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UNIFORM ESTIMATES FOR SOLUTIONS OF A CLASS OF NONLINEAR EQUATIONS IN A FINITE-DIMENSIONAL SPACE

The need to study boundary value problems for elliptic parabolic equations is dictated by numerous practical applications in the theoretical study of the processes of hydrodynamics, electrostatics, mechanics, heat conduction, elasticity theory, quantum physics.
Let H $(dimH \geq 1)$ – a finite-dimensional real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We will study the equation of the following form

$$u + L(u) = g \in H,$$

where $L(\cdot)$ is a non-linear continuous transformation, g is an element of the space H, u is the required solution of the problem from H.
In this paper, we obtain two theorems on a priori estimates for solutions of nonlinear equations in a finite-dimensional Hilbert space. The work consists of four items.
The conditions of the theorems are such that they can be used in the study of a certain class of initial-boundary value problems to obtain strong a priori estimates. This is the meaning of these theorems.
Key words: finite-dimensional Hilbert space, nonlinear equations, initial-boundary value problem, weak solution, strong solution, a priori estimates of the solution.

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Сызықты емес теңдеулердің бір классы шешімдерінің бірқалыпты бағалаулары

Эллиптикалық параболалық теңдеулер үшін шекаралық есептерді зерттеу қажеттілігі гидродинамика, электростатика, механика, жылу өткізгіштік, серпімділік теориясы және кванттық физика процестерін теориялық зерттеуде көптеген практикалық қолданулардан туындайды.
Скаляр көбейтіндісі $\langle \cdot, \cdot \rangle$ және нормасы $\| \cdot \|$ бар H $(dimH \geq 1)$ – ақырлы нақты Гильберт кеңістігінде келесі түрдегі теңдеу зерттеледі

$$u + L(u) = g \in H,$$

мұндағы $L(\cdot)$ – сызықты емес үзіліссіз бейнелеу, g – H -тың элементі, u – H -тағы ізделінді шешімі.

Бұл жұмыста біз ақырлы өлшемді кеңістіктегі сызықтық емес теңдеулерді шешуге арналған априорлық бағалаулар бойынша екі теореманы аламыз. Бұл теоремалар белгілі бір шарттарда дәлелденеді, олар сызықты емес бастапқы-шеттік есептердің бір класының соңғы өлшемді жуықтауларымен қанағаттандырылатын шарттардан алынған.
Теореманың шарттары күшті априорлық бағалаулар алу үшін бастапқы-шеттік есептердің белгілі бір класын зерттеуде қолдануға болады. Бұл теоремалардың негізгі мағынасы осында.
Түйін сөздер: ақырлы Гильберт кеңістігі, сызықтық емес теңдеу, бастапқы-шеттік есеп, әлсіз шешім, күшті шешім, шешімнің априорлық бағалануы.

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Равномерные оценки решений одного класса нелинейных уравнений в конечномерном пространстве

Необходимость исследования краевых задач для эллиптических параболических уравнений диктуется многочисленными практическими приложениями при теоретическом исследовании процессов гидродинамики, электростатики, механики, теплопроводности, теории упругости, квантовой физики.
В H – конечномерном ($dimH \geq 1$) действительном гильбертовом пространстве со скалярным произведением $\langle \cdot, \cdot \rangle$ и с нормой $\| \cdot \|$ исследуется уравнение следующего вида

$$u + L(u) = g \in H,$$

где $L(\cdot)$ – нелинейное непрерывное преобразование, g – элемент пространства H, u – искомое решение задачи из H.
В настоящей работе получены две теоремы об априорных оценках решений нелинейных уравнений в конечномерном пространстве. Эти теоремы доказаны при выполнении некоторых условий, которые заимствованы из условий которым удовлетворяют конечномерные аппроксимации одного класса нелинейных начально-краевых задач.
Условия теорем таковы, что их можно использовать при изучении определенного класса начально-краевых задач для получения сильных априорных оценок. В этом смысл этих теорем.
Ключевые слова: конечномерное гильбертовое пространство, нелинейные уравнения, начально-краевая задача, слабое решение, сильное решение, априорные оценки решения.

1 Introduction

The problem of describing the dynamics of an incompressible fluid is an urgent problem of our time.

In mid-2000, the Clay Mathematics Institute formulated several unsolved problems of mathematics in the millennium (The Millennium Prize Problems). One of the problems is the existence and smoothness of solutions to the Navier–Stokes equations for an incompressible viscous fluid [1].

Many mathematicians have worked on this problem and obtained significant results in [2]–[4]. This problem is solved in the two-dimensional case of O.A. Ladyzhenskaya in [3]. The work [4] provides a fairly complete analysis of the state of the problem and a review of the available literature.

Works [5]– [12] are devoted to the study of the solvability in the whole of equations of the Navier-Stokes type, the continuous dependence of the solution of a parabolic equation and the smoothness of the solution. In papers [13], [14] questions of the formulation and their solvability of boundary value problems for high-order quasi-hyperbolic equations were studied. The work [15], [16] is devoted to the dedication of Green's function type Dirichlet for a polyharmonic equation and the description of the correct boundary problem for the polyharmonic operator. In the works [17]– [19], studied the questions of the Fredholm solvability of the general problem Neumann for the elliptic equation of high order on the plane.

The works [20]– [22] is devoted to the study of the uniqueness of the solution of time-regular problems for some operator-differential equations of the form where the operator A is: a) an operator with a Wave Operator with Displacement, в) the Tricomi operator, c) an arbitrary self-adjoint high-order elliptic differential operator.

In the work [23] a complete proof of Theorem 2 is given in another form. This article is a continuation of the work [23].

In this article, we obtain two theorems on a priory estimates for solutions of nonlinear equations in a finite-dimensional space. These theorems are proved under certain conditions, which are borrowed from the conditions that are satisfied by finite-dimensional approximations of one class of nonlinear initial-boundary value problems.

2 Materials and methods

3 Used conditions and designations. Formulation of the main results

In this paper, we are engaged in the derivation of uniform estimates for solutions of nonlinear equations of the form

$$u + L(u) = g \in H, \tag{1}$$

where H is a finite-dimensional Hilbert space, $L(\cdot)$ is a non-linear continuous transformation, g is an element of the space H, the solution u of problem (1) is sought in H.

We aim at such finite-dimensional equations of the form (1), which are finite-dimensional approximations of infinite-dimensional problems of the form (1) in an infinite-dimensional Hilbert space. In this case, it will turn out to be very important to obtain estimates that are independent of the approximation number and allow one to pass to the limit and obtain an a priori estimate in the limit for solving the infinite-dimensional problem. The infinite-dimensional problems of the form (1) that we are aiming at are, as a rule, problems of mathematical physics written in a restricted form.

In this section, $f(u)$ will mean an operation of the form

$$f(u) = u + L(u). \tag{2}$$

If ξ is a parameter from $[0, +\infty)$ and the vector $u(\xi)$ is a vector function continuously differentiable with respect to the parameter ξ, then we assume that the vector function $L(u(\xi))$ is also continuously differentiable, as well as the expressions arising from $L(u)$ and $f(u)$.

Let's introduce the notation L_u :

$$\left(L\left(u(\xi)\right)\right)_\xi = L_{u(\xi)} u_\xi(\xi). \tag{3}$$

It is obvious that L_u (for each $u \in H$) will be a linear operator

$$L_u v = \left(L\left(u\left(\xi\right)\right)\right)_\xi \Big|_{u_\xi = v}. \tag{4}$$

We have

$$\left(f\left(u(\xi)\right)\right)_\xi = u_\xi + L_u u_\xi = \left(E + L_u\right) u_\xi.$$

Here and throughout what follows, E is the identity transformation.

Denote

$$D_u^* = E + L_u^*, \tag{5}$$

$$D_u^* f\left(u\right) = \left(E + L_u^*\right) f\left(u\right). \tag{6}$$

Let us present the conditions used U1–U4.

Condition U1. *For the transformation $L\left(\cdot\right)$ and the operators L_u, L_u^*, D_u and D_u^* conditions are met*

$$\|L(u) - L(v)\| + \|L_u^* - L_v^*\|_{H \to H} \leq \psi(\|u\|)\|u - v\|, \tag{7}$$

$$\|D_v u\| + \|D_u^*\| \leq \psi(\|v\|)\|u\|, \tag{8}$$

where $\psi(\cdot)$ is a continuous function on $[0, \infty)$.

Condition U2. *There are linear invertible operators T and G such that*

$$\|G\| \leq 1, \ \|T\| \leq 1, \ \|G^{-1}\| + \|T^{-1}\| < \infty, \tag{9}$$

and for any $u \in H$ the relations

$$\langle L(u), Tu\rangle \geq 0, \ \langle Tu, u\rangle \geq \|Gu\|^2 \geq \|Tu\|^2. \tag{10}$$

Condition U3. *If $u \in H$ is an eigenvector of the operator G^*G, then the inequality*

$$\|u\|^2 \leq \left(\|f(u)\|^2 + 2\right)^m, \ m \geq 1. \tag{11}$$

Condition U4. *If $D_u^* f(u) = \lambda u$, $\lambda > 0$, then*

$$\gamma(u) :\equiv \langle D_u^* f, u\rangle \|u\|^{-2} \geq \left(\|f(u)\|^2 + 2\right)^{-m} \|u\|^{-2}. \tag{12}$$

Theorem 1. *If conditions U1 and U2 are satisfied, then for any $g \in H$ problem*

$$f(u) = g$$

has a solution $u \in H$ such that the estimate

$$\|Gu\| \leq \|g\|. \tag{13}$$

Remark 1. We will see in the applications that Theorem 1 allows us to obtain the existence of a weak solution of a certain class of problems of mathematical physics written in restricted notation (integral form), for which the problem

$$u + L(u) = g$$

is a finite-dimensional approximation.

Theorem 2. *If conditions U1, U2, U3 and U4 are satisfied, then the problem*

$$u + L(u) = g$$

for any $g \in H$ has a solution satisfying the estimate

$$\|u\|^2 \leq C_0 exp\{-\{\|g\|^2\}\}, \tag{14}$$

where C_0 is a constant number independent of $g \in H$ (depending on $m-$ from condition U4).

Remark 2. This theorem allows one to obtain the existence of a strong solution to some problems of mathematical physics (written in a restricted form). Conditions U3 and U4 can be noticeably weakened, but the remaining conditions U1 and U2 are not sufficient to obtain estimate (14) from the theorem. It can be seen from the course of the proof of Theorem 2 that estimate (14) can be significantly improved. A complete proof of Theorem 2 in a slightly different form is given in [23].

4 Proof of the theorem 1

The equation $u+L(u)=g$ is scalarly multiplied by Tu. Then, using conditions U2, we obtain

$$\langle Tu, g\rangle = \langle u, Tu\rangle + \langle L(u), Tu\rangle \geq \langle u, Tu\rangle \geq \|Gu\|^2.$$

From this and condition U2 we get the estimate

$$\|Gu\|^2 \leq \langle Tu, g\rangle \leq \|Tu\|\|g\| \leq \|Gu\|\|g\|.$$

From this estimate we obtain the a priori estimate

$$\|Gu\| \leq \|g\|. \tag{15}$$

Denote

$$M = \{u : \langle Tu, u\rangle \leq 8\langle Tg, g\rangle\}. \tag{16}$$

Recall that $\langle Tu, u\rangle$ is positive (strictly!). Therefore, $\langle Tu, u\rangle$ and $\langle Tg, g\rangle$ can be taken as the squares of norms.

Let the equation $u + L(u) = g$ have no solution. Let us define the transformation $F(u)$:

$$F(u) = -\frac{u + L(u) - g}{\sqrt{\langle T(u + L(u) - g), u + L(u) - g\rangle}}\sqrt{8\langle Tg, g\rangle}. \tag{17}$$

Since the equation $u + L(u) = g$ has no solution, this transformation is continuous. But

$$\langle TF(u), F(u)\rangle \le 8\langle Tg, g\rangle.$$

Therefore, a continuous transformation takes the set M into itself. But then (since H is finite-dimensional) according to Browder's theorem, the transformation F has a fixed point, i.e.,

$$F(u_0) = u_0. \tag{18}$$

Let us act on (18) with the operator T, and then scalarly multiply the resulting equality by the vector $u_0 + L(u_0) = g$. Then using (17) we have

$$-\frac{\langle T(u_0 + L(u_0) - g), u_0 + L(u_0) - g\rangle}{\sqrt{\langle T(u_0 + L(u_0) - g), u_0 + L(u_0) - g\rangle}}\sqrt{8\langle Tg, g\rangle} =$$

$$\langle Tu_0, u_0 + L(u_0) - g\rangle$$

or

$$-\sqrt{8}\sqrt{\langle Tg, g\rangle}\sqrt{\langle T(u_0 + L(u_0) - g), u_0 + L(u_0) - g\rangle} = \langle u_0, T(u_0 + L(u_0) - g)\rangle. \tag{19}$$

Let us scalarly multiply equality (18) by the vector $T(u_0 + L(u_0) - g)$. Then using (17) instead of (19) we obtain

$$-\sqrt{8}\sqrt{\langle Tg, g\rangle}\sqrt{\langle T(u_0 + L(u_0) - g), u_0 + L(u_0) - g\rangle} = \langle Tu_0, u_0 + L(u_0) - g\rangle. \tag{20}$$

We add equalities (19) and (20), then we obtain

$$-\sqrt{8}\sqrt{\langle Tg, g\rangle}\sqrt{\langle T(u_0 + L(u_0) - g), u_0 + L(u_0) - g\rangle} =$$

$$\frac{1}{2}\left(\langle u_0, T(u_0 + L(u_0) - g)\rangle - \langle T(u_0 + L(u_0) - g), u_0\rangle\right). \tag{21}$$

Now, since $\langle Tx, x\rangle \ge \|Gx\|^2 > 0$, we can $\frac{1}{2}\left(\langle y, Tx\rangle + \langle Ty, takex\rangle\right)$ as the scalar product. Then $\langle Tx, x\rangle$ and $\langle y, Tx\rangle + \langle Ty, y\rangle$– will be norm squares. Then, since the right side of (21) must be negative, we get

$$-\sqrt{8}\sqrt{\langle Tg, g\rangle} \ge -\langle Tu_0, u_0\rangle$$

or

$$\sqrt{8}\sqrt{\langle Tg, g\rangle} \le -\langle Tu_0, u_0\rangle.$$

This inequality contradicts the membership of u_0 in the set M from (16). Therefore, the equation $u + L(u) = g$ has a solution u, for which, due to (15), estimate (13) is satisfied. Theorem 1 is proved.

Remark 3. Note that Theorem 1 can be proved under more general assumptions than conditions U1 and U2. The above follows from the proof of the theorem. The formulation of Theorem 1 given by us is convenient for us.

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