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A MAXIMUM PRINCIPLE FOR TIME-FRACTIONAL DIFFUSION EQUATION WITH MEMORY

One of the most beneficial techniques for studying partial differential equations of the parabolic and elliptic types is the use of the maximum and minimum principles. They enable the acquisition of specific solution attributes without the need for knowledge of the solutions' explicit representations. Despite the fact that the maximum principle for fractional differential equations has been studied since the 1970s, a particular interest in this field of study has just lately arisen.

In the present study, a maximum principle for the one-dimensional time fractional diffusion equation with memory is formulated and established. The proof of the maximal principle is based on a maximum principle for the Caputo fractional derivative. The initial boundary value problem for the time-fractional diffusion equation with memory has at most one classical solution, and the maximum principle is then used to show that this solution is continuous depends on the initial and boundary conditions.

Key words: time-fractional diffusion equation, fractional derivative, maximum principle, initial-boundary value problem.

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Жады бар уақыт бойынша бөлшек ретті диффузия теңдеуі үшін максимум қағидасы

Параболалық және эллиптикалық типтердің ішінара туындыларындағы теңдеулерді зерттеудің ең пайдалы әдістерінің бірі максимум мен минимум қағидаларын қолдану. Олар шешімдердің нақты көріністерін білуді қажет етпестен шешімнің нақты атрибуттарын алуға мүмкіндік береді. Бөлшек дифференциалдық теңдеулер үшін максимум қағидасы 1970 жылдардан бері зерттеліп келе жатқанына қарамастан, бұл зерттеу саласына ерекше қызығушылық жақында пайда болды.

Бұл зерттеу жадымен уақыт бойынша бөлшек диффузияның бір өлшемді теңдеуі үшін максимум қағидасын тұжырымдайды және белгілейді. Максимум қағидасының дәлелі сәйкесінше Капутоның бөлшек туындысы үшін максимум қағидасына негізделген. қолданба ретінде максимум қағидасы бөлшек уақыт жадымен диффузия теңдеуі үшін бастапқы-шеттік есептің бір ғана классикалық шешімі бар екенін көрсету үшін пайдаланылады және бұл шешім бастапқы және шекаралық шарттарға үздіксіз тәуелді болады.

Түйін сөздер: уақыт бойынша бөлшек ретті диффузия теңдеуі, бөлшек ретті туынды, максимум қаұидасы, бастапқы-шеттік есеп.

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Принцип максимума для уравнения дробной диффузии по времени с памятью

Одним из наиболее полезных методов изучения уравнений в частных производных параболического и эллиптического типов является использование принципов максимума и минимума. Они позволяют получать конкретные атрибуты решения без необходимости знания явных представлений решений. Несмотря на то, что принцип максимума для дробно-дифференциальных уравнений изучается с 1970-х годов, особый интерес к этой области исследований возник совсем недавно.

В этом исследовании сформулирован и установлен принцип максимума для одномерного уравнения дробной диффузии во времени с памятью. Доказательство принципа максимума основано на принципе максимума для дробной производной Капуто, соответственно. В качестве приложения принцип максимума используется для демонстрации того, что существует не более одного классического решения начально-краевой задачи для уравнения диффузии с дробной временной памятью, и это решение непрерывно зависит от начальных и граничных условий.
Ключевые слова: уравнение дробной диффузии по времени, дробное производное, принцип максимума, начально-краевая задача.

Introduction and statement of problem

The maximum-minimum principles are among the best techniques for studying partial differential equations of the parabolic and elliptic types. They allow one to obtain certain properties of solutions without resorting to information about their explicit representations. Although the maximal principle for fractional differential equations has been researched since the 1970s (see [1–4]), special interest in research in this area has appeared relatively recently.

In [5] Luchko obtained a maximal principle for $\partial^{\alpha}_{0|t}$ the Caputo fractional derivative of the form:

- let $g \in C^1((0,T)) \cap ([0,T])$ attains its maximum (minimum) over $[0,T]$ at $t_0 \in (0,T]$ and $0 < \alpha < 1$, then $\partial^{\alpha}_{0|t} g(t_0) \geq 0$ $(\partial^{\alpha}_{0|t} g(t_0) \leq 0)$.

He established a maximal using Caputo, the fundamentals of the fractional diffusion equation time derivative on a bounded domain based on the aforementioned findings. The maximum principle for time-fractional diffusion equations was demonstrated using these results (see [6, 7, 9–14].

We consider the following time-fractional diffusion equation with memory

$$\partial^{\alpha}_{0|t} u(x,t) = \frac{\partial^2}{\partial x^2} I^{\beta}_{0|t} u(x,t) + F(x,t) \text{ in } (0,a) \times (0,T], \tag{1}$$

supplemented with the initial and boundary conditions

$$\begin{cases} u(x,0) = \phi(x) \text{ on } [0,a], \\ u(0,t) = \psi_1(t), u(a,t) = \psi_2(t) \text{ for } 0 \leq t < T, \end{cases} \tag{2}$$

since a and T are real numbers that are positive, the functions F, ϕ, ψ_1 and ψ_2 are continuous in a way that $\phi(0) = \psi_1(0)$ and $\phi(a) = \psi_2(0)$. Here, $I_{0|t}^{\beta}$ is the Riemann-Liouville fractional integral of order $\beta > 0$, defined as (see [15, P. 69])

$$I_{0|t}^{\beta} u(x,t) = \frac{1}{\Gamma(\beta)} \int\limits_0^t (t-s)^{\beta-1} u(x,s)\, ds, \;\; t \in (0,T],$$

and the operator $\partial_{0|t}^{\alpha}$ is the left Caputo fractional derivative with $\alpha \in (0,1)$, given by (see [15, P. 92])

$$\partial_{0|t}^{\alpha} u(x,t) = I_{0|t}^{1-\alpha} \frac{\partial}{\partial t} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int\limits_0^t (t-s)^{-\alpha} \frac{\partial}{\partial s} u(x,s)\, ds.$$

The purpose of this article is to study the maximum principle of linear fractional diffusion equation (1).

If $\alpha \to 1$, $\beta \to 0$ then eq. (1) corresponds to the well-known heat equation. The subdiffusion equation is the equation of the type (1) with fractional derivatives with respect to the time variable [17]. The slow diffusion is described by this equation.

Below we present some well-known properties of fractional operators

Lemma 1 *[15, Lemma 2.21] If $0 < \mu < 1$ for $v(t) \in C[0,T]$, then*

$$\partial_{0|t}^{\mu} [I_{0|t}^{\mu} v(t)] = v(t),$$

holds true.

Lemma 2 *[16, Proposition 2.3] Let $v(t) \in C([0,T])$. If $\alpha + \beta < 1$, then*

$$\partial_{0|t}^{\alpha+\beta} [I_{0|t}^{\beta} v(t)] = \partial_{0|t}^{\alpha} v(t).$$

Lemma 3 *[8] (a) Let $v(t) \in C^1([0,T])$ attains its maximum at $t_0 \in (0,T)$,*

$$\partial_{0|t}^{\alpha} v(t_0) \geq \frac{t_0^{-\alpha}}{\Gamma(1-\alpha)} [v(t_0) - v(0)] \geq 0,$$

for all $0 < \alpha < 1$.

(b) Let $v(t) \in C^1([0,T])$ attain its minimum at $t_0 \in (0,T)$,

$$\partial_{0|t}^{\alpha} v(t_0) \leq \frac{t_0^{-\alpha}}{\Gamma(1-\alpha)} [v(t_0) - v(0)] \leq 0,$$

for all $0 < \alpha < 1$.

1 Main results

The main results of this article are presented in this section.

The existence of $u_t(x,t)$ is implied by the solutions to problems (1) and (2). Therefore, if $t>0$, then $\partial^{\alpha}_{0|t}$ occurs for any $0<\alpha<1$. This means that a solution $u(x,t)$ of the problem (1) and (2) in the region $[0,a]\times[0,T]$ is a (classical) solution in $C([0,a]\times[0,T])\cap C^{2,1}((0,a)\times(0,T])$.

Theorem 1 *Let $\alpha+\beta<1$. If $u(x,t)$ satisfies* (1),

$$u(x,0)=\phi(x)\geq 0,\ x\in[0,a],$$

$$u(0,t)=0=u(a,t),\ t\in[0,T),$$

and

$$F(x,t)\geq 0,(x,t)\in(0,a)\times(0,T],$$

then

$$u(x,t)\geq 0\ \ for\ \ (x,t)\in(0,a)\times(0,T].$$

Let us define the function

$$v(x,t)=u(x,t)+\epsilon t^{\gamma},$$

where $\epsilon>0$ and $\alpha<\gamma$.

From (2), we obtain $v(0,t)=v(a,t)=\epsilon t^{\gamma}>0$ for $t>0$, and $v(x,0)=\phi(x)$ for $x\in[0,a]$. Since

$$\partial^{\alpha}_{0|t}v(x,t)=\partial^{\alpha}_{0|t}u(x,t)+\partial^{\alpha}_{0|t}[\epsilon t^{\gamma}]=\partial^{\alpha}_{0|t}u(x,t)+\frac{\epsilon\Gamma\left(\gamma+1\right)}{\Gamma\left(\gamma-\alpha+1\right)}t^{\gamma-\alpha}$$

and

$$I^{\beta}_{0|t}v(x,t)=I^{\beta}_{0|t}u(x,t)+I^{\beta}_{0|t}[\epsilon t^{\gamma}]=I^{\beta}_{0|t}u(x,t)+\frac{\epsilon\Gamma\left(\gamma+1\right)}{\Gamma\left(\gamma+\alpha+1\right)}t^{\gamma+\alpha},$$

it follows that

$$\frac{\partial^2}{\partial x^2}I^{\beta}_{0|t}v(x,t)=\frac{\partial^2}{\partial x^2}I^{\beta}_{0|t}u(x,t).$$

Consequently, the function $v(x,t)$ satisfies the equation

$$\partial^{\alpha}_{0|t}v(x,t)=\frac{\partial^2}{\partial x^2}I^{\beta}_{0|t}v(x,t)+F(x,t)+\frac{\epsilon\Gamma\left(\gamma+1\right)}{\Gamma\left(\gamma-\alpha+1\right)}t^{\gamma-\alpha}\ \text{ in }\ (0,a)\times(0,T],$$

with the initial-boundary conditions

$$v(x,0)=\phi(x)\quad x\in[0,a]$$

$$v(a,t)=v(0,t)=\epsilon t^{\gamma}>0,\quad t>0.$$

Assume that there is some point $(x,t) \in [0,a] \times [0,T]$ such that $v(x,t) < 0$. Since

$$v(x,t) \geq 0 \text{ for } (x,t) \in \{0,a\} \times [0,T] \cup [0,a] \times 0,$$

there is a point $(x_0,t_0) \in (0,a) \times (0,T)$ such that $v(x_0,t_0)$ is the negative minimum of $v(x,t)$ over $(0,a) \times (0,T)$. In view of Lemma 3 (b), we have

$$\partial^{\alpha}_{0|t} v(x_0,t_0) \leq \frac{t_0^{-\alpha}}{\Gamma(1-\alpha)} [v(x_0,t_0) - v(x_0,0)] < 0.$$

Let us define $w(x,t) = I^{\beta}_{0|t} v(x,t)$. From Lemma 1 we conclude that

$$\partial^{\beta}_{0|t} \omega(x,t) = \partial^{\beta}_{0|t} [I^{\beta}_{0|t} v(x,t)] = v(x,t).$$

Using $v(x,t)$ is bounded in $[0,a] \times [0,T]$, we obtain

$$\omega(x,t) = I^{\beta}_{0|t} v(x,t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} v(x,s)\, ds \to 0 \text{ as } t \to 0.$$

Noting $\alpha + \beta < 1$ from Lemma 2, we have

$$\partial^{\alpha+\beta}_{0|t} \omega(x,t) = \partial^{\alpha+\beta}_{0|t} [I^{\beta}_{0|t} v(x,t)] = \partial^{\alpha}_{0|t} \omega(x,t).$$

At this point, we obtain for $\omega(x,t)$ the next initial and boundary conditions

$$\omega(x,t) = I^{\beta}_{0|t} v(x,t) = I^{\beta}_{0|t} u(x,t) + \frac{\epsilon \Gamma(1+\gamma)\, t^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)} > 0 \text{ as } t \to 0^{+}$$

and

$$I^{\beta}_{0|t} v(0,t) = I^{\beta}_{0|t} v(a,t) = I^{\beta}_{0|t} \epsilon t^{\gamma} = \frac{\epsilon \Gamma(1+\gamma)\, t^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)} > 0 \text{ for } t > 0.$$

Hence the function $\omega(x,t)$ satisfies the problem

$$\begin{cases} \partial^{\alpha+\beta}_{0|t} \omega(x,t) = \frac{\partial^2}{\partial x^2} \omega(x,t) + F(x,t) + \frac{\epsilon\Gamma(1+\gamma) t^{\gamma-\alpha}}{\Gamma(\gamma-\alpha)} \text{ in } (0,a) \times (0,T], \\ \omega(x,0) > 0 \text{ on } [0,a], \\ \omega(0,t) > 0, \omega(a,t) > 0 \text{ for } 0 < t \leq T. \end{cases}$$

From Lemma 3 (b), we have $\omega(x_0,t_0) < 0$. Due to $\omega(x,t) > 0$ on the boundary, there exists $(x_1,t_1) \in (0,a) \times (0,T]$ such that $\omega(x_1,t_1)$ is the negative minimum of $\omega(x,t)$ in $[0,a] \times [0,T]$. From Lemma (2) (b), it is evident that

$$\partial^{\alpha+\beta}_{0|t} \omega(x_1,t_1)] \leq \frac{t_1^{\alpha}}{\Gamma(1-\alpha)} [\omega(x_1,t_1) - \omega(x_1,0)] < 0.$$

Since $\omega(x_1,t_1)$ is a local minimum, it yields that $\dfrac{\partial^2}{\partial x^2} \omega(x_1,t_1) \geq 0$.
Therefore at the point (x_1,t_1), we obtain

$$0 > \partial_{0|t}^{\alpha+\beta}\omega(x_1,t_1) = \frac{\partial^2}{\partial x^2}\omega(x_1,t_1) + F(x_1,t_1) + \frac{\epsilon\Gamma\left(1+\gamma\right)t_1^{\gamma-\alpha}}{\Gamma\left(\gamma-\alpha\right)} > 0.$$

This contradiction demonstrates that $v(x,t) \geq 0$ on $[0,a]\times[0,T]$, and it follows that $u(x,t) \geq -\epsilon t^{\gamma}$ on $[0,a]\times[0,T]$ for any $\epsilon > 0$. Because of given ϵ is arbitrary, then $u(x,t) \geq 0$ on $[0,a]\times[0,T]$, which concludes the proof.

The outcome is comparable for the negativity of the solution $u(x,t)$ by considering $\phi(x) \leq 0$ and $F(x,t) \leq 0$.

Theorem 2 *Suppose that* $\alpha+\beta<1$. *If* $u(x,t)$ *satisfies* (1),

$$u(x,0) = \phi(x) \leq 0,\ x \in [0,a],$$

$$u(0,t) = 0 = u(a,t),\ t \in [0,T)$$

and

$$F(x,t) \leq 0, (x,t) \in [0,a]\times(0,T],$$

then

$$u(x,t) \leq 0, (x,t) \in [0,a]\times[0,T].$$

The results in Theorems 1 and 2 can be extended to obtain the next two theorems.

Theorem 3 *Let* $\alpha+\beta<1$. *Suppose* $u(x,t)$ *satisfies* (1),

$$u(x,0) = \phi(x),\ x \in [0,a],$$

$$u(0,t) = g_1(t), u(a,t) = g_2(t),\ t \in [0,T),$$

where $g_1(t)$ *and* $g_2(t)$ *are given real numbers. If*

$$F(x,t) \geq 0, (x,t) \in [0,a]\times[0,T],$$

then

$$u(x,t) \geq \min_{[0,a]}\{g_1, g_2, \phi(x)\}\ \textit{for}\ (x,t) \in [0,a]\times[0,T].$$

Let

$$M = \min_{[0,a]}\{g_1, g_2, \phi(x)\}$$

and

$$\bar{u}(x,t) = u(x,t) - M.$$

Then, $\bar{u}(0,t) = g_1 - M \geq 0$, $\bar{u}(a,t) = g_2 - M \geq 0$ for $t \in [0,T]$, and $\bar{u}(x,0) = \phi(x) - M \geq 0$ for $x \in [0,a]$, Since

$$\partial_{0|t}^{\alpha}\bar{u} = \partial_{0|t}^{\alpha}u,\ \ \frac{\partial^2}{\partial x^2}I_{0|t}^{\beta}\bar{u}(x,t) = \frac{\partial^2}{\partial x^2}I_{0|t}^{\beta}u(x,t),$$

then $\bar{u}(x,t)$ is satisfies (1), respectively. Hence, it emerges from a argument similar to the proof of Theorem 2 that

$$\bar{u}(x,t) \geq 0\ \text{ on }\ [0,a]\times[0,T].$$

Consequently,

$$u(x,t)=\min_{[0,a]}(g_1,g_2,\phi(x))\,\text{for}\,(x,t)\in[0,a]\times[0,T].$$

The proof is completed.

Using $\bar{u}(x,t)=-u(x,t)$, a similar proof to that of Theorem 3 gives the next conclusion.

Theorem 4 *Let $\alpha+\beta<1$. Suppose that $u(x,t)$ satisfies (1),*

$$u(x,0)=\phi(x),x\in[0,a],$$

$$u(0,t)=g_1,u(a,t)=g_2,$$

where g_1 and g_2 are given real numbers. If

$$F(x,t)\leq 0,(x,t)\in[0,a]\times[0,T],$$

then

$$u(x,t)\leq\max_{[0,a]}(g_1,g_2,\phi(x))\ \textit{for}\ (x,t)\in[0,a]\times[0,T].$$

The heat equations weak maximum principle is similar to Theorems 3 and 4.

The fractional variant is backed by the weak maximum principle a solution's uniqueness, as in the classical case.

Theorem 5 *Let $\alpha+\beta<1$. The problem (1)-(2) has at most one solution.*

Suppose that $u_1(x,t)$ and $u_2(x,t)$ two solutions of (1)-(2). Hence,

$$\partial^{\alpha}_{0|t}(u_1(x,t)-u_2(x,t))=\frac{\partial^2}{\partial x^2}I^{\beta}_{0|t}(u_1(x,t)-u_2(x,t)),$$

with zero initial and zero boundary conditions for $u_1(x,t)-u_2(x,t)$.
In view of Theorems 4 and 5, we have

$$u_1(x,t)-u_2(x,t)=0\ \text{on}\ [0,a]\times[0,T],$$

which completes the proof. A solution $u(x,t)$ of (1)-(2) depends constantly on the initial data $\phi(x)$, according to the Theorems 4 and 5.

Theorem 6 *Assume that $\alpha+\beta<1$. Let $u(x,t)$ and $\bar{u}(x,t)$ are the solutions of the problem (1)-(2) with the initial condition $\phi(x)$ and $\bar{\phi}(x)$, respectively. If*

$$\max_{[0,a]}\{|\phi(x)-\bar{\phi}(x)|\}\leq\epsilon,$$

then

$$|u(x,t)-\bar{u}(x,t)|\leq\epsilon.$$

The function

$$v(x,t) = u(x,t) - \bar{u}(x,t)$$

satisfies the problem

$$\partial^{\alpha}_{0|t} v(x,t) = \frac{\partial^2}{\partial x^2} I^{\beta}_{0|t} v(x,t),$$

with the initial data

$$v(x,0) = \phi(x) - \bar{\phi}(x)$$

and zero boundary conditions. Then, in view of Theorems 4 and 5

$$|v(x,t)| \leq \max_{[0,a]} \{|\phi(x) - \bar{\phi}(x)|\}$$

the desired result follows.

Conclusion

A maximum principle is formulated and established in this paper for the one-dimensional time fractional diffusion equation with memory. A maximum principle for the Caputo fractional derivative serves as the foundation for the proof of the maximal principle. There is only one classical solution to the initial-boundary value problem for the time-fractional diffusion equation with memory, and the maximum principle is then used to demonstrate that this solution is continuous and depends on the initial and boundary conditions.

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