t \in [0,T]$. From this estimate follows that $u(x,t) = 0 \ \ \Leftrightarrow \ \ u_1(x,t) = u_2(x,t)$.

3.1 Blow up of solution

Let us introduce the notation

$$\begin{aligned}&\varphi(t) = \|u\|^2_{2,\Omega} + \|\nabla u\|^2_{2,\Omega}, \\ &J(u) = \tfrac{1}{2}\textstyle\int_\Omega |\nabla u|^2dx - \tfrac{1}{2}\int_\Omega h(x)u^2dx - \int_0^t e^{-1}(\tau)e'(\tau)\int_\Omega uu_\tau dxd\tau + \int_0^t e^{-1}(\tau)\int_\Omega uu_\tau dx\int_\Omega h(x)udxd\tau.\end{aligned}$$

We calculate the derivative of the function $\varphi(t)$, and get

$$\begin{aligned}&\varphi'(t) = 2\textstyle\int_\Omega uu_tdx + \int_\Omega \nabla u\nabla u_tdx = -2\int_\Omega |\nabla u|^2dx + 2\int_\Omega h(x)u^2dx + \\ &+e^{-1}(t)e'(t)\textstyle\int_\Omega u^2dx - 2e^{-1}(t)\int_\Omega u^2dx\int_\Omega h(x)udx \ge \psi(t),\end{aligned} \tag{24}$$

where

$$\begin{aligned}\psi(t) = -2\alpha J(u) = -\alpha \int_\Omega |\nabla u|^2 dx + \alpha \int_\Omega h(x)u^2 dx + 2\alpha \int_0^t e^{-1}(\tau)e'(\tau) \int_\Omega uu_\tau dxd\tau - \\ -2\alpha \int_0^t e^{-1}(\tau) \int_\Omega uu_\tau dx \int_\Omega h(x)udxd\tau,\end{aligned} \tag{25}$$

where $\alpha > 2$.

Differentiating the last relation, we obtain

$$\begin{aligned}\psi'(t) = -2\alpha J(u) = -2\alpha \int_\Omega \nabla u \nabla u_t dx + 2\alpha \int_\Omega h(x)uu_t dx + 2\alpha e^{-1}(t)e'(t) \int_\Omega uu_t dx - \\ -2\alpha e^{-1}(t) \int_\Omega uu_t dx \int_\Omega h(x)udx = 2\alpha \left(\int_\Omega \Delta uu_t dx + \int_\Omega h(x)uu_t dx + e^{-1}(t)e'(t) \int_\Omega uu_t dx - \right. \\ \left. -e^{-1}(t) \int_\Omega uu_t dx \int_\Omega h(x)udx \right) = 2\alpha \left(\|u_t\|^2_{2,\Omega} + \|\nabla u_t\|^2_{2,\Omega} \right).\end{aligned} \tag{26}$$

Now consider the product $\varphi(t)\psi'(t)$where we can apply the Schwarz's inequality, as a result, we get

$$\begin{aligned}\varphi(t)\psi'(t) = 2\alpha \left(\|u\|^2_{2,\Omega} + \|\nabla u\|^2_{2,\Omega} \right) \left(\|u_t\|^2_{2,\Omega} + \|\nabla u_t\|^2_{2,\Omega} \right) \geq \\ \geq 2\alpha \left(\int_\Omega uu_t dx + \int_\Omega \nabla u \nabla u_t dx \right)^2 = \tfrac{\alpha}{2} [\varphi'(t)]^2.\end{aligned} \tag{27}$$

Moreover, from (26) we obtain that $\psi(t)$ is non-decreasing function, and if $J(u_0) < 0$, then $\psi(t) \geq 0$ for all $t > 0$. Taking into account the inequality (24), (27) takes the next form:

$$\varphi(t)\psi'(t) \geq \frac{\alpha}{2}\varphi'(t)\psi(t),$$

or

$$\frac{\psi'(t)}{\psi(t)} \geq \frac{\alpha}{2}\frac{\varphi'(t)}{\varphi(t)}. \tag{28}$$

We integrate (28) from 0 to t and using (24), we obtain

$$\frac{\varphi'(t)}{[\varphi(t)]^{\frac{\alpha}{2}}} \geq \frac{\psi(0)}{[\varphi(0)]^{\frac{\alpha}{2}}}.$$

Integrating the last inequality, we have

$$\frac{1}{[\varphi(t)]^{\frac{\alpha-2}{2}}} \leq \frac{1}{[\varphi(0)]^{\frac{\alpha-2}{2}}} - \frac{(\alpha-2)\psi(0)t}{2[\varphi(0)]^{\frac{\alpha}{2}}}. \tag{29}$$

It is obvious that the inequality (29) cannot be fulfilled for entire time t, and we conclude that the solution $u(x,t)$ blows up in a finite time T_*, where

$$T_* \leq \frac{2}{\alpha-2}\frac{\varphi(0)}{\psi(0)} = \frac{2\varphi(0)}{2\alpha(2-\alpha)J(u_0)}. \tag{30}$$

These results can be summarized as follows.

Theorem 2. Let the conditions (6) be satisfied, and also $h(x) \equiv (u_0(x))^{-1} [u_1(x) - \Delta u_1(x) - \Delta u_0(x)] > 0$,

$$J(u_0) = \frac{1}{2}\int_\Omega |\nabla u_0(x)|^2 dx - \frac{1}{2}\int_\Omega h(x)u_0^2(x)dx < 0,$$

then the solution $u(x,t)$ of the problem (11)-(13) blows up in a finite time T_*, that is $\lim\limits_{t\to T_0} \left(\|u\|^2_{2,\Omega} + \|\nabla u\|^2_{2,\Omega} \right) = \infty$, where T_* is limited to constant from (30).

3.2 Numerical solution

We consider the problem

$$\frac{\partial u}{\partial t}-\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial x^2}\right)-\frac{\partial^2 u}{\partial x^2}=u(x,t)\cdot h_1(x,\ t)-e^{-1}(t)\cdot u(x,t)\int_0^1 h_R(x)\cdot u(x,t)dx+R(x,t);$$

where

$$u_0(x)=e^{\frac{x^2}{2}-\frac{x^3}{3}};$$

$$e^{-1}(t)=e^{2t+t^2};$$

$$h_1(x,\ t)=-3+2x-\left(x-x^2\right)^2+(a_0+b_0)(2+2t);$$

$$h_R(x)=-3+2x-\left(x-x^2\right)^2;$$

$$R(x,t)=(2+2t)\left(1-2x+\left(x-x^2\right)^2\right)\cdot e^{-2t-t^2}\cdot e^{\frac{x^2}{2}-\frac{x^3}{3}};$$

$$a_0=\int_0^1 e^{\frac{x^2}{2}-\frac{x^3}{3}}dx; \qquad b_0=\int_0^1\left(1-2x+\left(x-x^2\right)^2\right)\cdot e^{\frac{x^2}{2}-\frac{x^3}{3}}dx.$$

For the given problem, the following initial and boundary conditions are used: $u(x,0)=u_0(x)$, $u(0,t)=0$, $u(1,t)=0$.

The problem is solved by using numerical methods. To check the obtained solutions, the numerical solution is compared with the analytical solution, so the following is analytical solution:

$$u(x,t)=e^{-2t-t^2}\cdot e^{\frac{x^2}{2}-\frac{x^3}{3}}.$$

The numerical solution of this problem is based on finite difference approximation method.

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}-\left(\frac{(u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1})}{\Delta t\Delta x^2}-\frac{(u_{i+1}^{n}-2u_i^{n}+u_{i-1}^{n})}{\Delta t\Delta x^2}\right)-\frac{(u_{i+1}^{n}-2u_i^{n}+u_{i-1}^{n})}{\Delta x^2}=$$

$$=u(x_i,t_n)\cdot h_1(x_i,t_n)-e^{-1}(t_n)\cdot u(x_i,t_n)\int_0^1 h_R(x_i)\cdot u(x_i,t_n)dx+R(x_i,t_n).$$

The basis of this numerical method is the sweep algorithm, where we can obtain analogue such as

$$Au_{i+1}^{n+1}+Bu_i^{n+1}+Cu_{i-1}^{n+1}=d_i;$$

where $A=(-1),\ B=(2+\Delta x^2),\ C=(-1)$;

$$d_i=\Delta x^2\cdot u_i^n+(u_{i+1}^n-2u_i^n+u_{i-1}^n)\cdot(\Delta t-1)+\Delta t\cdot\Delta x^2\cdot$$

$$\left(u(x_i,t_n)h_1(x_i,t_n)-e^{-1}(t_n)u(x_i,t_n)\int_0^1 h_R(x_i)\cdot u(x_i,t_n)dx+R(x_i,t_n)\right).$$

The problem is reduced to solving the system of linear equations:

$$\begin{cases} Bu_1^{n+1} + Au_2^{n+1} = d_1, \ i = 1, \\ Cu_{i-1}^{n+1} + Bu_i^{n+1} + Au_{i+1}^{n+1} = d_i, \ i = 1, 2, ..., N-1, \\ Cu_{N-1}^{n+1} + Bu_N^{n+1} = d_N, \ i = N. \end{cases}$$

Finally, the numerical solution is sought in the form $u_i^{n+1} = \alpha_i u_{i+1}^{n+1} + \beta_i$; where

$$\alpha_i = -\frac{A}{B + C \cdot \alpha_{i-1}}, \ \beta_i = \frac{d_i - C \cdot \beta_{i-1}}{B + C \cdot \alpha_{i-1}}.$$

To determine $\alpha_0, \ \beta_0, \ u_N^{n+1}$, we use boundary conditions.

Numerical results:

At $x = \frac{1}{40}$:

X	$Numerical$	$Analytical$
0.000000	0.041108	0.049787
0.025000	0.041108	0.049802
0.050000	0.041134	0.049847
0.075000	0.041182	0.049920
0.100000	0.041254	0.050020
0.125000	0.041347	0.050145
0.150000	0.041459	0.050294
0.175000	0.041591	0.050465
0.200000	0.041740	0.050658
0.225000	0.041906	0.050870
0.250000	0.042087	0.051101
0.275000	0.042282	0.051348
0.300000	0.042490	0.051612
0.325000	0.042710	0.051890
0.350000	0.042941	0.052181
0.375000	0.043180	0.052483
0.400000	0.043428	0.052795
0.425000	0.043683	0.053116
0.450000	0.043943	0.053444
0.475000	0.044208	0.053777
0.500000	0.044475	0.054114
0.525000	0.044744	0.054453
0.550000	0.045012	0.054792
0.575000	0.045280	0.055130

X	$Numerical$	$Analytical$
0.600000	0.045544	0.055465
0.625000	0.045804	0.055795
0.650000	0.046058	0.056118
0.675000	0.046305	0.056433
0.700000	0.046543	0.056737
0.725000	0.046769	0.057028
0.750000	0.046984	0.057305
0.775000	0.047184	0.057565
0.800000	0.047369	0.057806
0.825000	0.047537	0.058026
0.850000	0.047685	0.058224
0.875000	0.047813	0.058397
0.900000	0.047919	0.058543
0.925000	0.048000	0.058660
0.950000	0.048056	0.058745
0.975000	0.048085	0.058798
1.000000	0.048085	0.058816

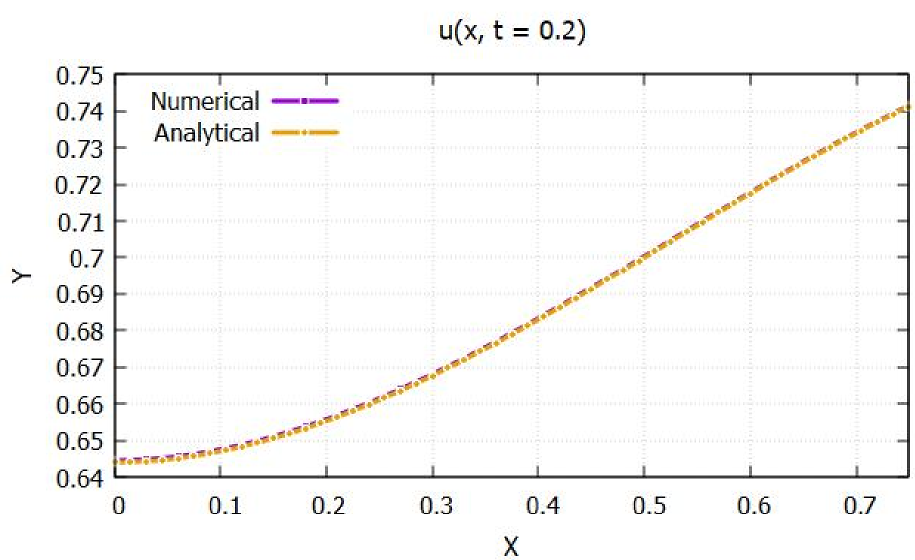

Figure 1: Comparing the analytical (exact) solution $u(x, t)$ and the numerical solution $w(x, t)$

Error Analysis:

$$z^i = |u(x_i, t_n) - w(x_i, t_n)|\,, \quad where\ u(x,t)\ is\ analytical\ solution,$$

$$w(x,t)\ is\ numerical\ solution, \quad n = 100.$$

Δt	Δx	$max_{0<i<N+1}\left[z^i\right]$
10^{-2}	1/10	0.027061
	1/20	0.0230092
	1/40	0.0183683
	1/80	0.0149184
	1/100	0.0141077
	1/160	0.0128041
10^{-3}	1/10	0.164578
	1/20	0.0860432
	1/40	0.043323
	1/80	0.0211665
	1/100	0.0166748
	1/160	0.00990058
10^{-4}	1/10	0.17669
	1/20	0.091649
	1/40	0.046468
	1/80	0.0232889
	1/100	0.0186095
	1/160	0.0115641
10^{-5}	1/10	0.177748
	1/20	0.0921916
	1/40	0.0468258
	1/80	0.0235722
	1/100	0.0188793
	1/160	0.0118145

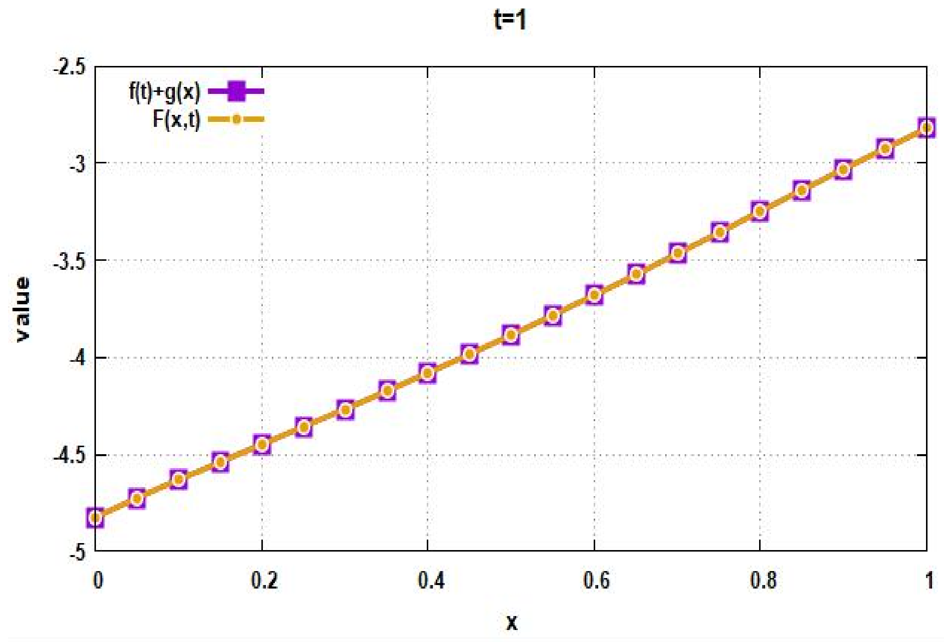

Figure 2: Comparing F(x,t) and f(t)+g(x)

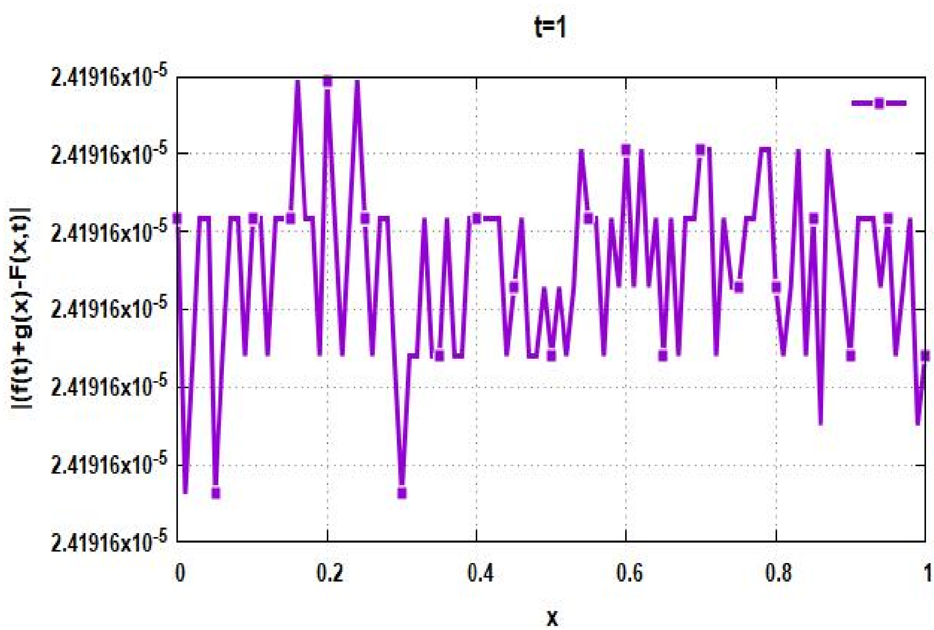

Figure 3: Absolute value change depending on the space

4 Conclusion

In this paper, the existence and uniqueness of the solution of the inverse coefficient problem are proved. The existence of the solution is proved by the Faedo-Galerkin method. The uniqueness of the solution is proved from the a priori estimates obtained. Sufficient conditions for the blow up of the solution in finite time in a bounded domain are established. An algorithm for the numerical solution of this problem is constructed and numerical experiments are carried out illustrating the theoretical calculations obtained in the work.

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