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1-бөлім **Раздел 1** **Section 1**

Математика **Математика** **Mathematics**

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SPECTRUM OF THE HILBERT TRANSFORM ON ORLICZ SPACES OVER $\mathbb{R}$

In this paper, we investigate the spectrum of the classical Hilbert transform on Orlicz spaces L_Φ over the real line $\mathbb{R}$, extending Widom's and Jörgens's results in the context of L^p spaces [3, 8], since the classical Lebesgue spaces are particular examples of Orlicz spaces when the N-function $\Phi = x^p/p$. Our motivation to do so is due to the classical result of Boyd [1] which says that the Hilbert transform is bounded on certain Orlicz spaces and the fact that the spectrum of the bounded linear operator is not an empty set. We first present an auxiliary result from the general theory of Banach algebras and results from general theory of Banach spaces, which further helps us to give a full decsription of the fine spectrum of the Hilbert transform on Orlicz spaces over the real line $\mathbb{R}$. We also present a resolvent set of the Hilbert transform on Orlicz spaces over the real line $\mathbb{R}$ as well as its resolvent operator.

Key words: Hilbert transform, spectrum, point spectrum, Orlicz space.

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$\mathbb{R}$ жиынындағы Орлич кеңістіктерінде анықталған Гильберт түрлендіруінің спектрі

Бұл мақалада біз, Уидом және Йоргенстің L^p кеңістіктерінің контекстіндегі нәтижелерін [3,8] кеңейте отырып, $\mathbb{R}$ нақты түзуінде анықталған L_Φ Орлич кеңістіктеріндегі классикалық Гильберт түрлендіруінің спектрін зерттейміз, себебі классикалық Лебег кеңістіктері Φ N-функциясы $\Phi = x^p/p$ болған кезде Орлич кеңістіктерінің ерекше мысалдары болып табылады. Зерттеу жүргізудегі негізгі мотивациямыз Бойдтың кейбір Орлич кеңістіктеріндегі Гильберт түрлендіруінің шенелгендігі туралы классикалық нәтижесі [1] және жалпы шенелген сызықты операторлардың спектрі бос жиын емес екендігімен байланысты. Біріншіден, біз Банах алгебраларының жалпы теориясынан көмекші нәтижені ұсынамыз, ол әрі қарай $\mathbb{R}$ нақты түзуінде анықталған Орлич кеңістіктеріндегі Гильберт түрлендіруінің дәл спектрін толық сипаттауға көмектеседі. Біз сондай-ақ $\mathbb{R}$ нақты түзуінде анықталған Орлич кеңістіктеріндегі Гильберт түрлендіруінің резольвентті жиынын, сонымен қатар оның резольвенттік операторын анықтаймыз.

Түйін сөздер: Гильберт түрлендіруі, спектр, нүктелік спектр, Орлич кеңістігі.

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Спектр преобразовании Гильберта в пространствах Орлича над $\mathbb{R}$

В данной статье мы исследуем спектр классического преобразования Гильберта в пространствах Орлича L_Φ над вещественной прямой $\mathbb{R}$, расширяя результаты Видома и Йоргенса в контексте L^p пространств [3, 8], поскольку классические пространства Лебега являются частными примерами пространств Орлича, когда N-функция $\Phi = x^p/p$. Наша мотивация в иследовании, обусловлена классическим результатом Бойда [1] об ограниченности преобразования Гильберта в некоторых пространствах Орлича и к тому, что спектр ограниченного линейного оператора не является пустым множеством.

Сначала приведем вспомогательный результат из общей теории банаховых алгебр, который в дальнейшем поможет нам дать полное описание тонкого спектра преобразования Гильберта в пространствах Орлича над вещественной прямой $\mathbb{R}$. Мы также представляем резольвентное множество преобразования Гильберта в пространствах Орлича над вещественной прямой $\mathbb{R}$, а также его резольвентный оператор.

Ключевые слова: преобразование Гильберта, спектр, точечный спектр, пространство Орлича.

1 Introduction

Let $L_\Phi(\mathbb{R})$ be an Orlicz space over the real line $\mathbb{R}$. Define the Hilbert transform $\mathcal{H}$ on the space $L_\Phi(\mathbb{R})$ by the formula

$$\mathcal{H}f(t) := \frac{1}{\pi i} p.v. \int_{-\infty}^{\infty} \frac{f(s)}{t-s} ds, \ t \in \mathbb{R}, \tag{1}$$

where the integral is understood as Caushy principal value. Boundedness of $\mathcal{H}$ acting on $L_\Phi(\mathbb{R})$ obtained by D.W.Boyd in [1, Theorem 5.8]. In the same work, Boyd demonstrated that the necessary condition for the boundedness of the Hilbert Transform is the reflexivity of the Orlicz space, which coincides with the condition of non-triviality of Boyd indices. According to the fundamental principles of spectral theory for linear operators on Banach spaces, it is established that if a linear operator is bounded, then its spectrum must necessarily be non-empty. Hence, the purpose of this paper is to investigate the spectrum of the Hilbert transform, including the classification of points within the spectrum. The spectrum of the Hilbert Transform on $L^p(-1,1), 1 < p < \infty$ was completely identified by Widom in 1960. Additionally, he provided a decomposition of the spectrum into its point spectrum, denoted as $\sigma_{pt}(\mathcal{H})$, continuous spectrum $\sigma_c(\mathcal{H})$, and residual spectrum $\sigma_r(\mathcal{H})$ [8, §5]. In 2021 Guillermo P.Curbera, Susumu Okada and Werner J.Ricker extended Widom's results to any rearrangement invariant Banach spaces over $(-1, 1)$ [9]. They investigated the spectrum and fine spectra of the finite Hilbert transform acting on rearrangement invariant spaces over $(-1, 1)$ with non-trivial Boyd indices. Jörgens demonstrated the spectrum and point spectrum of the Hilbert Transform in the context of $L^p(\mathbb{R})$ and $L^p(\mathbb{R}_+)$, where $1 < p < \infty$, as presented in [3]. In our research, we applied the identical methodology, which also relies on the theory of Banach algebras.

2 Preliminaries

In this section, we provide certain definitions and notations from the theory of Banach algebras and from the theory of linear bounded operators on Banach spaces.

Definition 1 *A complex normed space $\mathcal{A}$ is called a normed algebra if for any elements A, B of $\mathcal{A}$ there is defined a product $AB \in \mathcal{A}$ with the following properties*

- *associativity:* $A(BC) = (AB)C$;
- *distributivity:* $(A + B)C = AC + BC,\ A(B + C) = AB + AC$;
- *homogeneity:* $\alpha(AB) = (\alpha A)B = A(\alpha B), \forall \alpha \in \mathbb{C}$.

The product satisfies the inequality $\|AB\| \leq \|A\|\|B\|$, where $\|\cdot\|$ is a norm of $\mathcal{A}$. If $\mathcal{A}$ is a Banach space, then $\mathcal{A}$ is called a *Banach algebra*. Moreover, if there exists in $\mathcal{A}$ an element I with following properties $AI = IA = A, \forall A \in \mathcal{A}$ and $\|I\| = 1$ then $\mathcal{A}$ is called a normed algebra with unit (respectively Banach algebra with unit). An element $A \in \mathcal{A}$ is called *regular* if there exists $B \in \mathcal{A}$ such that $AB = BA = I$. The element B is the inverse of A and accordingly is denoted by A^{-1}. If an element is not regular, then it is called singular. For every $A \in \mathcal{A}$ we define the resolvent set $\rho(A)$ as the set of all $\lambda \in \mathbb{C}$ such that $(\lambda I - A)$ is regular. For all $\lambda \in \rho(A)$ we defined the *resolvent operator* $R_\lambda(A) = (\lambda I - A)^{-1}$. The complement of $\rho(A)$ is called the spectrum of A and denoted by $\sigma(A)$.

Definition 2 *An element $A \in \mathcal{A}$ is called algebraic if there exists a polynomial*

$$p(\lambda) = \sum_{i=0}^{m} \alpha_i \lambda^i \tag{2}$$

with coefficients $\alpha_i \in \mathbb{C}$ and $\alpha_m \neq 0$ such that

$$p(A) = \sum_{i=0}^{m} \alpha_i A^i = 0. \tag{3}$$

A minimal polynomial of A is a monic polynomial (whose highest-degree coefficient equals 1) $p(x)$ of the lowest degree such that $p(A) = 0$

First note that a minimal polynomial is unique. Indeed, if there are two minimal polynomials, denoted as $p(x)$ and $q(x)$, both of degree m, then, $(p-q)(A) = p(A)-q(A) = 0$. Note that, the degree of the resulting polynomial $(p - q)(x)$ is less than m, contradicting their minimality. Thus, it follows that $p = q$. Additionally, recognize that the minimal polynomial $p(x)$ is irreducible, i.e. it cannot be expressed as the product of two polynomials. To illustrate, if $p(x) = r(x)t(x)$, where both $r(x)$ and $t(x)$ have degrees lower than that of $p(x)$, then $0 = p(A) = r(A)t(A)$. Consequently, either $r(A) = 0$ or $t(A) = 0$, contradicting the minimality of $p(x)$. It is also known that any polynomial $q(x)$ with $q(A) = 0$ is divisible by the minimal polynomial of A. In other words, if there exists a polynomial $q(x)$ satisfying $q(A) = 0$, then $q(x) = p(x)r(x)$, where $p(x)$ is the minimal polynomial of A.

Now we introduce some definitions from the theory of bounded linear opearators on Banach spaces. Let X be a Banach space. For Banach space X we denote by $\mathcal{B}(X)$, the set of all bounded operators from X into itself. It is known that $\mathcal{B}(X)$ is a Banach algebra [2].

Definition 3 *Let $T \in \mathcal{B}(X)$. The spectrum of T is the set of all $\lambda \in \mathbb{C}$ for which the operator $\lambda I - T$ does not have an inverse that is a bounded linear operator, denoted as $\sigma(T)$. The resolvent set $\rho(T)$ of T in X is defined as complement of the spectrum: $\mathbb{C} \backslash \sigma(T)$. In other words, the resolvent set of T is the set of all $\lambda \in \mathbb{C}$ for which the operator $\lambda I - T$ have an inverse that is a bounded linear operator.*

Definition 4 *[7, Definition 1.13] Let $T \in \mathcal{B}(X)$. Define*

$$\sigma_{pt}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \quad \textit{is not injective} \quad \Leftrightarrow \quad Ker\,(\lambda I - T) \neq \{0\}\};$$

$$\sigma_c(T) = \left\{\lambda \in \mathbb{C} : \lambda I - T \quad \textit{is injective} \quad \overline{Im\,(\lambda I - T)} = X, \textit{ but } \quad Im\,(\lambda I - T) \neq X\right\};$$

$$\sigma_r(T) = \left\{\lambda \in \mathbb{C} : \lambda I - T \quad \textit{is injective, but} \quad \overline{Im\,(\lambda I - T)} \neq X\right\}.$$

$\sigma_{pt}(T)$, $\sigma_c(T)$ and $\sigma_r(T)$ are called respectively the point spectrum, the continuous spectrum and the residual spectrum of T in X.

It is known that $\sigma_{\mathrm{pt}}(T)$, $\sigma_{\mathrm{c}}(T)$ and $\sigma_{\mathrm{r}}(T)$ are disjoint [7] and

$$\sigma(T) = \sigma_{\mathrm{pt}}(T) \cup \sigma_{\mathrm{c}}(T) \cup \sigma_{\mathrm{r}}(T).$$

Since $\mathcal{B}(X)$ is a Banach algebra, the definition of the spectrum and resolvent set on Banach algebras and linear bounded operators on Banach spaces coincide. As usual, by $L^p(\mathbb{R})$ we denote the standard Lebesgue space.

Definition 5 *[4, 5] A mapping $\Phi : \mathbb{R} \to \mathbb{R}_+$ is called an N-function if*

- *$\Phi(x) = 0$ iff $x = 0$ and $\Phi(x) > 0$ for $x > 0$;*
- *Φ is convex, continuous and even;*
- $\lim\limits_{x\to 0} \frac{\Phi(x)}{x} = 0$, $\lim\limits_{x\to\infty} \frac{\Phi(x)}{x} = +\infty$.

We say that N-function Φ satisfies the Δ_2-condition, if and only if there exists a constant $k > 0$ such that

$$\Phi(2x) \leq k\Phi(x), \text{ for all } x > 0.$$

For every N-function Φ and for every measurable function f on $\mathbb{R}$ we can define a functional

$$M^{\Phi}(f) = \int_{\mathbb{R}} \Phi(|f|)dx$$

and set

$$\|f\|_{L_\Phi} = \inf\left\{a > 0 : M^{\Phi}\left(\frac{f}{a}\right) \leq 1\right\}.$$

Definition 6 *The set*

$$L_\Phi = \{f \in L : \|f\|_{L_\Phi} < \infty\}$$

equipped with the norm $\|\cdot\|_{L_\Phi}$ *is called an Orlicz function space.*

An Orlicz space is an example of rearrangement invariant spaces. Note that L^p spaces coincide with Orlicz spaces when the N-function has the form $\Phi(x) = x^p/p, p \in (1,\infty)$. For a more general information on Orlicz spaces L_Φ, see [2, 4, 5, 7].

Definition 7 *Let* $L_\Phi(\mathbb{R})$ *be reflexive. If* $f \in L_\Phi(\mathbb{R})$*, then the classical Hilbert transform* $\mathcal{H}$ *is defined by the principal-value integral*

$$\mathcal{H}f(t) := \frac{1}{\pi i} p.v. \int_{-\infty}^{\infty} \frac{f(s)}{t-s}\, ds, \quad \forall f \in L_\Phi(\mathbb{R}),$$

(see, e.g. [2, Chapter III. 4]).

The Hilbert transform $\mathcal{H}$ is bounded on $L_\Phi(\mathbb{R})$ if and only if $L_\Phi(\mathbb{R})$ is reflexive (see, for example, [1, 2, 6]).

3 Methods and materials. Spectrum of the Hilbert transform on $L_\Phi(\mathbb{R})$

In this section, we find the spectrum of the Hilbert transform on $L_\Phi(\mathbb{R})$. First, we present a preliminary result from the of theory of Banach algebras.

Proposition 1 *[3, Exercize 4.10] Let* A *be an algebraic element of* $\mathcal{A}$ *(cf. Definition 2). Then*

1. *The resolvent of* A *has the form*

$$R_\lambda(A) = \frac{1}{p(\lambda)} \sum_{i=0}^{m-1} \lambda^i B_i$$

where

$$B_j = \sum_{k=i+1}^{m} \alpha_k A^{k-i-1}.$$

2. *The spectrum* $\sigma(A)$ *is contained in the set of zeros of* p*:*

$$p_0 = \{\lambda \in \mathbb{C} \mid p(\lambda) = 0\} \supset \sigma(A)$$

3. *If* p *is a minimal polynomial for* A*, then*

$$p_0 = \sigma(A).$$

Proof. 1. We know that

$$R_\lambda(A) = \frac{1}{p(\lambda)}(B_0 + \lambda B_1 + \lambda^2 B_2 + ... + \lambda^{m-1} B_{m-1})$$

and

$$\begin{aligned}
&B_0 = \alpha_1 I + \alpha_2 A + ... + \alpha_m A^{m-1};\\
&B_1 = \alpha_2 I + \alpha_3 A + ... + \alpha_m A^{m-2};\\
&B_2 = \alpha_3 I + \alpha_4 A + ... + \alpha_m A^{m-3};\\
&...\\
&B_{m-2} = \alpha_{m-1} I + \alpha_m A;\\
&B_{m-1} = \alpha_m I.
\end{aligned}$$

Then we have

$$\begin{aligned}
p(\lambda)R_\lambda(A) &= (\alpha_1 I + \alpha_2 A + ... + \alpha_m A^{m-1})\\
&+ \lambda(\alpha_2 I + \alpha_3 A + ... + \alpha_m A^{m-2})\\
&+ \lambda^2(\alpha_3 I + \alpha_4 A + ... + \alpha_m A^{m-3})\\
&+ ... + \lambda^{m-2}(\alpha_{m-1} I + \alpha_m A) + \lambda^{m-1}\alpha_m I.
\end{aligned}$$

Therefore,

$$\begin{aligned}
p(\lambda)R_\lambda(A) &= \alpha_1 I + \alpha_2(\lambda I + A) + \alpha_3(\lambda^2 I + \lambda A + A^2)\\
&+ \alpha_4(\lambda^3 I + \lambda^2 A + \lambda A^2 + A^3) + ...\\
&+ \alpha_{m-1}(\lambda^{m-2} I + \lambda^{m-3} A + ... + \lambda A^{m-3} + A^{m-2}))\\
&+ \alpha_m(\lambda^{m-1} I + \lambda^{m-2} A + ... + \lambda A^{m-2} + A^{m-1}))\\
&= \alpha_1(\lambda I - A)(\lambda I - A)^{-1} + \alpha_2((\lambda I)^2 - A^2)(\lambda I - A)^{-1} + ...\\
&+ \alpha_{m-1}((\lambda I)^{m-1} - A^{m-1})(\lambda I - A)^{-1} + \alpha_m((\lambda I)^m - A^m)(\lambda I - A)^{-1}\\
&= (\lambda I - A)^{-1}(\alpha_1(\lambda I - A) + \alpha_2((\lambda I)^2 - A^2) + ...\\
&+ \alpha_{m-1}((\lambda I)^{m-1} - A^{m-1}) + \alpha_m((\lambda I)^m - A^m))\\
&= (\lambda I - A)^{-1}(p(\lambda) - p(A)) = p(\lambda)(\lambda I - A)^{-1}.
\end{aligned}$$

If $p(\lambda) \neq 0$, then

$$R_\lambda(A) = (\lambda I - A)^{-1},$$

and

$$R_\lambda(A)(\lambda I - A) = (\lambda I - A)R_\lambda(A) = I.$$

2. We demonstrate that $\sigma(A)$ is a subset of $p_0 = \{\lambda \in \mathbb{C} : p(\lambda) = 0\}$. Choose any $\lambda \in \sigma(A)$. It follows that $p(\lambda) = 0$, meaning that λ belongs to p_0. To argue this, assume the opposite, i.e., suppose $p(\lambda) \neq 0$. In such a case, based on the preceding reasoning, a resolvent $R_\lambda(A)$ exists, implying that $\lambda \in \rho(A)$. This assumption leads to a contradiction, compelling the conclusion that $p(\lambda) = 0$. Consequently, we establish that $\sigma(A)$ is a subset of p_0

3. Let $p(x)$ be the minimal polynomial of A. Assume that $\sigma(A)$ is a subset of p_0. Then choose $\lambda_0 \in p_0 \setminus \sigma(A)$. Then, by the preceding arguments, there would exist a resolvent $R_{\lambda_0}(A)$ which has the form

$$R_{\lambda_0}(A) = \frac{1}{q(\lambda_0)} \sum_{i=0}^{m-1} \lambda_0^i B_i,$$

for some polynomial $q(x)$ with $q(A) = 0$. Since $p(x)$ is the minimal polynomial, the polynomial $q(x)$ can be expressed as follows $q(x) = r(x)p(x)$. We know that $\lambda_0 \in p_0$, i.e. $p(\lambda_0) = 0$, therefore, $q(\lambda_0) = r(\lambda_0)p(\lambda_0) = 0$, which contradicts the existence of the resolvent $R_{\lambda_0}(A)$. The contradiction proves the fact that $p_0 = \sigma(A)$.

Before proceeding to the main result of this paper we present some technical lemmas. The proofs can be found in their respective references.

Lemma 1 *[10, Lemma 2.1] Let $X(\mathbb{R})$ be a separable Banach function space. Then the set $L^2(\mathbb{R}) \cap X(\mathbb{R})$ is dense in $X(\mathbb{R})$.*

Lemma 2 *[4, 5] The following statements are equivalent:*

(a) N-function Φ satisfies Δ_2-condition;

(b) $L_\Phi(\mathbb{R})$ is reflexive;

(c) $L_\Phi(\mathbb{R})$ is separable.

Let $L_\Phi(\mathbb{R})$ be a reflexive Orlicz space, then as stated earlier the Hilbert transform is bounded linear operator on $L_\Phi(\mathbb{R})$. In other words, we have that $\mathcal{H} \in \mathcal{B}(L_\Phi(\mathbb{R}))$. Since $L_\Phi(\mathbb{R})$ is a Banach space, $\mathcal{B}(L_\Phi(\mathbb{R}))$ is a Banach algebra.

Lemma 3 *Let $L_\Phi(\mathbb{R})$ be a reflexive Orlicz space, then $\mathcal{H}^2 f = f$, for any $f \in L_\Phi(\mathbb{R})$.*

Proof. It is a known [3], [11, Chapter 4] that $\mathcal{H}^2 = I$ for all $f \in L^p(\mathbb{R})$, $1 < p < \infty$. Let $p = 2$. Hence, one can obviously see that $\mathcal{H}^2 f = f$ for every $f \in L^2(\mathbb{R}) \cap L_\Phi(\mathbb{R})$. By Lemma 2, one has that $L_\Phi(\mathbb{R})$ is separable. Hence, by Lemma 1, $L^2(\mathbb{R}) \cap L_\Phi(\mathbb{R})$ is dense in $L_\Phi(\mathbb{R})$. Therefore noting that the integrals involved in the Hilbert transfrom are finite, passing to the limit, we have $\mathcal{H}^2 f = f$ for every $f \in L_\Phi(\mathbb{R})$, which completes the proof.

The following theorem is the main result of this paper, which extends Widom's result [3].

Theorem 1 *Let $L_\Phi(\mathbb{R})$ be a reflexive Orlicz space and let $\mathcal{H}$ be the Hilbert transform on $L_\Phi(\mathbb{R})$. Then,*

(i) $\sigma(\mathcal{H}) = \sigma_p(\mathcal{H}) = \{\pm 1\}$.

(ii) $\rho(\mathcal{H}) = \mathbb{C} \setminus \{\pm 1\}$ and the resolvent has the following form

$$R_\lambda(\mathcal{H}) = \frac{1}{2}(\lambda + 1)^{-1}(I - \mathcal{H}) + \frac{1}{2}(\lambda - 1)^{-1}(I + \mathcal{H}).$$

Proof. (i). Note that, by Lemma 3, $\mathcal{H}^2 = I$ and obviously there is no polynomial $q(\cdot)$ of degree 1 such that $q(\mathcal{H}) = 0$. Hence, the minimal degree of the polynomials is 2:

$$p(\mathcal{H}) = I - \mathcal{H}^2 = 0.$$

Therefore, by Proposition 1, we have

$$\sigma(\mathcal{H}) = \{\lambda \in \mathbb{C} \ : \ \lambda^2 - 1 = 0\} = \{\pm 1\}.$$

Moreover, for $\lambda = \pm 1$, one has

$$(I - \mathcal{H})(I + \mathcal{H})f = (\lambda^2 I - \mathcal{H}^2)f = 0, \ \ f \in L_\Phi(\mathbb{R}).$$

Hence, $g = (I + \mathcal{H})f \in \mathrm{Ker}\{I - \mathcal{H}\} \neq \varnothing$. Therefore, $\sigma(\mathcal{H}) = \sigma_p(\mathcal{H}) = \{\pm 1\}$.

(ii). By the definition of the resolvent, it easily follows that $\rho(\mathcal{H}) = \mathbb{C} \setminus \sigma(\mathcal{H}) = \mathbb{C} \setminus \{\pm 1\}$. Since

$$p(\lambda) = \lambda^2 - 1, \alpha_0 = -1, \alpha_2 = 1, \alpha_k = 0, \ k = 1, 3, 4, \ldots .$$

Then, by Proposition 1, we obtain

$$B_0 = \mathcal{H}, B_1 = I$$

and

$$R_\lambda(\mathcal{H}) = \frac{\lambda I + \mathcal{H}}{\lambda^2 - 1} = \frac{1}{2}(\lambda + 1)^{-1}(I - \mathcal{H}) + \frac{1}{2}(\lambda - 1)^{-1}(I + \mathcal{H}).$$

Remark 1 *Since the point spectrum, continuous spectrum and residual spectrum are disjoint sets, it easily follows from Theorem 1 that* $\sigma_c(\mathcal{H}) = \sigma_r(\mathcal{H}) = \emptyset$.

4 Conclusion

In this paper, we investigated the spectrum of the Hilbert transform on Orlicz spaces over the real line. Our findings revealed that the spectrum of $\mathcal{H}$ on $L_\Phi(\mathbb{R})$ consists two points, specifically $\sigma(\mathcal{H}_{L_\Phi(\mathbb{R})}) = \{-1, 1\}$, and this spectrum coincides with the point spectrum. We also determined the resolvent set $\rho(\mathcal{H})$ and the resolvent $R_\lambda(\mathcal{H})$ of the Hilbert transform on the spaces $L_\Phi(\mathbb{R})$.

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