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ON THE PROCESS OF TWO IMMISCIBLE LIQUIDS SEPARATED BY A CONTACT SURFACE WITHOUT SURFACE TENSION

The processes of immiscible liquids in porous media are one of the current topics in the modern world, where advanced technologies are used to obtain the most comprehensive information about the geological and geophysical properties of reservoirs. The process of separating two immiscible liquids at a contact surface without surface tension describes the phenomenon when two different types of liquids are in direct contact with each other without the formation of an interface boundary or surface tension between them. This phenomenon can be observed when certain conditions are met, and it is important in various scientific and engineering fields. It is well-known that all hydrodynamic processes are described by mathematical tools and models, and solving such problems allows for obtaining numerical solutions for practical applications in the future. The authors of the article present the problem statement of two immiscible liquids separated by a contact surface without surface tension. For the adequacy of this problem, the presence and singularity of a classical solution have been proven, which depend on the location of the unfixed boundary. The solution demonstrates the existence of a continuous boundary that divides the region into sections containing by two different liquids, where the initial density distribution is a smooth function.
Key words: mathematical modeling, porous media, filtration, immiscible liquids, Darcy's law.

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Беттік керілуі жоқ түйісілген бетімен бөлінген екі араласпайтын сұйықтар үдерісі туралы

Кеуекті орталардағы араласпайтын сұйықтықтардың процестері қазіргі заманғы әлемдегі өзекті тақырыптардың бірі болып табылады, мұнда қабаттардың геологиялық және геофизикалық қасиеттері туралы ең толық ақпаратты алу мүмкіндігі бар озық технологиялар қолданылады. Беттік керілусіз жанасу бетімен араласпайтын екі сұйықтықтың бөліну процесі екі түрлі сұйықтық түрлерінің олардың арасындағы шекаралық шекара немесе беттік керілу пайда болмай бір-бірімен тікелей жанасу құбылысын сипаттайды. Бұл құбылысты белгілі бір шарттар орындалған жағдайда байқауға болады және әртүрлі ғылыми және инженерлік салаларда маңызды. Барлық гидродинамикалық процестер математикалық аппараттар мен модельдермен сипатталатыны белгілі, дегенмен мұндай есептерді шешу болашақта практикалық қолдану үшін сандық шешім алуға мүмкіндік береді. Осы мақаланың авторлары беттік керілусіз жанасу бетімен бөлінген екі араласпайтын сұйықтық мәселесінің есебінің қойылымы ұсыналады. Бұл есептің қойылымының дұрыстығы үшін еркін шекараның позициясына тәуелді классикалық шешімнің бар болуы мен бірегейлігі дәлелденген. Шешім аймақты екі түрлі сұйықтықтар алып жатқан бөліктерге бөлетін тегіс беттің бар екенін дәлелдейді, мұнда бастапқы тығыздықтың таралуы тегіс функция болып табылады.

Түйін сөздер: математикалық модельдеу, кеуекті орта, сүзгілеу, араласпайтын сұйықтықтар, Дарси заңы

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О процессе двух несмешивающихся жидкостей разделенные поверхностью контакта без поверхностного натяжения

Процессы несмешивающихся жидкостей в пористых средах являются одним актуальных тем современной мире, где применяется передовые технологий имеющие возможности, чтобы получить наиболее полную информацию геолого-геофизические свойства пластов. Процесс разделения двух несмешивающихся жидкостей с поверхностью контакта без поверхностного натяжения описывает явление, когда два различных типа жидкостей находятся в прямом контакте друг с другом без образования интерфейсной границы или поверхностного натяжения между ними. Это явление может наблюдаться, если определенные условия удовлетворяются, и оно важно в различных научных и инженерных областях. Всеми известно, что все гидродинамические процессы описываются математическими аппаратами и моделями, а решение таких задач дает возможность получить в дальнейшем численное решение для практических применений. Авторами настоящей статьи приводиться постановка задачи двух несмешивающихся жидкостей, которые разделенные поверхностью контакта без поверхностного натяжения. Для адекватности данной задачи доказана существование и единственности классического решения, которая зависит от положения свободной границы. В решении доказывается существование гладкой поверхности, разделяющей область на части занятые двумя различными жидкостями, где начальное распределение плотности является гладкой функцией.

Ключевые слова: математическое моделирование, пористая среда, фильтрация, несмешивающиеся жидкости, закон Дарси.

1 Introduction

Existence theorems for a generalized solution of the Navier-Stokes system concerning inhomogeneous incompressible fluids can be located in various academic references, including those referenced as [1]- [5]. These works offer valuable insights into the problem at hand. However, it's important to note that while these references provide descriptions of solutions, they often do so without a comprehensive examination of the dataset. Furthermore, a notable limitation is that they do not consider the crucial aspect of density continuity in their analysis. In the quest to fully understand the dynamics of inhomogeneous incompressible fluids and the behavior of the Navier-Stokes system, it becomes evident that a more detailed investigation, which encompasses the continuous nature of density, is necessary. This continuity of density is a fundamental factor that can significantly influence the solutions and outcomes of the system. Consequently, future research endeavors should aim to bridge this gap by incorporating considerations for the continuity of density, as it is a crucial aspect in achieving a comprehensive understanding of these complex fluid dynamics.

In [6], the process of multi-fluid flow is considered, where a proof of weak solutions for the Navier-Stokes equation in the time domain is provided. In this problem, classical immiscibility conditions are imposed at the boundaries.

In [7], the process of flow of inhomogeneous viscous, incompressible liquids is considered, where the physical meaning of this problem is described by partial differential equations. For solving this problem, the authors state that there is no need for the existence of global solutions of this model to have initial density conditions with a positive lower bound.

In [8], the filtration of two immiscible viscous liquids with different densities is considered, formulating a problem with a free boundary, where the movement is characterized by the equations of Stokes. The authors provide proofs of the existence and uniqueness of classical solutions to the Stokes equation with homogeneous Dirichlet boundary conditions.

The matter of viscoelastic filtration in the microscopic domain within the context of the Muskat problem was discussed in detail in reference [9], utilizing suitable averaging techniques. The solvability of this mathematical model is proven, where the process of immiscible, incompressible two-fluid flow is derived from homogenization theory in the limit of dimensionless pore size.

2 The aim and objectives of the study

Examine the motion of two non-mixing viscous fluids characterized by distinct densities constants in a capillary $\Omega = \{\mathbf{x} \in \mathbb{R}^2 : -1 < x_1 < 1, -h < x_2 < h\}$. The motion is driven by external pressure and gravity forces. The moving interface, which naturally appears, separates subdomains $\Omega^+(t)$ and $\Omega^-(t)$filled with different liquids.

To be more precise, it is necessary to resolve the task of determining velocity $\mathbf{u} = (u_1, u_2) \in \mathbb{R}^2$, pressure $p \in \mathbb{R}$, and density $\rho \in \mathbb{R}$ by solving the system of equations pertaining to velocity and pressure

$$\mu \Delta \mathbf{u} - \nabla p + g\rho \mathbf{e} = 0, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{2}$$

where μ represents the fluid's viscosity, $\mathbf{e}$ is a specified unit vector, and g is the acceleration due to gravity force. Additionally, there's a density transport equation.

$$\frac{d\rho}{dt} \equiv \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = 0, \tag{3}$$

The parameter t is introduced into the velocity equation, making the initial condition unnecessary.

The boundary conditions applied to the lateral surfaces $S^0 = \{\mathbf{x} \in \mathbb{R}^2 : -1 < x_1 < 1, x_2 = \pm h\}$ of the boundary $S = \partial\Omega$ are expressed in the

$$\mathbf{u}(\mathbf{x}, t) = 0. \tag{4}$$

The boundary conditions at «input» $S^- = \{\mathbf{x} \in \mathbb{R}^2 : x_1 = -1, -h < x_2 < h\} \subset S$ and «output» boundaries $S^+ = \{\mathbf{x} \in \mathbb{R}^2 : x_1 = 1, -h < x_2 < h\} \subset S$ are as follows:

$$P(\mathbf{u}, p) \cdot \mathbf{n} = -p^0 \mathbf{n}, \ \mathbf{x} \in S^\pm. \tag{5}$$

Here $P(\mathbf{u}, p) = 2\mu D(\mathbf{u}) - pI$, $D(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^*)$,, where Irepresents the identity tensor., $p^0(\mathbf{x})$ is a known linear function, and $\mathbf{n} = (1, 0)$ represents the unit normal vector pointing towards to $S^\pm$

3 Materials and methods

At the initial time $t = 0$, the density is a piecewise-constant function assumed to be equal to two positive numbers describing different flow phases:

$$\rho(\mathbf{x}, 0) = \rho_0(\mathbf{x}) = \begin{cases} \rho^+, & \mathbf{x} \in \Omega^+(0), \\ \rho^-, & \mathbf{x} \in \Omega^-(0), \end{cases} \quad \rho^\pm = \text{const}, \ \rho^- > \rho^+ > 0. \tag{6}$$

Then, initial conditions for density are equivalent to the surface $\Gamma(0) = \Gamma_0$, which separates two subdomains $\Omega^\pm(0)$ initially occupied by different fluids. For simplicity, assume $\Gamma_0 = \{\mathbf{x} \in \mathbb{R}^2 : x_1 = 0, -h < x_2 < h\}$.

The problem is reduced to finding $\mathbf{u}$, p and density $\rho(\mathbf{x}, t)$ from equations $1 - 3$ satisfying initial and boundary conditions. Note that this problem is nonlinear due to the term $\mathbf{u} \cdot \nabla\rho$ in equation 3.

To simplify our reasoning, we will proceed to homogeneous boundary conditions by introducing a new $p \to p - p^0(\mathbf{x})$:

$$\mu\Delta\mathbf{u} - \nabla p = \mathbf{f} \equiv \nabla p^0 - g\rho\mathbf{e}, \tag{7}$$

$$(\mathbf{u}, p) \cdot \mathbf{n} = 0, \mathbf{x} \in S^\pm. \tag{8}$$

Below, it will be demonstrated that the dynamics depicted by the previously mentioned equations maintain the presence of two distinct subdomains $\Omega^\pm(t)$, each housing one of the fluids, and these are consistently divided by an unchanging free boundary $\Gamma(t)$ throughout any given timeframe $t > 0$. Consequently, the problem under investigation is analogous to the task of determining values for $\mathbf{u}$, p and the shifting boundary $\Gamma(t)$.

4 Results

A theorem establishing the existence and singularity of a classical solution.

Let $\Omega^{(m)} = \{\mathbf{x} \in \Omega : -1 + \frac{1}{m} < x_1 < 1 - \frac{1}{m}\}$, $m > 0$. The key findings are presented in the subsequent theorem.

Theorem. The problem 2-4, 6-8 has a unique solution on the interval $[0, T)$ for some $T > 0$, and the solution satisfies the subsequent characteristics: a) For any positive $m \in N$,$q > 2$ и $\lambda = 1 - \frac{2}{q}$, the velocity u satisfies the following relations

$$\mathbf{u} \in L_\infty(0, T; W^{2,q}(\Omega^{(m)})) \cap L_\infty(0, T; C^{1,\lambda}) \cap C^{0,\lambda}(0, T; C^{1,\lambda}).$$

b) Free boundary $\Gamma(t)$ is a sufface to the class $C^{1,\lambda}$ for any $t \in [0, T)$ and normal velocity at every location $\mathbf{x}$along the free boundary $V_n(x, t)$, the magnitude of the normal n direction is consistently limited,

$$\sup_{\substack{t \in (0,T) \\ \mathbf{x} \in \Gamma(t)}} |V_n(\mathbf{x}, t)| < \infty.$$

c) The density ρ has bounded variation,

$$\rho \in L_\infty(0, T; BV(\Omega^{(m)})) \cap BV(\Omega_T^{(m)}).$$

Here $\Omega_T = \Omega \times (0, T)$.

The existence time T of the classical solution depends on the position of free boundary $\Gamma(t)$. Specifically, let $\delta^{\pm}(t)$ be the distance between $\Gamma(t)$ and the boundaries $S^{\pm}$ and $\delta(t) = min(\delta^{-}(t), \delta^{+}(t))$. Then $\delta(t) > 0$ for all $0 < t < T$ and $\delta(t) \to 0$ for $t \to T$.

Entire this work, commonly accepted notations for functional spaces and norms are used [10]. And where $W^{2,q}(\Omega)$ is the Sobolev space of functions with derivatives summable with degree q, $C^{k,\lambda}$ represents the space of functions, the k^{th} derivatives of which conform the Hölder condition with degree λ.

5 Discussion of results

Proof of Theorem. Firstly, we will show that, given the initial density distribution as a smooth function $\rho_0 \in C^{\infty}(\Omega)$, the problem of determining $\mathbf{u}$ has at least one classical solution. This conclusion is derived from the application of the Schauder fixed-point theorem.

Next, a class of functions with specific regularity properties is specified, and compactness principles are used to show the convergence of smooth solutions to the solution of the original problem with piecewise-constant density ρ_0. The existence of a smooth surface separating the domain into regions occupied by two different fluids is proven.

In this work we fix the number $q > 2$, an integer m, and the initial density distribution $\rho_0^{(\varepsilon)} \in C^{\infty}(\Omega)$, $\varepsilon > 0$. Specifically we assume $\rho_0^{(\varepsilon)}(\mathbf{x}) = \rho^-$ for $-1 < x_1 < -\varepsilon$, $\rho_0^{(\varepsilon)}(\mathbf{x}) = \rho^+$ for $\varepsilon < x_1 < 1$, and $\rho^- \le \rho_0^{(\varepsilon)}(\mathbf{x}) \le \rho^+$.

For simplicity of the above entry, we omit the index ε.

The class of functions M consists of continuous functions

$$\tilde{\rho} \in C(\overline{\Omega_T})$$

satisfying the condition

$$\rho^- \le \tilde{\rho}(\mathbf{x}, t) \le \rho^+. \tag{9}$$

Next, we establish the linear operators as follows, where the initial operator converts the fixed density into its corresponding velocity field: $M \ni \tilde{\rho} \mapsto \mathbf{v} = \mathbf{U}[\tilde{\rho}] \in L_\infty(0, T; W^{2,q}(\Omega^{(m)}))$.

The second one describes the movement of density under the influence of the «frozen» velocity field:

$$L_\infty(0, T; W^{2,q}(\Omega^{(m)})) \ni \mathbf{U} \mapsto \rho = R[\rho_0, \mathbf{v}] \in L_\infty(\Omega_T).$$

The operator $\mathbf{U}$ transfers $\tilde{\rho}$ to the solution of the following problem

$$\mu \Delta \mathbf{v} - \nabla p = \tilde{\mathbf{f}} \equiv \nabla p^0 - g\tilde{\rho}\mathbf{e}, \mathbf{x} \in \Omega, 0 < t < T, \tag{10}$$

$$\nabla \cdot \mathbf{v} = 0, \mathbf{x} \in \Omega, 0 < t < T, \tag{11}$$

$$\mathbf{v}(\mathbf{x}, t) = 0, \mathbf{x} \in S^0, 0 < t < T, \tag{12}$$

$$(\mathbf{v}, p) \cdot \mathbf{n} = 0, \mathbf{x} \in S^{\pm}, 0 < t < T. \tag{13}$$

The operator $\rho = R[\rho_0, \mathbf{v}]$, contingent upon the initial density ρ_0, converts the velocity $\mathbf{v}$ into the solution for the Cauchy problem:

$$\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho = 0, \mathbf{x} \in \Omega, 0 < t < T, \tag{14}$$

$$\rho(\mathbf{x}, 0) = \rho_0(\mathbf{x}), \mathbf{x} \in \Omega. \tag{15}$$

In this problem we will limit ourselves to a time interval T_m, meeting the subsequent requirements:

$$\rho(\mathbf{x}, t) = \rho^-, for - 1 < x_1 < -1 + \frac{1}{m}, 0 < t < T_m,$$

$$\rho(\mathbf{x}, t) = \rho^+, for 1 - \frac{1}{m} < x_1 < 1, 0 < t < T_m, \tag{16}$$

where $\tilde{\rho} \in M$ and $\varepsilon > 0$ are occur in a way that

$$\rho(\mathbf{x}, t) \neq \rho^-, for - 1 < x_1 < -1 + \frac{1}{m}, t > T_m,$$

or

$$\rho(\mathbf{x}, t) \neq \rho^+, 1 - \frac{1}{m} < x_1 < 1, t > T_m \tag{17}$$

for $\rho(\mathbf{x}, t) = R[\rho_0^\varepsilon, \mathbf{v}]$, $\mathbf{v} = \mathbf{U}[\tilde{\rho}]$ and for any $\tilde{\rho} \in M$ and $\varepsilon > 0$.

It is obvious that

$$T_m \leq T_{m+1} \forall m > 0. \tag{18}$$

When dealing with a smooth initial density distribution represented by ρ_0, solving for $\mathbf{u}$ and ρ from equations 2 to 4, 6 to 8 involves identifying a fixed point within the composition of two linear operators. This composite operator, denoted as $F = R \circ \mathbf{U}$, is defined as:

$$M \ni \tilde{\rho} \mapsto F[\tilde{\rho}] = (R \circ \mathbf{U})[\tilde{\rho}] \stackrel{def}{=} R[\rho_0, \mathbf{U}[\tilde{\rho}]] \in M.$$

The Schauder fixed-point theorem's prerequisites are met for the operator F within the interval $(0, T_m)$, where $T_m > T_0$ and $T_0 > 0$, and these conditions remain unaffected by the values of m, $\tilde{\rho} \in M$ and ε.

Everywhere below C is a positive constant that does not depend on m and ε, K is a positive constant independent ofε.

For each $\tilde{\rho} \in M$, the linear problem $11 - 14$ has a unique solution

$$\mathbf{v} \in L_\infty(0, T; W^{1,2}(\Omega)) \cap L_\infty(0, T; W^{2,q}(\Omega^{(m)})),$$

$$p \in L_\infty(0, T; L_2(\Omega)) \cap L_\infty(0, T; W^{1,q}(\Omega^{(m)})),$$

For any given $q \in (1, \infty)$ and for every possible value of the parameter $t \in [0, T]$ the following estimates hold for the solution

$$\|p(t)\|_{L_2(\Omega)} + \|\mathbf{v}(t)\|_{W^{1,2}(\Omega)} \leq C \left\| \tilde{\mathbf{f}}(t) \right\|_{L_2(\Omega)} \leq C, \tag{19}$$

$$\|p(t)\|_{W^{1,q}(\Omega^{(m)})} + \|\mathbf{v}(t)\|_{W^{2,q}(\Omega^{(m)})} \leq K \left\| \tilde{\mathbf{f}}(t) \right\|_{L_q(\Omega)}. \tag{20}$$

6 Conclusion

To summarize, it can be affirmed that in this particular problem, there exists a unique classical solution, and its characteristics vary in accordance with the location of the free boundary, has been successfully demonstrated. In this case, during the solution process, the presence of a continuous surface was established, dividing the region into two parts filled with different liquids, and the initial density distribution remains a smooth function.

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