IRSTI 27.31.17 DOI: https://doi.org/10.26577/JMMCS2023v120i4a6

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INTEGRATING MULTIPERIODIC FUNCTIONS ALONG THE PERIODIC CHARACTERISTICS OF THE DIAGONAL DIFFERENTIATION OPERATOR

In this paper, trajectory of time changing along a helical line is represented by parametric equations in Cartesian coordinates of Euclidean space. On the basis of a cycloidal sweep of a cylindrical surface onto a plane, analytical form of a helix is determined. On its basis, integral surface is determined, which is called the periodic characteristic of the diagonal differentiation operator and its connection with its linear characteristic is established. a) elements of new approach related to the periodic characteristic of diagonal differentiation operator are proposed, b) method for reducing integral along the periodic characteristic to an integral with linear characteristic, c) conditions establishing structure of the integral as sum of linear and multiperiodic functions. Some consequences of these results and recommendations of an algorithmic nature for further expansion of research in this direction are given.
Key words: differentiation operator, periodic characteristic, vector field, infinite cylindrical surface, multiperiodicity, autonomous systems.

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Көппериодты функцияларды диагонал бойынша дифференциалдау операторының периодты характеристикалары бойында интегралдау

Жұмыста бұрандалы сызық бойымен өзгеретін уақыт траекториясы Евклид кеңістігінің декарттық координаттарындағы параметрлік теңдеулермен ұсынылған. Әрі қарай, цилиндрлік беттің жазықтықтағы циклоидты жазбасы негізінде бұрандалы сызықтың аналитикалық түрі анықталған. Оның негізінде диагонал бойынша дифференциалдау операторының периодты характеристикасы деп аталатын интегралдық бет анықталды және оның сызықтық характеристикасымен байланысы орнатылды. а) диагонал бойынша дифференциалдау операторының периодты характеристикамен байланысты жаңа тәсілдің элементтері, б) периодты характеристика бойымен интегралды сызықтық характеристикалы интегралға дейін келтіру әдісі және в) сызықты және көппериодты функцияның қосындысы түріндегі интегралдың құрылымын орнататын шарттар ұсынылған. Әрі қарай, осы нәтижелердің кейбір салдарлары және осы бағыттағы зерттеулерді одан әрі кеңейту бойынша алгоритмдік сипаттағы ұсыныстар келтірілген.
Түйін сөздер: дифференциалдау операторы, периодты характеристика, векторлық өріс, шексіз цилиндрлік бет, көппериодтылық, автономды жүйелер.

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Интегрирования многопериодических функций вдоль периодических характеристик операторы дифференцирования по диагонали

В работе траектория времени, изменяющиеся по винтовой линии представлена параметрическими уравнениями в декартовых координатах евклидового пространства. Далее, на основе циклоидальной развертки цилиндрической поверхности на плоскость определен аналитический вид винтовой линии. На ее основе определена интегральная поверхность, которая названа периодической характеристикой оператора дифференцирования по диагонали и установлена её связь с его линейной характеристикой. Предложены а) элементы нового подхода, связанного с периодической характеристикой оператора дифференцирования по диагонали, б) метод сведения интеграла вдоль периодической характеристики к интегралу с линейной характеристикой и в) условия, устанавливающие структуру интеграла в виде суммы линейной и многопериодической функции. Далее, приведены некоторые следствия этих результатов и рекомендации алгоритмического характера по дальнейшему расширению исследований такого направления.

Ключевые слова: оператор дифференцирования, периодическая характеристика, векторное поле, бесконечная цилиндрическая поверхность, многопериодичность, автономные системы.

1 Introduction

The main objective of this study was to reduce the integration of multiperiodic functions along the non-periodic characteristics of the operator D to the integration of their along δ-characteristics, that is, along the diagonal of the space of independent variables and to establish the structure of the integral of multiperiodic functions. To solve these issues, the necessary information was provided about the periodic β-characteristics of the operator D and about their scans on the plane. The connection between β and δ characteristics has been established. On this basis, the solution of the main problem, which is important in the theory of multi-frequency oscillations, is given. These questions are studied in detail in the two-dimensional case of time variables, and then their ideas are extended to the multidimensional case. Next, the related a) differentiation operators in the directions of a constant vector are given, b) the vector form of such operators, the components of which are operators with constant vector fields, and c) operators that are compositions of two operators of the form a). Methods for constructing the β-characteristics of these operators are indicated and algorithms for integrating multiperiodic functions along the β-characteristics are given.

Along with the concept of the periodic characteristic of the differentiation operator D, the only novelty of the study is the result on establishing the structure of the integral of a multiperiodic function, which has an application to solving the problem of integrals of quasi-periodic functions. These innovations have become a reality thanks to some methods of work [1–20].

2 Information about periodic characteristics and their scans

2.1. Obviously, if the variable τ changes on the numeric axis R, then the values $\tau + j\theta$ for $j \neq 0$, $\theta = const > 0$ and τ are two different points of the numeric axis (Figure 1):

$$\tau + j\theta \neq \tau, \ \tau \in R, \ j \neq 0, \ \theta = const > 0.$$

Now consider the variable t, which changes on the circle S centered at the origin of the plane xOy with radius r and length $2\pi r = \theta$ (Figure 2).

The peculiarity of the points t of the circle S is that two analytical representations: t and $t + k\theta$, $k \in Z$, the same point corresponds to the circle. Therefore, the geometric identity of two points t and t_0 on a circle (Figure 3) is represented as

$$t = t^0 + k\theta \Leftrightarrow t = t_0,\ t \in S,\ t^0 \in S. \tag{1}$$

2.2. Next, consider the characteristic equation

$$\frac{dt}{d\tau} = 1 \tag{2}$$

of the diagonal differentiation operator of the form D:

$$D = \frac{\partial}{\partial\tau} + \frac{\partial}{\partial t}, \tag{3}$$

acting on a plane R^2 with Cartesian coordinates (τ, t).

The characteristic of operator (3) originating from the origin defined by equation (2) is the main diagonal

$$t = \tau$$

coordinate systems, and other characteristics parallel to it are straight:

$$t = t^0 + \tau - \tau^0 \equiv \delta(\tau, \tau^0, t^0),\ (\tau^0, t^0) \in R \times R = R^2. \tag{4}$$

As can be seen from (4), equation (2) has no periodic period θ solutions, and therefore operator (3) on R^2 does not have periodic characteristics.

2.3. Now we associate equation (2) with the circle S, assuming that τ is a time variable, that is, it remains as a parameter, and the variable t, respectively, with τ changes along the circle S, as the solution of the equation under consideration.

This reasoning is justified by the fact that the vector field $v(t) \equiv 1$, given by equation (2), has a periodicity of t with an arbitrarily selected period $\theta = const > 0$. So we have every right to consider equation (2) given at $(\tau, t) \in R \times S$.

Then, according to (1), the solution $t = \beta(\tau, \tau^0, t^0)$ of equation (2) with the starting point $(\tau^0, t^0) \in R \times S$ is periodic:

$$\beta(\tau + \theta, \tau^0, t^0) = \beta(\tau, \tau^0, t^0) = t,\ \tau \in R,\ (\tau^0, t^0) \in R \times S \tag{5}$$

and $t = \beta\,(\tau, \tau^0, t^0) \in S$.

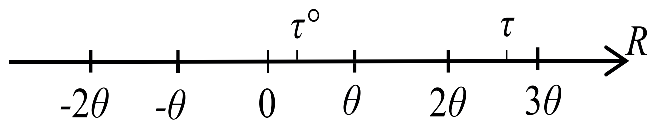

Figure 1: Time of straightline flow

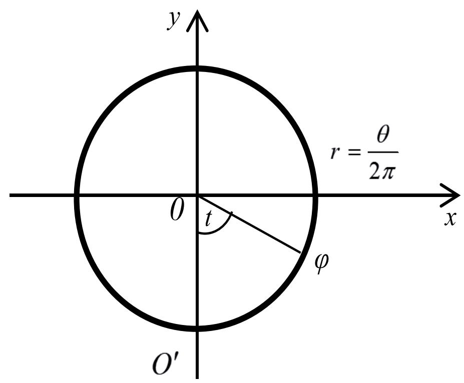

Figure 2: Time of the periodic change

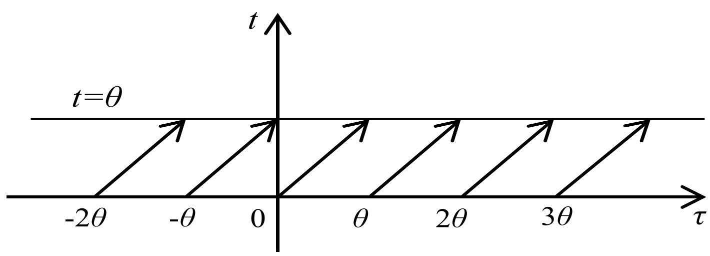

Figure 3: A cycloidal sweep of helical lines

Lemma 1 *Characteristic* (5) *of operator* (3) *defined at* $\tau \in R$, $\tau^0 \in R$ *and* $t^0 \in S$ *with the domain of change* S *has the properties of* θ *periodicity in* τ *and* τ^0, *linearity in* t^0 *and groups as a one-parameter family of motions with parameter* $\sigma = \tau - \tau^0$ *defined by equation* (2).

2.4. To get information about the solution $t = \beta(\tau, 0, 0) \equiv \beta^*(\tau)$ by breaking the circle S at the point O', we position its sweep on a straight line $t = -\tau$ and subject it to a vertical τ-shift.

Then all the arcs on the circle S on the sweep turn into segments, and when combining the plane xOy with the plane τOt from the graph θ-periodic function $t = \beta^*(\tau)$ is represented by the formula

$$t = \theta \left\{\theta^{-1}\tau\right\} \equiv s^*(\tau), \ \tau \in R, \tag{6}$$

where $\{\tau\}$ is the fractional part of the number τ. The graph of this function is shown in figure 3. Note that the sweep function (6) is the projection of the circular helix (Figure 4 on the plane τOt, where $O(0,0,0) = O$, $O^{'}(0,0,r) = O^{'}$, $P^{'}(0,t,\sigma) = P^{'}$, $P(\tau,t,\sigma) = P$, $\psi = \tan\varphi, [0,\psi] \subset Ot = R$, $\varphi = \angle OO^{'}P^{'}$.

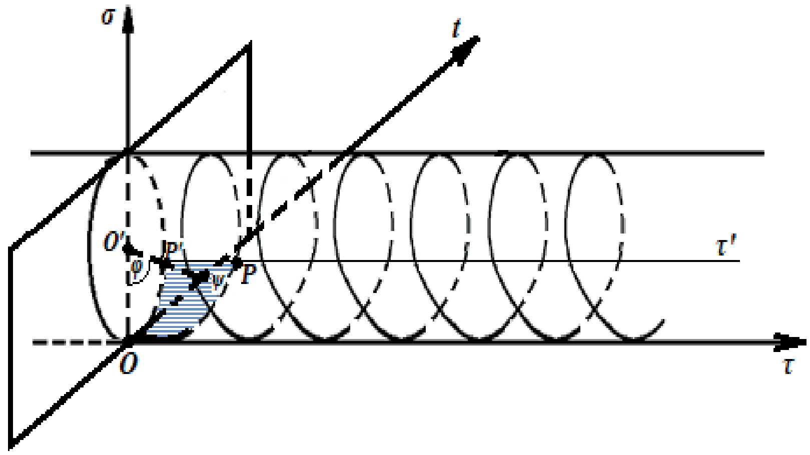

Figure 4: The projection of the circular helix on the plane τOt

Obviously, the function $t = s^*(\tau)$ tolerates discontinuities at points $\tau = k\theta$, $k \in Z$ and in addition to the continuity of the function $t = \beta(\tau)$ on the circle, it represents all other properties concerning the lengths of arcs, periodicity and smoothness between neighboring discontinuity points, and

$$\frac{ds^*(\tau)}{d\tau} = 1 - \sum_{k\in Z}\delta(\tau - k\theta), \quad \frac{ds^*(k\theta - 0)}{d\tau} = \frac{ds^*(k\theta + 0)}{d\tau} = 1, \ k \in Z, \tag{7}$$

where $\delta(\tau)$ is the Dirac delta function. Therefore, from the property (7), additions of the form are suggested

$$\frac{ds^*(k\theta)}{d\tau} = 1, \ k \in Z. \tag{8}$$

Thus, combining (7) and (8), we have a continuous derivative

$$\frac{ds^*(\tau)}{d\tau} = 1, \ \tau \in R \tag{9}$$

of the sweep $t = s^*(\tau)$ of θ-periodic characteristic $t = \beta^*(\tau)$ of the operator D with a range of values S and a definition area R.

Obviously,

$$t = t^0 + s^*(\tau - \tau^0) \equiv s(\tau, \tau^0, t^0) \tag{10}$$

there is a sweep of the solution (5) of equation (2) on the plane τOt.

By virtue of (9), the solution of the general form (10) satisfies equation (2). Due to the autonomy of (2), the solution (10) as a mapping R into itself has the properties of a one-parameter group a) identity, b) inversion and c) compositionality:

$$a)\, s(\tau^0, \tau^0, t^0) = t^0,$$

$b)\ s(\tau^0, \tau, t) = t^0,$ (11)

$c)\ s(\sigma, \tau^0, s(\tau^0, \tau, t)) = s(\sigma, \tau, t).$

Note that functions (4) and (5) also have properties *a*)-*b*). Thus, the following lemma is proved.

Lemma 2 *The sweep* (10) *of the* θ*-periodic characteristic* (5) *satisfies equation* (2) *and has the properties of* θ*-periodicity in* τ, τ^0, *linearity in* t^0 *and the properties of group a)-b).*

In accordance with lemma 2, it can be shown that a helix originating from a point ($\widetilde{\tau} = 0, t = 0, \sigma = r$) in Euclidean space $(\widetilde{\tau}, x, y)$ has a parametric equation $\widetilde{\tau} = r\tau$, $x = r\beta^*(\tau)$, $y = r + r\beta^*\left(\tau - \dfrac{\theta}{4}\right)$ where $\varphi = \tau$ is taken into account.

2.5. Thus, along with the natural rectilinear time $t = \tau$, time $t = \beta^*(\tau)$ was introduced, where $\beta^*(\tau)$ changes θ-rotationally and has the property of continuity on the circle S, but it is discontinuous on a flat sweep in the form of a function $t = s^*(\tau)$, where $\beta^*(\tau) = \beta(\tau, 0, 0)$. The transition from $t = \beta^*(\tau)$ to $t = s^*(\tau)$ makes it possible to measure quantities on a manifold $M = S$, for example, associated with the integral of a function $f(\tau, t)$ along β-characteristics a given on R^2 or on a part of $G \subset R^2$. In this regard, it is necessary to consider the difference between $\sigma(\tau)$ rectilinear time τ and rotational time $s^*(\tau)$:

$$\sigma^*(\tau) = \tau - s^*(\tau),\ \tau \in R. \tag{12}$$

Obviously, for $\tau \in [k\theta, (k+1)\theta]$, $k \in Z$ we have

$$\sigma(\tau) = \begin{cases} k\theta, k\theta \le \tau < (k+1)\theta,\ k \in Z, \\ (k+1)\theta, \tau = (k+1)\theta,\ k \in Z. \end{cases}$$

Hence, $\sigma(\tau)$ is a discontinuous θ-step function with derivative

$$\frac{d\sigma(\tau)}{d\tau} = \theta \sum_{k \in Z} \delta(\tau - k\theta), \quad \frac{d\sigma(k\theta+)}{d\tau} = \frac{d\sigma(k\theta - 0)}{d\tau} = 0.$$

Adding the values $\dfrac{\sigma(k\theta)}{d\tau} = 0$, $k \in Z$ by virtue of the last derivative we have

$$\frac{d\sigma^*(\tau)}{d\tau} = 0, \tau \in R.$$

By analogy with the relations (10), we introduce the function

$$\sigma(\tau, \tau^0, t^0) = t^0 + \sigma^*(\tau - \tau^0) = t, \tag{13}$$

moreover, by virtue of (12) it has the properties of group a)-b) and the difference of functions (4) and (13) is determined by

$$\delta(\tau, \tau^0, t^0) - \beta(\tau, \tau^0, t^0) = \delta(\tau, \tau^0, t^0) - s(\tau, \tau^0, t^0) = \sigma(\tau - \tau^0) = -\sigma(\tau^0 - \tau). \tag{14}$$

Since the function $\delta(\xi, \tau, t)$ with respect to the argument t has the property of linearity then from (14) it follows

$$\beta(\xi, \tau, t) = \delta(\xi, \tau, t) + \sigma(\tau - \xi) \equiv \delta(\xi, \tau, t + \sigma(\tau - \xi)),\ \xi \in R. \tag{15}$$

Lemma 3 *Linear* δ*-characteristics* (4) *and* θ*-periodic* β*-characteristics* (5) *of operator* (3) *are related by relation* (15).

3 Integrals of multiperiodic functions along periodic characteristics

3.1. Further, we note that along with θ-rotational time, we consider ω-rotational time t at $\omega < \theta$. It can be viewed along a circle S_0 by matching the points of a small circle S_0 to the points of an arc of a circle S of length $2\pi\theta > 2\pi\omega$. In the case $\theta k = \omega k_0$, then t along S makes a periodic movement of the period $p = k_0\theta = k\omega$, $(k_0, k) \in Z \times Z$. If there are no such integers k_0, k, then t performs a rotational movement, but non-periodic. Such a circumstance does not violate the (θ, ω)-periodicity of the composition of functions $f(\xi, \eta)$ and $\eta = \beta(\xi, \tau, t)$ by (τ, t), since $f(\xi, \eta)$ is ω-periodic by η, and $\beta(\xi, \tau, t)$ has the property $\beta(\xi, \tau, t + \omega) = \beta(\xi, \tau, t) + \omega$ according to lemma 1.

Thus, we have

$$f(\xi, \beta(\xi, \tau + \theta, t + \omega)) = f(\xi, \beta(\xi, \tau, t) + \omega) = f(\xi, \beta(\xi, \tau, t)), \tag{16}$$

where it is taken into account that $\beta(\xi, \tau + \theta, t) = \beta(\xi, \tau, t)$.

Also note that such a composition is θ-periodic and by ξ:

$$f(\xi + \theta, \beta(\xi + \theta, \tau, t)) = f(\xi, \beta(\xi, \tau, t)),\ \xi \in R,\ (\tau, t) \in R \times S. \tag{17}$$

3.2. According to the Cauchy characteristic method, the equation

$$Dx = f(\tau, t), \quad f(\tau + \theta, t + \omega) = f(\tau, t) \in C_{\tau,t}^{(0,1)}(R \times R) \tag{18}$$

with operator (3) is equivalent to a system of characteristic equations

$$\begin{cases} \dfrac{dt}{d\tau} = 1 \\ \dfrac{dx}{d\tau} = f(\tau, t), \end{cases} \tag{19}$$

where the first equation (2) does not depend on the second equation of the system (19). Therefore, based on expediency, the first equation can be considered on any variety. In this case, we consider it on the direct product of a straight line (Figure 1) and a circle (Figure 2), which in three-dimensional Euclidean space represents an infinite cylindrical surface with generators parallel to the axis $O\tau$. On this manifold, as we have shown above, equation (2)has the first integral: $\beta(\xi, \tau, t)$-characteristic. Then the second equation along this first integral has the form of the equation

$$\frac{dx}{d\xi} = f(\xi, \beta(\xi, \tau, t)), \tag{20}$$

which is defined when $\xi \in I \subset R$, $(\tau, t) \in R \times S =$ Ц is a cylindrical surface, I is a gap enclosed by the points τ^0 and τ of the axis R, and the right part, according to (16) and (17), has the properties of θ-periodicity by ξ and (θ, ω)-periodicity by (τ, t).

Thus, the problem of (θ, ω)-periodic solutions for equation (18) were reduced to the definition of θ–periodic solutions of the integral equation

$$x(\tau, t) = u(\beta(\tau^0, \tau, t)) + \int_{\tau^0}^{\tau} f(\xi, \beta(\xi, \tau, t))d\xi,\ (\tau, t) \in R \times S, \tag{21}$$

which is derived from equation (20) taking into account the initial condition

$$x|_{\tau=\tau^0} = u(t), \quad u(t+\omega) = u(t) \in C_t^{(1)}(R) \tag{22}$$

for equation (18).

As noted above, the calculation of the integral can be carried out according to lemma 3, using the transition from β–characteristics to their sweeps, and under integrals to δ–characteristics.

Then the solution of the problem of the integral of a multiperiodic continuously differentiable functions along periodic characteristics is represented as

$$x(\tau,t) = u(s(\tau^0,\tau,t)) + \int_{\tau^0}^{\tau} f(\xi,\delta(\xi,\tau,t+\sigma))d\xi, \tag{23}$$

where $\tau^0 \in R$, $(\tau,t) \in R \times R$, $\sigma = \sigma(\tau-\xi)$ and in the process of integration σ behaves as a parameter.

Thus, the following theorem is proved.

Theorem 1 *Multiperiodicity of the solution of the problem* (18), (22) *on integrals smooth (θ,ω)–periodic functions are uniquely solved by the integral equation* (21) *along β-periodic characteristics, and the representation of its solution in Euclidean space is determined by the ratio* (23) *along δ-characteristics.*

Regarding the proof of theorem 1, we note that all its statements are justified by lemmas 1, 2, as well as by the relations (19)-(23). The uniqueness of the solution follows from the equivalence of the problem with the system (19) with the corresponding initial conditions.

3.3. It is possible to determine the structure of the integral of a multiperiodic function along a periodic characteristic.

Theorem 2 *If $f(\tau,t)$ is a continuously differentiable multiperiodic function of periods (θ,ω):*

$$f(\tau+\theta,t+\omega) = f(\tau,t) \in C_{\tau,t}^{(0,1)}(R \times R), \tag{24}$$

then its integral along θ–periodic β–characteristics has the following structure:

$$\int_0^{\tau} f(\xi,\beta(\xi,\tau,t))d\xi = \tau \cdot c(\tau,t) + \varphi(\tau,t), \tag{25}$$

where $c(\tau,t)$ is a constant along β–characteristics, a smooth function:

$$c(\tau+\theta,t+\omega) = c(\tau,t) \in C_{\tau,t}^{(1,1)}(R \times R), \quad Dc(\tau,t) = 0. \tag{26}$$

and $\varphi(\tau,t)$ is a multiperiodic of periods (θ,ω) a function with the smoothness property:

$$\varphi(\tau+\theta,t+\omega) = \varphi(\tau,t) \in C_{\tau,t}^{(1,1)}(R \times R). \tag{27}$$

Proof. Let's put

$$c(\tau,t)=\frac{1}{\theta}\int_0^\theta f(\xi,\beta(\xi,\tau,t))d\xi. \tag{28}$$

By virtue of the condition (24) and the smoothness of $\beta(\xi,\tau,t)$ we have continuity over ξ and smoothness by (τ,t) of the composition of the function $f(\xi,\eta)$ and $\eta=\beta(\xi,\tau,t)$ of the form of a subintegral function $f=(\xi,\beta(\xi,\tau,t))$. This implies the smoothness of $c(\tau,t)$ by (τ,t). From the linearity of $\beta(\xi,\tau,t)$ with respect to t, the multiperiodicity of f by (τ,t) and θ–periodicity $\beta(\tau,t)$ by τ we have (θ,ω) –periodicity of the function $c(\tau,t)$.

Since $D\beta(\xi,\tau,t)=0$, we have

$$Dc(\tau,t)=\frac{1}{\theta}\int_0^\theta \frac{\partial f(\xi,\beta)}{\partial\beta}\cdot D\beta(\xi,\tau,t)d\xi=0.$$

Thus, it is proved that the function (28) satisfies the condition (26).

Next, let's put

$$\varphi(\tau,t)=\int_0^\tau f(\xi,\beta(\xi,\tau,t))d\xi-\frac{\tau}{\theta}\int_0^\theta f(\xi,\beta(\xi,\tau,t))d\xi. \tag{29}$$

The periodicity of $\varphi(\tau,t)$ by t with the period ω follows from the property $\beta(\xi,\tau,t+\omega)=\beta(\xi,\tau,t)+\omega$ and conditions (24). Now let's check θ-periodicity by τ directly:

$$\varphi(\tau+\theta,t)=\int_0^{\tau+\theta} f(\xi,\beta(\xi,\tau+\theta,t))d\xi-\frac{\tau+\theta}{\theta}\int_0^\theta f(\xi,\beta(\xi,\tau+\theta,t))d\xi=$$

$$=\int_0^\theta f(\xi,\beta(\xi,\tau,t))d\xi+\int_\theta^{\tau+\theta} f(\xi,\beta(\xi,\tau,t))d\xi-\frac{\tau}{\theta}\int_0^\theta f(\xi,\beta(\xi,\tau,t))d\xi-\int_0^\theta f(\xi,\beta(\xi,\tau,t))d\xi=$$

$$=\int_0^\theta f(\xi+\theta,\beta(\xi+\theta,\tau,t))d\xi-\frac{\tau}{\theta}\int_0^\theta f(\xi,\beta(\xi,\tau,t))d\xi=\varphi(\tau,t).$$

Since $\beta(\xi+\theta,\tau,t)=\beta(\xi,\tau+\theta,t)=\beta(\xi,\tau,t)$ and $f(\xi,\eta)$ satisfies the condition (24).

Thus, the function (29) satisfies the requirement (27). Smoothness follows from the fact that τ is the upper limit of the integral and integral functions are continuously differentiable due to the smoothness of f and β.

From (28) and (29) follows the identity (25). Theorem 2 is proved.

By putting $t=\tau$, from (25) one can obtain the structure of the integral of the quasi-periodic function $f(t,\tau)=\varphi(\tau)$, generated by the multiperiodic function (24).

4 Periodic characteristics and integration of a function along them in the multidimensional case

Now consider the differentiation operator D in the multidimensional case, which has the form

$$D=\frac{\partial}{\partial\tau}+\sum_{j=1}^m\frac{\partial}{\partial t_j}=\frac{\partial}{\partial\tau}+\left\langle e,\frac{\partial}{\partial t}\right\rangle, \tag{30}$$

where $e = (1, \ldots, 1)$ is m–vector, $\dfrac{\partial}{\partial t} = \left(\dfrac{\partial}{\partial t_1}, \ldots, \dfrac{\partial}{\partial t_m}\right)$ is vector operator, $\tau \in R$, $t = (t_1, \ldots, t_m) \in R \times \ldots \times R = R^m$.

The operator (30) corresponds to the characteristic equation

$$\frac{dt}{d\tau} = e, \tag{31}$$

which is a direct product of m equations

$$\frac{dt_j}{d\tau} = 1, \ j = \overline{1, m}, \tag{32}$$

which, with a fixed j, were studied in points 1.1-1.5.

Consequently, the periodic characteristics of equation (31) are determined by the vector function

$$\beta(\tau, \tau^0, t^0) = \left(\beta(\tau, \tau^0, t_1^0), ..., \beta(\tau, \tau^0, t_m^0)\right), \tag{33}$$

which is a direct product of the characteristics of the systems of equations (32), where $t^0 = (t_1^0, \ldots, t_m^0)$ and the vector notation β is left unchanged, as in the scalar case.

According to (5) the characteristic (33) θ-periodic with respect to τ and τ^0, linear with respect to t^0, and $\tau \in R$, $\tau^0 \in R$, $t^0 \in S^m$.

The system (32), therefore, the equation (30) is autonomous, and $\eta = \beta\left(\tau, \tau^0, t^0\right)$, as a family of transformations with the parameter $\sigma = \tau - \tau^0$ has the properties of a) identity, b) reversibility, and c) group. These properties are represented as

$$a)\beta(\tau^0, \tau^0, t^0) = t, \qquad b)t^0 = \beta(\tau^0, \tau, t), \qquad c)\beta(\sigma, \tau^0, \beta(\tau^0, \tau, t)) = \beta(\sigma, \tau, t), \tag{34}$$

$$\beta(\tau + \theta, \tau^0, t^0) = \beta(\tau, \tau^0 + \theta, t^0) = \beta(\tau, \tau^0, t^0), \quad \beta(\tau, \tau^0, t^0 + \omega) = \beta(\tau, \tau^0, t^0) + \omega, \tag{35}$$

where $\omega = (\omega_1, \ldots, \omega_m)$ is const, $\tau \in R, \tau^0 \in R, \sigma \in R, t_0 \in S^m, \omega \in S^m$;

$$\frac{d\beta(\tau, \tau^0, t^0 n)}{d\tau} = e, \quad D\beta(\tau^0, \tau, t) = 0. \tag{36}$$

Similarly, based on (6)-(11) we have a net of $s(\tau, \tau^0, t^0)$ periodic characteristics $\beta(\tau, \tau^0, t^0)$ in Euclidean space with m-dimensional Cartesian coordinates, having properties similar to (34)-(36):

$$a)s(\tau^0, \tau^0, t^0) = t^0, \qquad b)s(\tau^0, \tau, t) = t^0, \qquad c)s(\sigma, \tau^0, s(\tau^0, \tau, t)) = s(\sigma, \tau, t), \tag{37}$$

$$s(\tau^0 + \theta, \tau, t) = s(\tau^0, \tau + \theta, t) = s(\tau^0, \tau, t), \quad s(\tau^0, \tau, t + \omega) = s(\tau^0, \tau, t) + \omega, \tag{38}$$

$$\frac{ds(\tau, \tau^0, t^0)}{d\tau} = e, \quad Ds(\tau^0, \tau, t) = 0, \ \tau^0 \in R, \ \tau \in R, \ t \in R^m. \tag{39}$$

About the properties of the characteristics of $\delta(\tau, \tau^0, t^0) = t^0 + \tau - \tau^0$ of the operator (30), defined in Euclidean space can be found from the work [1–3]. They also have properties similar to (34) and (36). The relationship between these characteristics is established by the relations

$$\beta(\xi, \tau, t) = \delta(\xi, \tau, t + \sigma(\tau - \xi)), \tag{40}$$

where the vector function $\sigma(\tau)$ is determined by the relation

$$\sigma(\tau) = e(\tau - s^*(\tau)), \quad es^*(\tau) = s(\tau, 0, 0). \tag{41}$$

Thus, we come to a theorem generalizing lemmas 1, 2, 3.

Theorem 3 *The characteristic $\eta = \beta(\xi, \tau, t)$ with the parameter $\xi \in R$, defined at $(\tau, t) \in R \times S^m$ has the properties of the group* (34)*, periodicity* (35) *and the characteristics of* (36) *for the operator* (30)*, and its sweep in Euclidean space $\zeta = s(\xi, \tau, t)$, $\xi \in R$, $(\tau, t) \in R \times R^m$, related to it by the relations* (40) *and* (41) *also has properties* (37)-(39)*, similar to the property of the characteristic.*

Further, in order to generalize theorems 1, 2 to the multidimensional case, we consider a vector equation of the form

$$\begin{aligned} &Dx = f(\tau, t), \\ &f(\tau + \theta, t + \omega) = f(\tau, t) \in C^{(0,e)}_{\tau,t}(R \times R^m), \\ &x = (x_1, ..., x_n), f = (f_1, ..., f_n), \tau \in R, t = (t_1, ..., t_m) \in R \times ... \times R = R^m \end{aligned} \tag{42}$$

with the operator (30). It is obvious that the vector equation (42) consists of a direct product of scalar equations (18). Therefore, it is possible to formulate the following two theorems without proof.

For certainty, we put to be specific, we set

$$\begin{aligned} &x|_{\tau=\tau^0} = u(t), \\ &u(t + \omega) = u(t) \in C^e_t(R^m). \end{aligned} \tag{43}$$

Theorem 4 *Solution of the problem* (42)-(43) *in the space of multiperiodic functions of periods* (θ, ω) *is equivalent to the vector integral equation*

$$x(\tau, t) = u(\beta(\tau^0, \tau, t)) + \int_{\tau^0}^{\tau} f(\xi, \beta(\xi, \tau, t))d\xi, (\tau, t) \in R \times S^m,$$

which on the Euclidean net $R \times R^m = R^{m+1}$ of the cylindrical space $R \times S^m$ is represented by equation

$$x(\tau, t) = u(s(\tau^0, \tau, t)) + \int_{\tau^0}^{t} f(\xi, \delta(\xi, \tau, t + \sigma))d\xi,$$

where $\tau^0 \in R, \sigma = \sigma(\tau - \xi)$.

Since the vector function $f(\tau, t)$, the expressions (42), the expressions (24)-(29) acquire a vector form. Therefore, in the following formulation, for brevity, we use the same notation.

Theorem 5 *Under the condition* (24) *for the vector function $f(\tau, t)$, its integral* (25) *is represented using the vector functions* (28) *and* (29)*, which respectively have the properties* (26) *and* (27)*.*

Theorem 5, which defines the structure of the integral of a multiperiodic vector function, is essential in the theories of multiparticle oscillations.

5 Integrals of multiperiodic functions along periodic characteristics of some other differentiation linear operators over vector fields

5.1. The linear operator of differentiation D n the directions of the vector field $v(\tau, t)$ is an operator of the form

$$D = \frac{\partial}{\partial \tau} + \left\langle v(\tau, t), \frac{\partial}{\partial t} \right\rangle, \tag{44}$$

where $\tau \in R$, $t = (t_1, \ldots, t_m) \in R \times \ldots \times R = R^m$, $\dfrac{\partial}{\partial t} = \left(\dfrac{\partial}{\partial t_1}, \ldots, \dfrac{\partial}{\partial t_m} \right)$ is vector operator, $v(\tau, t) = (v_1(\tau, t), \ldots, v_m(\tau, t))$ is vector-function of variables $(\tau, t) \in G \subset R^{m+1}$, $\langle \, , \rangle$ is the sign of the scalar product.

When a system defined with the operator D does not depend on (τ, t), then it is called autonomous. Therefore, in this case v is a constant vector: $v = c = (c_1, \ldots, c_m)$, where $c_j \neq 0, j = \overline{1, m}$ are constant coordinates. Then the differentiation operator D has the form

$$D = \frac{\partial}{\partial t} + \left\langle c, \frac{\partial}{\partial t} \right\rangle. \tag{45}$$

If each equation of the system is given by the same differentiation operator (44), then it is said that the system has the same main part or a system with one differentiation operator by (τ, t) in the directions of the vector field

$$\frac{dt}{d\tau} = v(\tau, t). \tag{46}$$

In particular, the vector field (46) can be given by the constant vector $\nu = c$. Then we have a vector field

$$\frac{dt}{d\tau} = c,$$

corresponding to the operator (46).

Each equation of a system consisting of two or more equations can have its differentiation operator of the form (44) or (46). For example, if we restrict ourselves to constant vector fields and the cases $m = 1, n = 2$, then we have two operators D_1 and D_2 with two variables $\tau \in R$ and $t \in R$ of the form

$$\begin{aligned} D_1 &= \frac{\partial}{\partial t} + \nu_1 \frac{\partial}{\partial t}, \\ D_2 &= \frac{\partial}{\partial \tau} + \nu_2 \frac{\partial}{\partial \tau} \end{aligned} \tag{47}$$

with constants ν_1 and ν_2.

Of particular interest is the second-order operator $D^2_{1,2}$, which is a composition of two linear operators (47) of the form

$$D^2_{1,2} = D_1 D_2.$$

Now let's define the periodic characteristics of these operators.

5.2. The case of systems with a one differentiation operator in the direction of a constant vector. In this case, we are dealing with the operator (45). The operator of autonomous systems also applies to this case. In particular, the operator (45) can have the form (30), where $c = e$. Then θ-periodic by τ characteristic of the structure of the form

$$t = \beta(\tau, \tau^0, t^0) \equiv t^0 + \beta^*(\tau - \tau^0), \quad \beta^*(\tau + \theta) = \beta^*(\tau) \tag{48}$$

corresponds to this operator in accordance with 1, where $(\tau^0, t^0) \in R \times S$, $(\tau, t) \in R \times S$.

Now, in order to build a θ-periodic by τ characteristic, we use the relation (48). To do this, the operator (45) by linear replacement

$$\widetilde{t}_j = c_j^{-1} t_j; \quad t_j = c_j \widetilde{t}_j, \ j = \overline{1, m} \tag{49}$$

is reduced to the form of the operator (30):

$$D_c = \frac{\partial}{\partial \tau} + \sum_{j=1}^{m} c_j \frac{\partial}{\partial t_j} = \frac{\partial}{\partial \tau} + \sum_{j=1}^{m} c_j \frac{\partial}{\partial \widetilde{t}_j} \cdot \frac{\partial \widetilde{t}_j}{\partial t_j} = \frac{\partial}{\partial \tau} + \sum_{j=1}^{m} c_j \frac{\partial}{\partial \widetilde{t}_j} \cdot c_j^{-1} = \frac{\partial}{\partial \tau} + \sum_{j=1}^{m} c_j \frac{\partial}{\partial \widetilde{t}_j} = D_e.$$

According to (48), the operator D_e has a θ-periodic characteristic

$$\widetilde{t}_j = \widetilde{t}_j^0 + \beta^*(\tau - \tau^0), \ j = \overline{1, m}$$

which, according to (49), we represent as

$$c_j^{-1} = c_j^{-1} t_j^0 + \beta^*(\tau - \tau^0), \ j = \overline{1, m}$$

or as

$$t_j = t_j^0 + c_j \beta^*(\tau - \tau^0), \ j = \overline{1, m}, \tag{50}$$

we have a characteristic system of the operator D_c.

Therefore, in the vector form (50) we write in the form

$$t = t^0 + c\beta^*(\tau - \tau^0), \quad \beta^*(\tau + \theta) = \beta^*(\tau). \tag{51}$$

Thus, (51) is a characteristic of the operator (45), periodic with respect to τ and τ^0 of the period θ, and with respect to t^0 is linear.

And so, next statement is proved.

Statement 1 *The operator* (45) *has a characteristic of the form* $t = t^0 + c\beta^*(\tau - \tau^0) \equiv \widetilde{\beta}(\tau, \tau^0, t^0)$, *which has the properties*

$$\beta(\tau + \theta, \tau^0, t) = \beta(\tau, \tau^0 + \theta, t) = \beta(\tau, \tau^0, t^0), \ \ \beta(\tau, \tau^0, t^0 + \omega) = \beta(\tau, \tau^0, t^0) + \omega, \tag{52}$$

as well as the properties of the group, as a one-parameter family of transformations of a) identity, b) reversibility and c) compositionality.

Note that the properties (52) are a consequence of the property (51), and the properties a)-b) are known from lemma 1.

5.3. The case of systems with two differentiation operators in the directions of constant vectors. Operators of the form (47) belong to this case. According to statement 1, these operators have θ-periodic characteristics of the form

$$t = t^0 + \nu_1 \beta^*(\tau, \tau^0) \equiv \beta_1(\tau, \tau^0, t^0), \quad t = t^0 + \nu_2 \beta^*(\tau, \tau^0) \equiv \beta_2(\tau, \tau^0, t^0). \tag{53}$$

From statement 1 and the relations (53) we get statement 2:

Statement 2 *Operator $D = (D_1, D_2)$ has θ- periodic by τ and τ^0 characteristics of the form $\bar{t} = (\beta_1(\tau, \tau^0, t^0), \beta_2(\tau, \tau^0, t^0))$, defined by the relations* (53)*, having the properties of linearity in t^0 and groups a)-b).*

5.4. The case of the canonical second-order differentiation operator. First, we give an explanation regarding the name of this case It is well known that the canonical hyperbolic equation

$$\frac{\partial^2 z}{\partial \tau^2} = a^2 \frac{\partial^2 z}{\partial t^2}, \; a = const > 0$$

with the time variable τ and the spatial variable t Euler by linear substitution

$$\beta_1 = \tau + at, \beta_2 = \tau - at$$

led to another canonical form of the equation of this type:

$$\frac{\partial^2 z}{\partial \beta_1 \partial \beta_2} = 0.$$

And immediately obtained the general integral

$$z = f(\beta_1) + \varphi(\beta_2) \equiv f(\tau + at) + \varphi(\tau - at).$$

It is obvious that there is a certain interest of applied scientists about multiperiodic solutions of equation

$$\frac{\partial^2 z}{\partial \beta_1 \partial \beta_2} + c_1 \frac{\partial z}{\partial \beta_1} + c_2 \frac{\partial z}{\partial \beta_2} + c_3 z = f(\tau, t), \; f(\tau + \theta, t + \omega) = f(\tau, t)$$

with constants or (θ, ω)-periodic by (τ, t) by the coefficients c_1, c_2 and c_3. The integration of this equation in the homogeneous case was handled by Laplace and we propose the cascade method.

The study of such a problem, combining a classical question with a modern problem of the theory of oscillations, on the basis of periodic characteristics is important in the development of the theory of multiperiodic systems. The existence of periodic characteristics of this operator is proved by statement 2.

5.5. Integration of multiperiodic functions along the periodic characteristics of the corresponding differentiation operators. It is carried out in accordance with the methodology used in the proofs of lemma 3 and theorem 1. To implement this technique, 1) it is necessary to determine the δ-characteristic of the operator in Euclidean space, 2) it is necessary to establish a connection between the δ-characteristic and the β-characteristic in accordance with 3, and then 3) prove analogs of theorems 1, 2. The described algorithm for integrating many periodic functions is used to create the foundations of the theory of multiperiodic systems with corresponding differentiation operators in the directions of vector fields.

Each differentiation operator along the characteristics passes to an ordinary differentiation operator (finding the full derivative), and the corresponding system becomes a system of ordinary differential equations. Consequently, multiperiodic solutions based on characteristics are converted into quasi-periodic solutions of a system of ordinary differential equations. Thus, we have applications of the developed theory to the study of many particular oscillations in physical and technical systems.

Acknowledgment

This research has been funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (grant No. AP19676629).

References

[1] Харасахал В.Х. Почти периодические решения обыкновенных дифференциальных уравнений. – Алма-Ата: Наука, 1970.

[2] Умбетжанов Д.У. Почти многопериодические решения дифференциальных уравнений в частных производных. – Алма-Ата: Наука, 1979.

[3] Умбетжанов Д.У. Почти периодические решения эволюционных уравнений. – Алма-Ата: Наука, 1990.

[4] Умбетжанов Д.У., Сартабанов Ж.А. О необходимом и достаточном условии многопериодичности решения одной системы уравнений в частных производных с одинаковой главной частью // В книге: Математика и механика. – 1972. – Т. 7, Ч. 2. – С. 22-27.

[5] Сартабанов Ж.А. Периодты функциялар және кейбір қарапайым дифференциалдық теңдеулердің периодты шешімдері. – Алматы: РБК, 2001.

[6] Сартабанов Ж.А. Об одном способе изучения периодических решений уравнений в частных производных специального вида // Известия АН КазССР. серия физ.-мат. – 1989. – №1. – С. 42-48.

[7] Мухамбетова А.А., Сартабанов Ж.А. Устойчивость решений систем дифференциальных уравнений. – Актобе: ПринтА, 2007.

[8] Кульжумиева А.А., Сартабанов Ж.А. Периодические решения системы дифференциальных уравнений с многомерным временем. – Уральск: РИЦ ЗКГУ, 2013.

[9] Малкин И.Г. Методы Ляпунова и Пуанкаре в теории нелинейных колебаний. – М: Едиториал УРСС, 2004.

[10] Андронов А.А., Вит А.А., Хайкин С.Э. Теория колебаний. – М: Наука, 1981.

[11] Понтрягин Л.С. Обыкновенные дифференциальные уравнения. – М: Наука, 1965.

[12] Арнольд В.И. Дополнительные главы теории обыкновенных дифференциальных уравнений. – М: Наука, 1978.

[13] Sartabanov Zh.A., Omarova B.Zh., Kerimbekov A. Research of multiperiodic solutions of perturbed linear autonomous systems with differentiation operator on the vector field // News of the National Academy of Sciences of the Republic of Kazakhstan-series Physico-mathematical. – 2020. – 329, No. 1. – Pp. 5-13.

[14] Omarova B.Zh., Sartabanov Zh.A. On multiperiodic solutions of perturbed nonlinear autonomous systems with the differentiation operator on a vector field // Eurasian Math. J. – 2021. – 12:1. – Pp. 68-81.

[15] Zhumagaziyev A.Kh., Sartabanov Zh.A., Sultanaev Ya.T. On a new method for investigation of multiperiodic solutions of quasilinear strictly hyperbolic system // Azerbaijan Journal of Mathematics. – 2022. – 12:1. – Pp. 32-48.

[16] Sartabanov Z.A., Zhumagaziyev A.K., Abdikalikova G.A. Multiperiodic solution of linear hyperbolic in the narrow sense system with constant coefficients // Bulletin of the Karaganda University-Mathematics. – 2020. – 98, No. 2. – Pp. 125-140.

[17] Sartabanov Z.A., Zhumagaziyev A.K., Abdikalikova G.A. On one method of research of multiperiodic solution of block-matrix type system with various differentiation operators // News of the National Academy of Sciences of the Republic of Kazakhstan-Series Physico-Mathematical. – 2020. – 330, No. 2. – Pp. 149-158.

[18] Sartabanov Z.A., Aitenova G.M., Abdikalikova G.A. Multiperiodic solutions of quasilinear systems of integro-differential equations with D_c-operator and ϵ-period of hereditarity // Eurasian Math. J. – 2022. – 13:1. – Pp. 86–100.

[19] Sartabanov Z.A., Aitenova G.M., Abdikalikova G.A. Multiperiodic solution of the initial-boundary value problem for an integro-differential equation of the parabolic type // Russian Mathematics. – 2022. – 66:8. – Pp. 46–55.

[20] Aitenova G.M., Sartabanov Zh.A., Abdikalikova G.A. Multiperiodic bounded oscillations in quasilinear finite-hereditary integro-differential systems convection-diffusion type // Lobachevskii Journal of Mathematics. – 2022. – 43:8. – Pp. 2046-2056.

References

[1] Kharasakhal V.Kh., *Pochti periodicheskie reshenija obyknovennyh differencialnyh uravnenij* [Almost-periodic solutions of ordinary differential equations] (Alma-Ata: Nauka, 1970).

[2] Umbetzhanov D.U., *Pochti mnogoperiodicheskie reshenija differencialnyh uravnenij v chastnyh proizvodnyh* [Almost multiperiodic solutions of partial differential equations] (Alma-Ata: Nauka, 1979).

[3] Umbetzhanov D.U., *Pochti periodicheskie reshenija ievoljucionnyh uravnenij* [Almost periodic solutions of evolution equations] (Alma-Ata: Nauka, 1990).

[4] Umbetzhanov D.U., Sartabanov Zh.A., "O neobhodimom i dostatochnom uslovii mnogoperiodichnosti reshenija odnoj sistemy uravnenij v chastnyh proizvodnyh s odinakovoj glavnoj chastju. [On the necessary and sufficient condition for the multiperiodicity solutions of one system of partial differential equations with the same main part]", *Mathematics and Mechanics* V. 7, No 2 (1972): 22-27.

[5] Sartabanov Zh.A., *Periodty funkcijalar zhane kejbir karapajym differencialdyk tendeulerdi? periodty sheshimderi* [Periodic functions and periodic solutions of some ordinary differential equations] (Almaty: RBK, 2001).

[6] Sartabanov Zh.A., "Ob odnom sposobe izuchenija periodicheskih reshenij uravnenij v chastnyh proizvodnyh special'nogo vida. [On one way to study periodic solutions of partial differential equations of a special form]", *Izvestiya RAN. ser. matem.* No 1 (1989): 42-48.

[7] Mukhambetova A.A., Sartabanov Zh.A., *Ustoichivost reshenji sistem differentsialnykh uravnenij s mnogomernym vremenem* [Stability of solutions of the systems of differential equations with multidimensional time] (Aktobe: Print A, 2007).

[8] Kulzhumieva A.A., Sartabanov Zh.A., *Periodicheskie reshenija sistem differentsialnykh uravnenij s mnogomernym vremenem* [Periodic solutions of the systems of differential equations with multidimensional time] (Uralsk: RIC WKSU, 2013).

[9] Malkin I.G., *Metody Ljapunova i Puankare v teorii nelinejnyh kolebanij* [The methods of Lyapunov and Poincare in the theory of nonlinear oscillations] (M.: Editorial URSS, 2004).

[10] Andronov A.A., Vitt A.A., Khaikin S.E., *Teorija kolebanij* [Theory of oscillators] (M.: Nauka, 1981).

[11] Pontryagin L.S., *Obyknovennye differencial'nye uravnenija* [Ordinary differential equations] (M.: Nauka, 1965).

[12] Arnold V.I., *Dopolnitelnye glavy teorii obyknovennyh differencial'nyh uravnenij* [Additional chapters in the theory of ordinary differential equations] (M.: Nauka, 1978).

[13] Sartabanov Zh.A., Omarova B.Zh., Kerimbekov A., "Research of multiperiodic solutions of perturbed linear autonomous systems with differentiation operator on the vector field", *News of the National Academy of Sciences of the Republic of Kazakhstan-series Physico-mathematical.* 329, No. 1 (2020): 5-13.

[14] Omarova B.Zh., Sartabanov Zh.A., "Lagrange formula for differential operators self-adjoint restrictions of the maximal operator on a tree" , *Eurasian Math. J.,* 12:1 (2021): 68-81.

[15] Zhumagaziyev A.Kh., Sartabanov Zh.A., Sultanaev Ya.T, "On a new method for investigation of multiperiodic solutions of quasilinear strictly hyperbolic system" , *Azerbaijan Journal of Mathematics,* 12:1 (2022): 32-48.

[16] Sartabanov Z.A., Zhumagaziyev A.K., Abdikalikova G.A., "Multiperiodic solution of linear hyperbolic in the narrow sense system with constant coefficients" , *Bulletin of the Karaganda University-Mathematics.* 98, No. 2 (2020): 125-140.

[17] Sartabanov Z.A., Zhumagaziyev A.K., Abdikalikova G.A., "On one method of research of multiperiodic solution of block-matrix type system with various differentiation operators" , *News of the National Academy of Sciences of the Republic of Kazakhstan-series Physico-mathematical.* 330, No. 2 (2020): 149-158.

[18] Sartabanov Z.A., Aitenova G.M., Abdikalikova G.A., "Multiperiodic solutions of quasilinear systems of integro-differential equations with D_c-operator and ϵ-period of hereditarity" , *Eurasian Math. J.,* 13:1 (2022): 86-100.

[19] Sartabanov Z.A., Aitenova G.M., Abdikalikova G.A., "Multiperiodic solution of the initial-boundary value problem for an integro-differential equation of the parabolic type" , *Russian Mathematics,* 66:8 (2022): 46–55.

[20] Aitenova G.M., Sartabanov Zh.A., Abdikalikova G.A., "Multiperiodic bounded oscillations in quasilinear finite-hereditary integro-differential systems convection-diffusion type" , *Lobachevskii Journal of Mathematics,* 43:8 (2022): 2046-2056.