IRSTI 27.31.00; 27.31.15 DOI: https://doi.org/10.26577/JMMCS2023v119i3a3

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CONDITIONS FOR THE EXISTENCE OF AN "ISOLATED"SOLUTION OF A BOUNDARY VALUE PROBLEM FOR A SEMILINEAR LOADED HYPERBOLIC EQUATION

Boundary value problems for hyperbolic equations are an important area of mathematical physics and science in nature. They arise in various physical and engineering contexts and have a wide range of applications, including wave propagation in elastic media, electromagnetic waves, as well as problems related to fluid and gas motion. In this article, we will focus on one of the significant subclasses of hyperbolic equations, namely, semi-linear loaded hyperbolic equations, and examine the conditions for the existence of isolated solutions to boundary value problems for such equations.Semi-linear loaded hyperbolic equations are equations in which nonlinear terms depend on the solutions themselves. This makes their study more complex and mathematically intriguing. Our task is to find conditions under which such equations have isolated solutions, meaning solutions that exist in a bounded region of space and time and remain bounded themselves.Studying the conditions for the existence of isolated solutions for semi-linear loaded hyperbolic equations is of significant importance both in theory and practical applications. In this article, we will explore various approaches and methods used to analyze.
Key words: isolated solution, boundary value problem, loaded hyperbolic equation, semilinear hyperbolic equation, semi-periodic boundary value problem.

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Жартылай сызықтық жүктелген гиперболалық теңдеу үшін шекаралық есептің оқшауланған"шешімінің бар болуының шарттары

Гиперболалық теңдеулер үшін шеттік есептер математикалық физика мен жаратылыстанудың маңызды саласы болып табылады. Олар әртүрлі физикалық және серпімді толқындардың таралуын, электромагниттік толқындарды және сұйық пен газдардың қозғалысы мәселелерімен бірге кең ауқымды қолданбаларға ие. Бұл мақалада гиперболалқ теңдеулердің маңызды класстарының бірі жартылай сызықты жүктелген гиперболалық теңдеулер үшін шеттік есебі қарастырылады. Атап айтқанда олардың оқшауланған шешімдерінің бар болу шарттары зерттелінеді. Жартылай сызықты жүктелген гиперболалық теңдеулер сызықты емес мүшелері шешімдердің өзіне тәуелді болатын теңдеулер. Бұл олардың математикалық зерттеу тұрғысынан күрделірек және қызықты етеді. Біздің міндетіміз мұндай теңдеулердің "оқшауланған"шешімінің, яғни шектеулі аймақта шешімдері бар және шектеулі болып табылатын шарттарын алу. Жартылай сызықты жүктелген гиперболалық теңдеулер үшін оқшауланған шешімдердің бар болу шарттарын зерттеу теория және практикалық тұрғыдан да, қолдану тұрғысынан да маңызды. Мақалада осы мәселелерді талдау үшін қолданылатын әртүрлі әдістер мен тәсілдер қарастырылады.
Түйін сөздер: оқшауланған шешім, шекаралық есеп, жүктелген гиперболалық теңдеу, жартылай сызықты гиперболалық теңдеу, жартылай периодты шекаралық есеп.

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Условия существования "изолированного"решения краевой задачи для полулинейного нагруженного гиперболического уравнения

Краевые задачи для гиперболических уравнений являются важной областью математической физики и науки о природе. Они возникают в различных физических и инженерных контекстах и имеют широкий спектр приложений, включая распространение волн в упругих средах, электромагнитные волны, а также в задачах, связанных с движением жидкости и газа. В данной статье мы сосредотачим внимание на одной из важных подклассов гиперболических уравнений, а именно полулинейных нагруженных гиперболических уравнениях, и рассмотрим условия существования изолированных решений краевых задач для таких уравнений. Полулинейные нагруженные гиперболические уравнения представляют собой уравнения, в которых нелинейные члены зависят от самих решений. Это делает их изучение более сложным и интересным с математической точки зрения. Наша задача — найти условия, при которых такие уравнения имеют изолированные решения, то есть решения, существующие в ограниченной области пространства и времени и сами остающиеся ограниченными. Изучение условий существования изолированных решений для полулинейных нагруженных гиперболических уравнений имеет значительное значение как с точки зрения теории, так и с точки зрения практических приложений. В этой статье мы рассмотрим различные подходы и методы, используемые для анализа таких задач.
Ключевые слова: изолированное решение, краевая задача, нагруженное гиперболическое уравнение, полулинейное гиперболическое уравнение, полупериодическая краевая задача.

1 Introduction

In many instances, hyperbolic equations prove to be complex and challenging to comprehend, especially when considering loaded cases where external factors or boundary conditions affect the system. It is crucial to understand that loaded hyperbolic equations have a broad spectrum of applications, from weather forecasting to aerodynamic system design, and their comprehension plays a critical role in solving complex engineering and scientific challenges.Loaded points are points in space or on the boundary of the domain in which a hyperbolic equation is considered, subject to the influence of external factors such as sources or flows. These points can have a significant impact on the dynamics of the system and the formulation of the solution to the hyperbolic equation. It is important to understand that loaded points can arise in both natural and engineering and scientific contexts, and their consideration is a key aspect when modeling and analyzing complex physical phenomena.Several ways in which loaded points affect the solution of hyperbolic equations.Sources and Sinks: Loaded points can represent sources or sinks that introduce or remove mass, energy, or other physical quantities from the system. This can alter the local distribution of parameters and, consequently, the dynamics of the system.Systems with External Influences: In real systems, such as aerodynamic systems or electromagnetic waves, loaded points can represent external objects or actions that affect the field of variables within the system. Modeling and Analysis: Including loaded points in mathematical models allows for a more accurate representation of real conditions and influences, which can be important for more precise forecasting and analysis. In [1], issues related to loaded equations and their applications are investigated.The computational method for solving boundary value problems for loaded integro-differential equations and the correct solvability of boundary value problems for loaded differential equations were studied in works [2],[3]. Various problems for loaded differential equations and methods for finding their solutions are considered in [4-9].

1.1 Problem statement

In the domain $\bar{\Omega} = [0,\omega] \times [0,T]$ we considered semi-periodic boundary value problem for a semi-linear loaded hyperbolic equations:

$$\frac{\partial^2 u}{\partial x \partial t} = A(x,t)\frac{\partial u(x,t)}{\partial x} + A_0(x,t)\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=x_0} + f\Big(x,t,u(x,t),\frac{\partial u(x,t)}{\partial t}\Big), \tag{1}$$

$$u(x,0) = u(x,T), \qquad x \in [0,\omega], \tag{2}$$

$$u(0,t) = \psi(t), \qquad t \in [0,T], \tag{3}$$

where $f : \overline{\Omega} \times R^2 \to R$, continuous on $\bar{\Omega}$, $\psi(t)-$ continuously differentiable on $[0,T]$ and satisfying the condition $\psi(0) = \psi(T)$ functions.

Function $u(x,t) \in C(\overline{\Omega})$, having partial derivatives $\dfrac{\partial u(x,t)}{\partial x} \in C(\overline{\Omega})$, $\dfrac{\partial u(x_0,t)}{\partial x} \in C(\overline{\Omega})$, $\dfrac{\partial u(x,t)}{\partial t} \in C(\overline{\Omega})$, $\dfrac{\partial^2 u(x,t)}{\partial x \partial t} \in C(\overline{\Omega})$, is called a classical solution of problem (1)-(3), if it satisfies equation (1) for all $(x,t) \in \overline{\Omega}$ and the boundary conditions (2)-(3).

We take the function $u^{(0)}(x,t) \in C^{1,1}_{x,t}(\overline{\Omega})$ having a continuous mixed derivative of the second order, a number $\rho > 0$ and construct set

$$\begin{aligned} G(u^{(0)},\rho) &= \{(x,t,u,w,v) : (x,t) \in \overline{\Omega}, |u - u^{(0)}(x,t)| < \rho, |w - u^{(0)}_t(x,t)| < \rho\}, \\ S(u^{(0)},\rho) &= \{u(x,t) \in C^{1,1}_{x,t}(\overline{\Omega}) : \|u - u^{(0)}(x,\cdot)\|_1 < \rho\}. \end{aligned}$$

Through $V^0(f, L_1(x,t), L_2(x,t))$ defining totality $(u^{(0)}(x,t),\rho)$, under which the function $f(x,t,u,w)$ in $G(u^{(0)},\rho)$ has uniformly continuous partial derivatives $f_u(x,t,u,w), f_w(x,t,u,w)$ and execute the inequalities: $|f_u(x,t,u,w)| \leq L_1(x,t)$,
$|f_w(x,t,u,w)| \leq L_2(x,t)$, where $L_i(x,t)$, inequalities to $\overline{\Omega}$ function, $i = 1,2$.

Suppose, that the function f into the set $G(u^{(0)},\rho)$ has uniformly continuous partial derivatives with respect to u, w.

Definition 1. *The solution $u^*(x,t)$ tasks (1)-(2) called "isolated if exists continuous on $[0,\omega]$ function $\rho_0(x) > 0$, for which function f in $G(u^*,\rho_0) = \{(x,t,u,w,v) : (x,t) \in \overline{\Omega}$, $|u - u^*(x,t)| \leq \rho_0(x), \big|w - \dfrac{\partial u^*(x,t)}{\partial t}\big| \leq \rho_0(x)\}$ has uniformly continuous partial derivatives f_u, f_w and linear field periodic boundary value problem*

$$\frac{\partial^2 u}{\partial x \partial t} = A(x,t)\frac{\partial u}{\partial x} + A_0(x,t)\left.\frac{\partial u}{\partial x}\right|_{x=x_0} + f_w\Big(x,t,u^*(x,t),\frac{\partial u^*(x,t)}{\partial t}\Big)\frac{\partial u(x,t)}{\partial t} +$$

$$+ f_u\Big(x,t,u^*(x,t),\frac{\partial u^*(x,t)}{\partial t}\Big)u(x,t) + \widetilde{f}(x,t),\ (x,t) \in \overline{\Omega}, \tag{4}$$

$$u(0,t) = \psi(t), \quad t \in [0,T] \tag{5}$$

$$u(x,0) = u(x,T), \quad x, x_0 \in [0,\omega] \tag{6}$$

for any $\widetilde{f} \in C(\overline{\Omega})$, $\psi \in C^1\big([0,T]\big)$ has one solution.

1.2 Linearized problem

We introduce the concept of a linearizer of the operator E at the point $\hat{x} \in D(E)$, generalizing the derivative of Freshet [10, P. 637] to unbounded nonsmooth operators [11, P. 638].

Definition 2. *The linear operator $C : \overline{X} \to \overline{Y}$ is called the linearizer of the operator $E : \overline{X} \to \overline{Y}$ at the point $\hat{x} \in D(E)$, if $D(E) \subseteq D(C)$ and existing number $\varepsilon \geq 0$, $\delta > 0$, such that*

$$\|E(x) - E(\hat{x}) - C(x - \hat{x})\|_Y \leq \varepsilon \|x - \hat{x}\|_X,$$

for all $\hat{x} \in D(E)$, satisfying the inequality $\|x - \hat{x}\|_X < \delta$, where $D(E)$, $D(C)-$ domains operator E and C, $\overline{X}-$ Banach Space, $\overline{Y}-$ linear normalized space.

Consider the boundary value problem (1)-(2) with the condition on the characteristic

$$u(0,t) = 0, \quad t \in [0,T]. \tag{7}$$

Let $X-$ be the space of functions from $C^{1,1}_{x,t}(\overline{\Omega})$, satisfying the boundary conditions (2), (3), Y - the space of continuous functions (x,t) on $\overline{\Omega}$ functions with norm $\|\widetilde{f}(x,t)\|_3 = \max_{\overline{\Omega}} |\widetilde{f}(x,t)|$, $H = \dfrac{\partial^2}{\partial x \partial t}$, $F(u) = -A(x,t)u_x(x,t) - A_0(x,t)\, u_x(x,t)|_{x=x_0} - f(x,t,u(x,t),u_t(x,t))$. then the boundary value problem (1), (2), (7) is equivalent to the operator equation

$$A(u) \equiv Hu + F(u) = 0, \qquad u \in X, \tag{8}$$

$H : X \to Y$ linear unbounded operator, a $F(u)$ has a derivative of Freshet in $S(u^{(0)}, \rho)$, then linear operator $H + F'(u)$ will be a linearizer of the operator A at the point $\hat{u} \in S(u^{(0)}, \rho)$.

We introduce the notation

$$\hat{L} = f_u(x,t,\hat{u}(x,t),\hat{u}_t(x,t)) + f_w(x,t,\hat{u}(x,t),\hat{u}_t(x,t))\frac{\partial}{\partial t}. \tag{9}$$

The linear operator $\hat{L}$ maps X in Y and

$$\hat{L}u \;=\; f_u(x,t,\hat{u}(x,t),\hat{u}_t(x,t))u(x,t) + f_w(x,t,\hat{u}(x,t),\hat{u}_t(x,t))\frac{\partial u(x,t)}{\partial t}.$$

Let's demonstrate the boundedness of the operator $\hat{L}$ and compute its norm

$$\|\hat{L}u\|_Y \leq \max_{\overline{\Omega}} \left\{ \Big| f_u(x,\hat{u}(x,t),\hat{u}_t(x,t)) \cdot u(x,t) + f_w(x,t,\hat{u}(x,t),\hat{u}_t(x,t)) \cdot u_t(x,t) \Big| \right\} \leq$$

$$\leq \max_{\overline{\Omega}} \left\{ |f_u(x,t,\hat{u}(x,t),\hat{u}_t(x,t))| \cdot |u(x,t)| + |f_w(x,t,\hat{u}(x,t),\hat{u}_t(x,t))| \cdot |\frac{\partial u(x,t)}{\partial t}| \right\} \leq$$

$$\leq \max_{\overline{\Omega}} \left\{ c_1|u(x,t)| + c_2|u_t(x,t)| \right\} \leq c\|u\|_X,$$

when $c = c_1 + c_2$, $c_1 = \max_{\Omega} |f_u(x,t,\hat{u}(x,t),\hat{u}_t(x,t))|$, $c_2 = \max_{\Omega} |f_w(x,t,\hat{u}(x,t),\hat{u}_t(x,t))|$.

Let's show that $F(u)$ has a derivative of Frechet , i.e. for any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ and inequality

$$\|F(u) - F(\hat{u}) - \hat{L}(u - \hat{u})\|_Y \leq \varepsilon \|u - \hat{u}\|_X \tag{9}$$

holds for all u, $\hat{u} \in S(u^{(0)}, \rho)$, as soon as $\|u - \hat{u}\| < \delta$.
Using the formula of finite differences of Lagrange, we have

$$\|F(u) - F(\hat{u}) - \hat{L}(u - \hat{u})\|_Y =$$

$$= \max_{(x,t)\in\overline{\Omega}} \Big| - A(x,t)u_x(x,t) - A_0(x,t)\ u_x(x,t)|_{x=x_0} + A(x,t)\hat{u}_x(x,t) + A_0(x,t)\ \hat{u}_x(x,t)|_{x=x_0} -$$

$$-f(x,t,u(x,t),u_t(x,t)) + f(x,t,\hat{u}(x,t),\hat{u}_t(x,t)) -$$

$$-\{f_u(x,t,\hat{u}(x,t),\hat{u}_t(x,t))(u(x,t) - \hat{u}(x,t)) + f_w(x,t,\hat{u}(x,t),\hat{u}_t(x,t))(u_t(x,t) - \hat{u}_t(x,t))\}\Big| \leq$$

$$\leq (\alpha + \alpha_0) \max\Big(|\hat{u}_x(x,t) - u_x(x,t)|, |\ \hat{u}_x(x,t)|_{x=x_0} - u_x(x,t)|_{x=x_0}\ |\Big) +$$

$$\leq \max_{(x,t)\in\overline{\Omega}} \Big\{ \int_0^1 |\{f_u(x,t,\hat{u}(x,t) + \theta(u(x,t) - \hat{u}(x,t)), u_t(x,t)) -$$

$$-f_u(x,t,\hat{u}(x,t),\hat{u}_t(x,t))\}|d\theta +$$

$$+ \max_{(x,t)\in\overline{\Omega}} \int_0^1 |\{f_w(x,t,\hat{u}(x,t),\hat{u}_t(x,t) + \theta(u_t(x,t) - \hat{u}_t(x,t))) -$$

$$-f_w(x,t,\hat{u}(x,t),\hat{u}_t(x,t))\}|d\theta,$$

where $\alpha = \max\limits_{(x,t)\in\overline{\Omega}} |A(x,t)|$, $\alpha_0 = \max\limits_{(x,t)\in\overline{\Omega}} |A(x_0,t)|$.

Due to the uniform continuity in $G(u^{(0)}, \rho)$ partial derivatives f_u, f_w for any $\varepsilon > 0$ there exists a number $\delta_\varepsilon > 0$ such that, the inequalities $|u - \hat{u}| < \delta_\varepsilon$, $|u_t - \hat{u}_t| < \delta_\varepsilon$ lead to inequalities

$$|f_u(x,t,u,w,v) - f_u(x,t,\hat{u},\hat{w})| < \frac{\varepsilon}{3},$$

$$|f_w(x,t,u,,w,v) - f_w(x,t,\hat{u},\hat{w})| < \frac{\varepsilon}{3}.$$

Since from the inequality $\|u - \hat{u}\| < \delta_\varepsilon$ it follows, that $\max\limits_{\overline{\Omega}} |u(x,t) - \hat{u}(x,t)| < \delta_\varepsilon$, $\max\limits_{\overline{\Omega}} |u_t(x,t) - \hat{u}_t(x,t)| < \delta_\varepsilon$, then form the uniform continuity in the $G(u^{(0)}, \rho)$ functions f_u, f_w and from the inequalities

$$\theta \max_{\overline{\Omega}} |(u(x,t) - \hat{u}(x,t)| < \delta_\varepsilon,\ \theta \max_{\overline{\Omega}} |(u(x,t) - \hat{u}_x(x,t)| < \delta_\varepsilon,\ \theta \max_{\overline{\Omega}} |u(x,t) - \hat{u}_t(x,t)| < \delta_\varepsilon$$

the validity of the estimates follows

$$\max \int_0^1 |\{f_u(x,t,\hat{u}(x,t) + \theta(u(x,t) - \hat{u}(x,t)), u_t(x,t)) - f_u(x,t,\hat{u}(x,t),\hat{u}_t(x,t))\}|d\theta < \frac{\varepsilon}{3},$$

$$\max \int_0^1 |\{f_w(x,t,\hat{u}(x,t),\hat{u}_t(x,t) + \theta(u_t(x,t) - \hat{u}_t(x,t))) - f_w(x,t,\hat{u}(x,t),\hat{u}_t(x,t))\}|d\theta < \frac{\varepsilon}{3},$$

From here, it follows that inequality (9).To establish the invertibility of the linearizer $H + L_1(u) + F'(u)$ and estimate $\|[H + L_1(u) + F'(u)]^{-1}\|_{L(Y,X)}$, consider the operator equation

$$[H + L_1(u) + F'(\hat{u})]\widetilde{u} = \widetilde{f}, \quad \widetilde{f} \in Y, \quad \widetilde{u} \in X.$$

This problem is equivalent to the linear boundary value problem

$$\frac{\partial^2 \widetilde{u}}{\partial x \partial t} = A(x,t)\frac{\partial \widetilde{u}}{\partial x} + A_0(x,t)\left.\frac{\partial \widetilde{u}}{\partial x}\right|_{x=x_0} + \hat{b}(x,t)\frac{\partial \widetilde{u}}{\partial t} + \hat{c}(x,t)\widetilde{u} + \widetilde{f}(x,t),\ \widetilde{u}(x,t) \in X \tag{10}$$

$$\widetilde{u}(x,0) = \widetilde{u}(x,T),\ x \in [0,\omega], \tag{11}$$

$$\widetilde{u}(0,t) = 0,\ t \in [0,T] \tag{12}$$

where $\hat{b}(x,t) = f_w(x,t,\hat{u}(x,t),\hat{u}_t(x,t))$, $\hat{c}(x,t) = f_u(x,t,\hat{u}(x,t),\hat{u}_t(x,t))$.
Suppose, that problem (10)-(12) for any $\widetilde{f}$ has a unique solution $\widetilde{u}(x,t) \in C^{1,1}_{x,t}(\overline{\Omega})$ and for the estimate is valid

$$\|\widetilde{u}(x,\cdot)\|_1 \leq \gamma \|\widetilde{f}(x,\cdot)\|_1,$$

where $\gamma - const$, not depending on $\widetilde{f}$. Then, hence reversibility follows linearizer $H + L_1(u) + F^{'}(\hat{u})$ and the estimate

$$\|[H + L_1(u) + F^{'}(\hat{u})]^{-1}\|_{L(Y,X)} \leq \gamma.$$

The boundary value problem (10)-(12) is called correctly solvable, if for any $\widetilde{f}(x,t)$ it has a unique classical solution $\widetilde{u}(x,t)$ and valid inequality

$$\|\widetilde{u}\|_0 \leq K\|\widetilde{f}\|_3, \tag{13}$$

where $K-$ a number independent of $\widetilde{f}(x,t)$, ($K-$ constant of correct solvability of problem (10)-(12).

If problem (10)-(12) is correctly solvable with constant γ, then the linearizer $H + L_1(u) + F(u_x) + F'(\hat{u})$ is invertible and grade

$$\|[H + L_1(u) + F'(\hat{u})]^{-1}\|_{L(Y,X)} \leq \gamma, \tag{14}$$

where $L(Y,X)-$ the space of linear bounded operators $\Lambda : Y \to X$ with induced norm.

We introduce new unknown functions $v(x,t) = \dfrac{\partial u(x,t)}{\partial x}$, $w(x,t) = \dfrac{\partial u(x,t)}{\partial t}$, and we must take into account that $v(x_0,t) = \left.\dfrac{\partial u}{\partial x}\right|_{x=x_0}$, and problem (1)-(3) is reduced to the following equivalent problem:

$$\frac{\partial v}{\partial t} = A(x,t)v(x,t) + A_0(x,t)v(x_0,t) + f(x,t,u(x,t),w(x,t)), \tag{15}$$

$$v(x,0) = v(x,T), \quad x \in [0,\omega], \tag{16}$$

$$u(x,t) = \psi(t) + \int_0^x v(\xi,t)d\xi,\ w(x,t) = \dot{\psi}(t) + \int_0^x v_t(\xi,t)d\xi \tag{17}$$

A triple of functions $\{u(x,t), w(x,t), v(x,t)\}$ continuous on $\bar{\Omega}$ is called a solution to problem (15)-(17), if the function $v(x,t) \in C(\bar{\Omega})$ is continuous on $\bar{\Omega}$ derivative with respect to t and satisfies the family periodic boundary value problems (15),(17), where the functions $u(x,t)$, $w(x,t)$ are related to $v(x,t)$, $\dfrac{\partial v(x,t)}{\partial t}$ by functional relations (6).

2 Materials and methods

2.1 Correct solvability of the linearized problem

Problems (1)-(3) and (15)-(17) are equivalent in that sense, that if the triple function $\{u(x,t), w(x,t), v(x,t)\}$ is solution of problem (15)-(17), then the function $u(x,t)$ is problem solution (1)-(3) and, conversely, if the triple $\{u^*(x,t), w^*(x,t), v^*(x,t)\}-$ problem solution (15)-(17), then $u^*(x,t)$ will be a solution to the problem (1)-(3). The following statement takes place, establishing the correct solvability of the linear problem (10)-(12).

Lemma 1. *Let the functions $\hat{a}(x,t), \hat{b}(x,t), \hat{c}(x,t)$ continuous on $\overline{\Omega}$ and*

a) $\left|\int_0^T \hat{a}(x,t)dt\right| \geq \delta_1 > 0,$

b) $\left|\int_0^T \left[\hat{a}(x_0,\tau) + \hat{a_0}(x_0,\tau)\right]d\tau\right| \geq \delta_0 > 0, \delta_0, \delta_1 -$ *const for all* $x \in [0,\omega]$. *Then the linear boundary value problem (10)-(12) s correctly solvable with constant* $\gamma = \gamma_3 \max(1,\omega)$, *where*

$$\gamma_1 = \left\{1 + \frac{e^{\alpha T}\cdot e^{\delta_1}}{e^{\delta_1}-1} + \alpha_0\cdot\left(1+\frac{e^{\delta_1}}{e^{\delta_1}-1}\right)\cdot\left(1+\frac{e^{\delta_0}}{e^{\delta_0}-1}\right)\cdot\frac{e^{(\alpha+\alpha_0)T}-1}{\alpha+\alpha_0}\right\}\frac{e^{\alpha T}-1}{\alpha}$$

$$\gamma_0 = \frac{e^{(\alpha+\alpha_0)T}-1}{\alpha+\alpha_0}\left(\frac{e^{(\alpha+\alpha_0)T}\cdot e^{\delta_0}}{e^{\delta_0}-1}+1\right),\ \gamma_2 = (\alpha+\alpha_0)\Big(1+max(\gamma_1,\gamma_2)\Big), \hat{\alpha} = \max_{(x,t)\in\overline{\Omega}}|\hat{a}(x,t)|,$$

$$\gamma_3 = \max(\gamma_1,\gamma_2)\Big[1+\max(\gamma_1,\gamma_2)(\max_{(x,t)\in\overline{\Omega}}|\hat{b}(x,t)| + \max_{(x,t)\in\overline{\Omega}}|\hat{c}(x,t)|)\times$$

$$\times\int_0^\omega \exp\Big(\int_\xi^\omega \max(\gamma_1,\gamma_2)(\max_{t\in[0,T]}|\hat{b}(s,t)| + \max_{t\in[0,T]}|\hat{c}(s,t)|)ds\Big)d\xi\Big].$$

Proof. Consider the boundary problem (10)-(12). Let's introduce new unknown functions $\widetilde{v}(x,t) = \dfrac{\partial\widetilde{u}(x,t)}{\partial x}, \widetilde{w}(x,t) = \dfrac{\partial\widetilde{u}(x,t)}{\partial t}$ and problem (10)-(12) reduce to the following equivalent problem

$$\frac{\partial\widetilde{v}}{\partial t} = A(x,t)\widetilde{v} + A_0(x_0,t)\widetilde{v}(x_0,t) + \widetilde{\Phi}(x,t,\widetilde{w}(x,t),\widetilde{u}(x,t)),\ \ (x,t)\in\overline{\Omega}, \tag{18}$$

$$\widetilde{v}(x,0) = \widetilde{v}(x,T),\ \ x\in[0,\omega], \tag{19}$$

$$\widetilde{u}(x,t) = \int_0^x \widetilde{v}(\xi,t)d\xi,\ \ \widetilde{w}(x,t) = \int_0^x \widetilde{v}_t(\xi,t)d\xi, \tag{20}$$

Where $\widetilde{\Phi}(x,t,\widetilde{w}(x,t),\widetilde{u}(x,t)) = \hat{b}(x,t)\widetilde{w}(x,t) + \hat{c}(x,t)\widetilde{u}(x,t) + \widetilde{f}(x,t)$.

For fixed $\widetilde{u}(x,t), \widetilde{w}(x,t)$ from problem (18), (19) we obtain a family of periodic boundary problems for ordinary differential equations

$$\frac{\partial\widetilde{v}}{\partial t} = A(x,t)\widetilde{v} + A_0(x_0,t)\widetilde{v}(x_0,t) + \Phi(x,t),\quad (x,t)\in\overline{\Omega}, \tag{21}$$

$$\widetilde{v}(x,0) = \widetilde{v}(x,T),\quad x\in[0,\omega], \tag{22}$$

with continuous on $\overline{\Omega}$ function $\Phi(x,t)$.

Consider the problem

$$\frac{d\widetilde{v}(x_0,t)}{dt} = \big[A(x_0,t) + A_0(x_0,t)\big]\widetilde{v}(x_0,t) + \Phi(x_0,t),\ \ (x_0,t)\in\overline{\Omega}, \tag{23}$$

$$\widetilde{v}(x_0,0) = \widetilde{v}(x_0,T), \quad x_0\in[0,\omega]. \tag{24}$$

The second condition of this theorem ensures the fulfillment of condition 1 of Theorem 1 from [1, P.6], then problem (23)-(24) has a solution and it can be written explicitly. Also, under the conditions of the lemma, problem (21),(22) has a unique solution. Substituting the found solution of problem (23),(24) into the right side of equation (21), we find the solution of problem (21),(22).

$$v(x,t) = \frac{\exp\Big(\int_0^t A(x,\tau)d\tau\Big)}{1-\exp\Big(\int_0^T A(x,\tau)d\tau\Big)}\int_0^T \Phi(x,\tau)\exp\Big(\int_\tau^t A(x,\tau_1)d\tau_1\Big)d\tau +$$

$$+\int_0^t \Phi(x,\tau)\exp\Big(\int_\tau^t A(x,\tau_1)d\tau_1\Big)d\tau +$$

$$+\frac{\exp\Big(\int_0^t A(x,\tau)d\tau\Big)}{1-\exp\Big(\int_0^T A(x,\tau)d\tau\Big)}\int_0^T A_0(x,\tau)\exp\Big(\int_\tau^t A(x,\tau_1)d\tau_1\Big)\times$$

$$\times\Bigg\{\frac{\exp\Big(\int_0^\tau \big[A(x_0,\tau_1)+A_0(x_0,\tau_1)\big]d\tau_1\Big)}{1-\exp\Big(\int_0^T \big[A(x_0,\tau_1)+A_0(x_0,\tau_1)\big]d\tau_1\Big)}\times$$

$$\times\int_0^T \Phi(x_0,\tau)\exp\Big(\int_\tau^T \big[A(x_0,\tau_1)+A_0(x_0,\tau_1)\big])d\tau_1\Big)d\tau +$$

$$+\int_0^\tau \Phi(x,\tau_1)\exp\Big(\int_{\tau_1}^\tau \big[A(x_0,\tau_2)+A_0(x_0,\tau_2)\big]d\tau_2\Big)d\tau_1 d\tau\Bigg\} +$$

$$+\int_0^t A_0(x,\tau)\exp\Big(\int_\tau^t A(x,\tau_1)d\tau_1\Big)\Bigg\{\frac{\exp\Big(\int_0^\tau \big[A(x_0,\tau_1)+A_0(x_0,\tau_1)\big]d\tau_1\Big)}{1-\exp\Big(\int_0^T \big[A(x_0,\tau_1)+A_0(x_0,\tau_1)\big]d\tau_1\Big)}\times$$

$$\times\int_0^T \Phi(x,\tau)\exp\Big(\int_\tau^T \big[A(x_0,\tau_1)+A_0(x_0,\tau_1)\big])d\tau_1\Big)d\tau +$$

$$+\int_0^\tau \Phi(x_0,\tau_1)\exp\Big(\int_{\tau_1}^\tau \big[A(x_0,\tau_2)+A_0(x_0,\tau_2)\big]d\tau_2\Big)d\tau_1 d\tau\Bigg\},\ t\in[0,T] \tag{25}$$

and the estimate is fair for it

$$\max_{t\in[0,T]}|\widetilde{v}(x,t)| \le$$

$$\le\Bigg\{1+\frac{e^{\alpha T}\cdot e^{\delta_1}}{e^{\delta_1}-1}+\alpha_0\cdot\Big(1+\frac{e^{\delta_1}}{e^{\delta_1}-1}\Big)\cdot\Big(1+\frac{e^{\delta_0}}{e^{\delta_0}-1}\Big)\cdot\frac{e^{(\alpha+\alpha_0)T}-1}{\alpha+\alpha_0}\Bigg\}\times$$

$$\times\frac{e^{\alpha T}-1}{\alpha}\cdot\max(\max_{t\in[0,T]}|\Phi(x,t)|,\max_{t\in[0,T]}|\Phi(x_0,t)|), \tag{26}$$

where $\alpha=\max_{(x,t)\in\overline{\Omega}}|A(x,t)|$, $\alpha_0=\max_{(x,t)\in\overline{\Omega}}|A_0(x,t)|$,

$$\gamma_1=\left\{1+\frac{e^{\alpha T}\cdot e^{\delta_1}}{e^{\delta_1}-1}+\alpha_0\cdot\Big(1+\frac{e^{\delta_1}}{e^{\delta_1}-1}\Big)\cdot\Big(1+\frac{e^{\delta_0}}{e^{\delta_0}-1}\Big)\cdot\frac{e^{(\alpha+\alpha_0)T}-1}{\alpha+\alpha_0}\right\}\cdot\frac{e^{\alpha T}-1}{\alpha}.$$

Hence follows

$$\max_{t\in[0,T]}|\widetilde{v}(x,t)|\le\gamma_1\max(\max_{t\in[0,T]}|\Phi(x,t)|,\max_{t\in[0,T]}|\Phi(x_0,t)|). \tag{27}$$

And for $\widetilde{v}(x_0,t)$ the following estimate holds

$$\max_{t\in[0,T]}|\widetilde{v}(x_0,t)|\le\frac{e^{(\alpha+\alpha_0)T}-1}{\alpha+\alpha_0}\Big(\frac{e^{(\alpha+\alpha_0)T}\cdot e^{\delta_0}}{e^{\delta_0}-1}+1\Big)\le\gamma_0\max_{t\in[0,T]}|\Phi(x_0,t)|.$$

Then, by Theorem 1 from [6], problem (18)-(20) correctly resolved. Let $\widetilde{u}(x,t)$ be its the only solution. From differential equation (21) and estimate (27) it follows, that

$$\max_{t\in[0,T]}|\widetilde{v}_t(x,t)|\le$$

$$\le\Big(\max_{(x,t)\in\overline{\Omega}}|A(x,t)|+\max_{(x,t)\in\overline{\Omega}}|A(x_0,t)|\Big)max\Big(\max_{t\in[0,T]}|\widetilde{v}(x,t)|,\max_{t\in[0,T]}|\widetilde{v}(x_0,t)|\Big)+\max_{t\in[0,T]}|\Phi(x,t)|\le$$

$$\le(\alpha+\alpha_0)\max\Big(\gamma_1\max_{t\in[0,T]}|\Phi(x,t)|,\gamma_0\max_{t\in[0,T]}|\Phi(x_0,t)|\Big)+\max_{t\in[0,T]}|\Phi(x,t)|\le$$

$$\le(\alpha+\alpha_0)\Big(1+max\big(\gamma_0,\gamma_1\big)\Big)\max_{t\in[0,T]}|\Phi(x,t)|=\gamma_2\max_{t\in[0,T]}|\Phi(x,t)|. \tag{28}$$

From the assumption, that the functions $\widetilde{u}(x,t),\widetilde{w}(x,t)$ belong to the space $C(\overline{\Omega})$ we obtain the membership of the function $\Phi(x,t)=\hat{b}(x,t)\widetilde{w}(x,t)+\hat{c}(x,t)\widetilde{u}(x,t)+\widetilde{f}(x,t)$ in space $C(\overline{\Omega})$. Using relation (20) and inequalities (27), (28), we establish

$$\max_{t\in[0,T]}|\widetilde{v}(x,t)|\le\gamma_1\Big(\max_{t\in[0,T]}|\hat{b}(x,t)|\int_0^x\max_{t\in[0,T]}|\widetilde{v}_t(\xi,t)|d\xi+$$

$$+\max_{t\in[0,T]}|\hat{c}(x,t)|\int_0^x\max_{t\in[0,T]}|\widetilde{v}(\xi,t)|d\xi+\max_{t\in[0,T]}|\widetilde{f}(x,t)|\Big),$$

$$\max_{t\in[0,T]}|\widetilde{v}_t(x,t)|\le\gamma_2\Big(\max_{t\in[0,T]}|\hat{b}(x,t)|\int_0^x\max_{t\in[0,T]}|\widetilde{v}_t(\xi,t)|d\xi+$$

$$+\max_{t\in[0,T]}|\hat{c}(x,t)|\int_0^x\max_{t\in[0,T]}|\widetilde{v}(\xi,t)|d\xi+\max_{t\in[0,T]}|\widetilde{f}(x,t)|\Big),$$

$$\max(\max_{t\in[0,T]}|\widetilde{v}(x,t)|,\max_{t\in[0,T]}|\widetilde{v}_t(x,t)|)\le\max(\gamma_1,\gamma_2)(\max_{t\in[0,T]}|\hat{b}(x,t)|+\max_{t\in[0,T]}|\hat{c}(x,t)|)\times$$

$$\times\int_0^x\max(\max_{t\in[0,T]}|\widetilde{v}(\xi,t)|,\max_{t\in[0,T]}|\widetilde{v}_t(\xi,t)|)d\xi+\max(\gamma_1,\gamma_2)\max_{t\in[0,T]}|\widetilde{f}(x,t)|. \tag{29}$$

Applying the Gronwall-Bellman lemma to inequality (29), we obtain

$$\max(\max_{t\in[0,T]}|\widetilde{v}(x,t)|,\max_{t\in[0,T]}|\widetilde{v}_t(x,t)|)\leq\max(\gamma_1,\gamma_2)\max_{t\in[0,T]}|\widetilde{f}(x,t)|+$$

$$+\max(\gamma_1,\gamma_2)(\max_{t\in[0,T]}|\hat{b}(x,t)|+\max_{t\in[0,T]}|\hat{c}(x,t)|)\int_0^x\max(\gamma_1,\gamma_2)\max_{t\in[0,T]}|\widetilde{f}(\xi,t)|\times$$

$$\times\exp\Big(\max(\gamma_1,\gamma_2)\int_\xi^x(\max_{t\in[0,T]}|\hat{b}(s,t)|+\max_{t\in[0,T]}|\hat{c}(s,t)|)ds\Big)d\xi,$$

whence it follows

$$\max(\max_{(x,t)\in\overline{\Omega}}|\widetilde{v}(x,t)|,\max_{(x,t)\in\overline{\Omega}}|\widetilde{v}_t(x,t)|)\leq\gamma_3\max_{(x,t)\in\overline{\Omega}}|\widetilde{f}(x,t)|, \tag{30}$$

Where

$$\gamma_3=\max(\gamma_1,\gamma_2)\Big[1+\max(\gamma_1,\gamma_2)(\max_{(x,t)\in\overline{\Omega}}|\hat{b}(x,t)|+\max_{(x,t)\in\overline{\Omega}}|\hat{c}(x,t)|)\times$$

$$\times\int_0^\omega\exp\Big(\int_\xi^\omega\max(\gamma_1,\gamma_2)(\max_{(x,t)\in\overline{\Omega}}|\hat{b}(s,t)|+\max_{(x,t)\in\overline{\Omega}}|\hat{c}(s,t)|)ds\Big)d\xi\Big].$$

Then

$$\max(\|\widetilde{u}\|_3,\|\widetilde{v}\|_3,\|\widetilde{w}\|_3)\leq\max\{\gamma_3\|\widetilde{f}\|_3,\gamma_3\omega\|\widetilde{f}\|_3\}\leq\gamma_3\max(1,\omega)\|\widetilde{f}\|_3=\gamma\|\widetilde{f}\|_3.$$

Lemma 1 is proved.

2.2 Necessary and sufficient conditions for the existence of an "isolated" solution semiperiodic boundary value problem for a nonlinear loaded hyperbolic equation

The following theorem establishes necessary and sufficient conditions for the existence of an "isolated" solution semiperiodic boundary value problem for a nonlinear loaded hyperbolic equation with a mixed derivative.

Theorem 1. *Problem (1), (2), (7) has an "isolated" solution if and only if, When there exists a pair* $(u^{(0)}(x,t),\rho)\in V^0(f,L_1(x,t),L_2(x,t))$, *under which for any* $(x,t,u,w)\in G(u^{(0)},\rho)$ *the inequality*

a) $\Big|\int_0^T A(x,t)dt\Big|\geq\delta_1>0,$

b) $\Big|\int_0^T\big[A(x_0,\tau)+A_0(x_0,\tau)\big]d\tau\Big|\geq\delta_0>0,\delta_0,\delta_1-const$ *and the relation holds*

c) $\widetilde{\gamma}\max_{(x,t)\in\overline{\Omega}}|u_{xt}^{(0)}(x,t)-A(x,t)u_x-A_0(x,t)u_x(x_0,t)-f(x,t,u^{(0)}(x,t),u_t^{(0)}(x,t))|<\rho,$

$$\textit{where}\quad\widetilde{\gamma}=\widetilde{\gamma_3}\max(1,\omega),\widetilde{\gamma_3}=\max(\widetilde{\gamma_1},\ \widetilde{\gamma_2})\Big[1+\max(\widetilde{\gamma_1},\widetilde{\gamma_2})(\overline{L}_1+\overline{L}_2)\times$$

$$\times\int_0^\omega\exp\Big(\int_\xi^\omega\max(\widetilde{\gamma_1},\widetilde{\gamma_2})(\max_{t\in[0,T]}L_1(s,t)+\max_{t\in[0,T]}L_2(s,t))ds\Big)d\xi\Big],\ \widetilde{\gamma_2}=1++\widetilde{\gamma_1}\cdot\overline{L}_1,$$

$$\widetilde{\gamma_1}=\frac{e^{\overline{L}_0T}-1}{\overline{L}_0}\Big(\frac{e^\delta\cdot e^{\overline{L}_0T}}{e^\delta-1}+1\Big),\ L_0(x,t)=\max\big(A(x_0,\tau),A(x_0,\tau)+A_0(x_0,\tau)\big),$$

$\overline{L}_i = \max_{\overline{\Omega}} L_i(x,t),\ i = 0,1,2.$

Proof. Necessity. Let $u^*(x,t)$-be an "isolated" solution of the boundary value problem (1), (2), (7). Then there exists a number ρ_0, for which the function $f(x,t,u,w)$ in $G(u^*,\rho_0)$ has uniformly continuous partial derivatives with respect to u, w and a linear semi-periodic boundary value problem

$$\frac{\partial^2 u}{\partial x \partial t} = a^*(x,t)\frac{\partial u}{\partial x} + a_0^*(x,t)\left.\frac{\partial u}{\partial x}\right|_{x=x_0} + b^*(x,t)\frac{\partial u}{\partial t} + c^*(x,t)u + \widetilde{f}(x,t), \tag{30}$$

$$u(x,0) = u(x,T), \quad x \in [0,\omega], \tag{31}$$

$$u(0,t) = 0, \quad t \in [0,T] \tag{32}$$

it is correctly solvable. This implies, in particular, the existence continuous on $\overline{\Omega}$ functions $L_i^*(x,t)$, $i = 1,2$, under which the inequalities $|f_u(x,t,u^*(x,t),u_t^*(x,t))| \le L_1^*(x,t)$, $|f_w(x,t,u^*(x,t),u_t^*(x,t))| \le L_2^*(x,t)$. Boundary problem

$$\frac{\partial v}{\partial t} = a^*(x,t)v + a_0^*(x,t)v(x_0,t) + F(x,t), \quad (x,x_0,t) \in \overline{\Omega} \tag{33}$$

$$v(x,0) = v(x,T). \quad x \in [0,\omega] \tag{34}$$

In Theorem 2 of [12] it was proved, that the problem (33), (34) is correctly solvable if and only if, when $\left|\int_0^T A(x,\tau)d\tau\right| \ne 0$ and $\left|\int_0^T \left[A(x_0,\tau) + A_0(x_0,\tau)\right]d\tau\right| \ne 0$ for all $x, x_0 \in [0,\omega]$. Since the function $a^*(x,t) = A(x,t), a_0^*(x,t) = A_0(x_0,t)$ are continuous on $\overline{\Omega}$, then $\widetilde{a}(x) = \int_0^T a^*(x,t)dt$, $\widetilde{a}_0(x) = \left|\int_0^T \left[A(x_0,\tau) + A_0(x_0,\tau)\right]d\tau\right|$ are continuous functions on $[0,\omega]$. This implies, that there exists $\delta^* > 0, \delta_1^* > 0$ for which the inequalities $\left|\int_0^T a^*(x,t)dt\right| \ge \delta^*$, $\left|\int_0^T \left[a^*(x_0,\tau) + a_0^*(x_0,\tau)\right]d\tau\right| \ge \delta_1^*$ for all $x, x_0 \in [0,\omega]$.

In the force of uniform continuity in $G(u^*,\rho_0)$ functions $f_w(x,t,u,w), f_u(x,t,u,w)$ for $\varepsilon = \dfrac{\delta^*}{2T}$, exists $\rho_\varepsilon^* \in (0,\rho_0)$ such that, the inequalities

$|u - u^*(x,t)| < \rho_\varepsilon^*$, $|w - u_t^*(x,t)| < \rho_\varepsilon^*$, entail the fulfillment of the relations

$|f_u(x,t,u,w) - f_u(x,t,u^*(x,t),u_t^*(x,t))| < \dfrac{\delta^*}{2T}$,

$|f_w(x,t,u,w) - f_w(x,t,u^*(x,t),u_t^*(x,t))| < \dfrac{\delta^*}{2T}$,

Due to the choice of $\varepsilon = \frac{\delta^*}{2T}$, ρ_ε^*,

$$|f_u(x,t,u,w)| \le |f_u(x,t,u,w) - f_u(x,t,u^*(x,t),u_t^*(x,t))| +$$

$$+|f_u(x,t,u^*(x,t),u_t^*(x,t))| \le \frac{\delta^*}{2T} + L_1^*(x,t),$$

$$|f_w(x,t,u,w)| \le |f_w(x,t,u,w) - f_w(x,t,u^*(x,t),u_t^*(x,t))| +$$

$$+|f_w(x,t,u^*(x,t),u_t^*(x,t))| \le \frac{\delta^*}{2T} + L_2^*(x,t)$$

and $(u^*(x,t),\rho_\varepsilon^*) \in V(x,x_0,t,\dfrac{\delta^*}{2T} + L_1^*(x,t), \dfrac{\delta^*}{2T} + L_2^*(x,t))$.

Moreover, for any $(x,t) \in G(u^*,\rho^*)$ the inequalities $\Big|\int_0^T a^*(x,t)dt\Big| \geq \frac{\delta^*}{2}$, $\Big|\int_0^T \big[a^*(x_0,\tau)+a_0^*(x_0,\tau)\big]d\tau\Big| \geq \frac{\delta_1^*}{2}$ and conditions a) and b) theorem is satisfied for $\delta = \frac{\delta^*}{2}$, $\delta_1 = \frac{\delta_1^*}{2}$. When found $\delta = \frac{\delta^*}{2}$, $L_1(x,t) = \frac{\delta^*}{2T} + L_1^*(x,t)$, $L_2(x,t) = \frac{\delta^*}{2T} + L_2^*(x,t)$, using Lemma 1 we calculate

$$\widetilde{\gamma} = \widetilde{\gamma}_3 \max(1,\omega),\ \widetilde{\gamma}_3 = \max(\widetilde{\gamma}_1,\widetilde{\gamma}_2)\Big[1+\max(\widetilde{\gamma}_1,\widetilde{\gamma}_2)\Big(\frac{\delta^*}{T}+\overline{L}_1^*+\overline{L}_2^*\Big)\times$$

$$\times\int_0^\omega \exp\Big(\int_\xi^\omega \max(\widetilde{\gamma}_1,\widetilde{\gamma}_2)\Big(\frac{\delta^*}{T}+\max_{t\in[0,T]} L_1^*(s,t)+\max_{t\in[0,T]} L_2^*(s,t)\Big)ds\Big)d\xi\Big],$$

$$\widetilde{\gamma}_2 = 1+\widetilde{\gamma}_1\Big(\frac{\delta^*}{2T}+\overline{L}_1^*\Big),\ \ \widetilde{\gamma}_1 = \frac{e^{\overline{L}_0^*T}-1}{\overline{L}_0^*}\Big(\frac{e^\delta\cdot e^{\overline{L}_0^*T}}{e^\delta-1}+1\Big).$$

Thus, when choosing as $(u^{(0)}(x,t),\rho)$ pairs $(u^*(x,t),\rho_\varepsilon^*)$ all conditions are met theorems, and including the condition
c)$\widetilde{\gamma}\max_{(x,t)\in\overline{\Omega}}|u_{xt}^{(0)}(x,t)-A(x,t)u_x-A_0(x,t)u_x(x_0,t)-f(x,t,u^{(0)}(x,t),u_t^{(0)}(x,t))| = 0 < \rho^*$.

Sufficiency. Under the conditions of the theorem, we show the existence of an "isolated" solution to the boundary value problem (1), (2), (7). Since $(u^{(0)}(x,t),\rho) \in V(f,L_1(x,t),L_2(x,t))$, then the function $f(x,t,u,w,v)$ has uniformly continuous partial derivatives with respect to u,w,v and the inequalities

$$|f_w(x,t,u,w)| \leq L_1(x,t),\ |f_u(x,t,u,w)| \leq L_2(x,t)$$

for all $(x,t,u,w) \in G(u^{(0)},\rho)$. Hence, it follows that the operator $F(u)$ in (11) has a uniformly continuous Frechet derivative в $S(u^{(0)},\rho)$. If $\hat{u}(x,t) \in S(u^{(0)},\rho)$, then $(x,t,\hat{u}(x,t),\hat{u}_t(x,t),\hat{u}_x(x,t)) \in G(u^{(0)},\rho)$ for all $(x,t) \in \overline{\Omega}$ and functions $\hat{b}(x,t) = f_u(x,t,\hat{u}(x,t),\hat{u}_t(x,t))$, $\hat{c}(x,t) = f_w(x,t,\hat{u}(x,t),\hat{u}_t(x,t))$ and $\hat{a}(x,t) = A(x,t)$, $\hat{a}_0(x,t) = A_0(x,t)$ are continuous on $\overline{\Omega}$. Hence, by Lemma 1, the boundary value problem (10)-(12) is correctly solvable with the constant γ and the estimate (14). From the condition c) theorem we have $\gamma\|Hu^{(0)}+F(u^{(0)})\|_{L(Y,X)} < \rho$. According to Theorem 3 from [11] operator equation (11) in $S(u^{(0)},\rho)$ has solution. Due to the equivalence of the boundary value problem (1), (2), (14) and equation (11) it follows, that in $S(u^{(0)},\rho)$ exists $u^*(x,t)-$ solution to the boundary value problem (1), (2), (7). Let us show that this solution is "isolated" in the sense of the definition. For this, consider linear boundary value problem (10)-(12) with coefficients $a^*(x,t)$, $a_0^*(x,t)$, $b^*(x,t)$, $c^*(x,t)$. Since $u^*(x,t) \in S(u^{(0)},\rho)$, then $(x,t,u^*(x,t),u_t^*(x,t)) \in G(u^{(0)},\rho)$ $(x,t) \in \overline{\Omega}$ and then, by Lemma 1 the linear boundary value problem (10)-(12) with coefficients $a^*(x,t), b^*(x,t), c^*(x,t)$ has he only classical solution and estimate is valid (14), i.e. is correctly resolvable. By definition 2, the function $u^*(x,t)$ is an "isolated " solution to the problem (10)-(12). Theorem 1 is proved.

3 Conclusion

In conclusion, we introduced the concept of an isolated solution and presented an approach for finding an isolated solution for a nonlinear loaded boundary value problem. To achieve this goal, we successfully reduced the nonlinear boundary value problem to a linear one, and then linearized the operator equation describing our problem using the Frechet derivative.

This approach allowed us to formulate the necessary and sufficient conditions for the existence of an isolated solution for the second-order nonlinear loaded hyperbolic equation in terms of the original data. Our findings are of significant importance for understanding and solving complex problems related to isolated solutions and boundary value problems in the context of hyperbolic equations.

4 Acknowledgement

The work was financed by grant financing of projects by the Ministry of Education and Science of the Republic of Kazakhstan (grant № AP09058457).

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