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INTERPOLATION THEOREM FOR DISCRETE NET SPACES

In this paper, we study discrete net spaces $n_{p,q}(M)$, where M is some fixed family of sets from the set of integers $\mathbb{Z}$. Note that in the case when the net M is the set of all finite subsets of integers, the space $n_{p,q}(M)$ coincides with the discrete Lorentz space $l_{p,q}(M)$. For these spaces, the classical interpolation theorems of Marcinkiewicz-Calderon are known. In this paper, we study the interpolation properties of discrete network spaces $n_{p,q}(M)$,in the case when the family of sets M is the set of all finite segments from the class of integers $\mathbb{Z}$, i.e. finite arithmetic progressions with a step equal to 1. These spaces are characterized by such properties that for monotonically nonincreasing sequences the norm in the space $n_{p,q}(M)$ coincides with the norm of the discrete Lorentz space $l_{p,q}(M)$. At the same time, in contrast to the Lorentz spaces, the given spaces $n_{p,q}(M)$ may contain sequences that do not tend to zero. The main result of this work is the proof of the interpolation theorem for these spaces with respect to the real interpolation method. It is shown that the scale of discrete net spaces $n_{p,q}(M)$ is closed with respect to the real interpolation method. As a corollary, an interpolation theorem of Marcinkiewicz type is presented. These assertions make it possible to obtain strong estimates from weak estimates.
Key words: net spaces, discrete net spaces, Marcinkiewicz type interpolation theorem.

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Дискретті торлы кеңістіктердегі интерполяциялық теорема

Бұл жұмыста $n_{p,q}(M)$ дискретті торлы кеңістіктері зерттеледі, мұндағы M - $\mathbb{Z}$ бүтін сандар жиынынан алынған жиындардың тұрақты тобы. M торы бүтін сандардың барлық ақырлы ішкі жиындарының жиыны болған жағдайда $n_{p,q}(M)$ кеңістігі $l_{p,q}(M)$ дискретті Лоренц кеңістігімен сәйкес келетінін ескеріңіз. Бұл кеңістіктер үшін Марцинкевич-Кальдеронның классикалық интерполяциялық теоремалары белгілі. Жұмыста M жиындар тобы $\mathbb{Z}$ бүтін сандар класындағы барлық ақырлы сегменттердің жиыны, яғни қадамы 1-ге тең ақырлы арифметикалық прогрессиялар болған жағдайдағы, $n_{p,q}(M)$ дискретті торлы кеңістіктерінің интерполяциялық қасиеттері қарастырылады. Бұл кеңістіктер монотонды өспейтін тізбектер үшін $n_{p,q}(M)$ кеңістігіндегі нормасы $l_{p,q}(M)$ дискретті Лоренц кеңістігінің нормасымен сәйкес келетін қасиеттермен сипатталады. Сонымен қатар, аталмыш $n_{p,q}(M)$ кеңістіктерді? Лоренц кеңістігінен айырмашылығы бұл кеңістіктерде нөлге ұмытылмайтын тізбектер жатады. Бұл жұмыстың негізгі нәтижесі нақты интерполяциялық әдіске қатысты осы кеңістіктер үшін интерполяциялық теоремасын дәлелдеу болып табылады. Нақты интерполяциялық әдісіне қатысты $n_{p,q}(M)$ дискретті торлы кеңістіктерінің шкаласы тұйықталғаны көрсетілген. Салдар ретінде Марцинкевич типіндегі интерполяциялық теорема ұсынылған. Бұл теорема әлсіз бағалаулардан күшті бағалаулар алуға мүмкіндік береді.
Түйін сөздер: торлы кеңістіктер, дискретті торлы кеңістіктер, Марцинкевич типті интерполяциялық теоремасы.

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Интерполяционная теорема для дискретного сетевого пространства

В работе исследуется дискретные сетевые пространства $n_{p,q}(M)$, где M - некоторое фиксированное семейство множеств из множества целых чисел $\mathbb{Z}$. Отметим, что в случае когда сеть M есть множество всех конечных подмножеств целых чисел пространства $n_{p,q}(M)$ совпадает с дискретным пространством Лоренца $l_{p,q}(M)$. Для этих пространств известны классические интерполяционные теоремы Марцинкевича-Кальдерона.
В работе изучаются интерполяционные свойства дискретных сетевых пространств $n_{p,q}(M)$, в случае когда семейство множеств M является множеств всех конечных отрезков из класса целых чисел $\mathbb{Z}$, т.е. конечных арифметических прогрессии с шагом равным 1. Данные пространства характеризуется такими свойствами, что для монотонно не возрастающих последовательности норма в пространстве $n_{p,q}(M)$ совпадает с нормой дискретного пространства Лоренца $l_{p,q}(M)$. В то же время в отличие от пространств Лоренца данные пространства $n_{p,q}(M)$ может содержат последовательности нестремящиеся к нулю. Основным результатом данной работы является доказательство интерполяционной теоремы для этих пространств относительно вещественного интерполяционного метода. Показано, что шкала дискретных сетевых пространств $n_{p,q}(M)$ замкнута относительно вещественного интерполяционного метода. Как следствие приведена интерполяционная теорема типа Марцинкевича. Данные утверждения позволяют получить из слабых оценок сильные оценки.
Ключевые слова: сетевые пространства, дискретные сетевые пространства, интерполяционная теорема типа Марцикевича.

1 Introduction

Let S be the set of all finite sets of indices from $\mathbb{Z}^n$. For a fixed set $M \subset S$ we define the space $n_{p,q}(M)$ $(0 < p, q \leq \infty)$ as the set of sequences $a = \{a_m\}_{m \in \mathbb{Z}^n}$ with quasinorm for $0 < p < \infty$, $0 < q < \infty$

$$\|a\|_{n_{p,q}(M)} = \left(\sum_{k=1}^{\infty} k^{\frac{q}{p}-1} (\bar{a}_k(M))^q \right)^{\frac{1}{q}},$$

and for $q = \infty$, $0 < p \leq \infty$

$$\|a\|_{n_{p,\infty}(M)} = \sup_{1 \leq k < \infty} k^{\frac{1}{p}} \bar{a}_k(M),$$

where

$$\bar{a}_k(M) = \sup_{\substack{e \in M \\ |e| \geq k}} \frac{1}{|e|} \left| \sum_{m \in e} a_m \right|,$$

where $|e|$ is the number of indices in e.

These spaces were introduced in [6], and they were called net spaces.

Net spaces have found important applications in various problems of harmonic analysis, operator theory and theory of stochastic processes [1–3, 7–11]. In this paper, we study the interpolation properties of these spaces. It should be noted here that net spaces are in a sense close to the discrete Morrey spaces:

$$m_p^\lambda = \left\{ a = \{a_k\}_{k\in\mathbb{Z}} : \quad \sup_{m\in\mathbb{N}} \sup_{k\in\mathbb{Z}} \frac{1}{m^\lambda} \left(\sum_{r=k}^{k+m} |a_r|^p \right)^{\frac{1}{p}} < \infty \right\}.$$

In the case when $a = \{a_k\}_{k\in\mathbb{Z}},\ a_k \geq 0$, for $\lambda = n\left(1 - \frac{1}{p}\right)$

$$\|a\|_{n_{p,\infty}(M)} \asymp \|a\|_{m_1^\lambda}.$$

The question of interpolation of Morrey spaces was considered in the works [5, 12] and it was shown that this scale of spaces is not closed with respect to the real interpolation method.

In this paper we show that if M is the set of all segments from $\mathbb{Z}$ the scale of spaces is closed with respect to the real interpolation method, i.e. the following relation holds

$$(n_{p_0,q_0}(M), n_{p_1,q_1}(M))_{\theta,q} = n_{p,q}(M). \tag{1}$$

Given functions F and G, in this paper $F \lesssim G$ means that $F \leq c\, G$ (or $c\, F \geq G$), where c is a positive number, depending only on numerical parameters, that may be different on different occasions. Moreover, $F \asymp G$ means that $F \lesssim G$ and $G \lesssim F$.

2 Interpolation theorem

Let (A_0, A_1) be a compatible pair of Banach spaces [4]. Let

$$K(t, a; A_0, A_1) = K(t, a) = \inf_{a=a_0+a_1} (\|a_0\|_{A_0} + t\|a_1\|_{A_1}),\ a \in A_0 + A_1,$$

be the functional Petre. For $0 < q < \infty$, $0 < \theta < 1$

$$(A_0, A_1)_{\theta,q} = \left\{ a \in A_0 + A_1 : \|a\|_{(A_0,A_1)_{\theta,q}} = \left(\int_0^\infty (t^{-\theta} K(t, a))^q \frac{dt}{t} \right)^{1/q} < \infty \right\},$$

and for $q = \infty$

$$(A_0, A_1)_{\theta,q} = \left\{ a \in A_0 + A_1 : \|a\|_{(A_0,A_1)_{\theta,q}} = \sup_{0<t<\infty} t^{-\theta} K(t, a) < \infty \right\}.$$

Theorem 1 *Let $1 \leq p_0 < p_1 < \infty$ and $0 < q_0, q_1, q \leq \infty$. Let M be the set of all segments from $\mathbb{Z}$. Then*

$$\left(n_{p_0,q_0}(M), n_{p_1,q_1}(M)\right)_{\theta,q} = n_{p,q}(M),$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},\ \theta \in (0, 1)$.

Proof. Let us prove first

$$(n_{1,\infty}(M), n_{\infty,\infty}(M))_{\theta,q} = n_{p,q}(M), \tag{2}$$

where $\frac{1}{p} = 1-\theta$, $\theta \in (0,1)$. Let $\tau \in \mathbb{N}$ and M be the set of all segments from $\mathbb{Z}$. Divide our entire axis into disjoint segments $\{I_k\}_{k\in\mathbb{Z}}$, where $I_k = (2^k, 2^{k+1})$ of the measure $|I_k| = \tau$. It is obvious that $\tau = 2^{k+1} - 2^k = 2^k$. Let $a = \{a_m\}_{m\in\mathbb{Z}} \in n_{p,q}(M)$, define the sequence $b_0 = \{b_n^0\}$:

$$b_n^0 = \frac{1}{|I_k|}\left|\sum_{m\in I_k} a_m\right| \quad \text{for } n \in I_k.$$

Note that $|b_n^0| \le \bar{a}_\tau(M)$ and

$$\sum_{n\in I_k}(a_n - b_n^0) = 0.$$

Let $D_\tau(a) = \{b = \{b_n\}_{n\in\mathbb{Z}} : |b_n| \le \bar{a}_\tau(M)\}$. For the Petre functional, we have the following

$$K(t, a; n_{1,\infty}(M), n_{\infty,\infty}(M)) = \inf_{a=a_0+a_1}\left(\|a_0\|_{n_{1,\infty}(M)} + t\|a_1\|_{n_{\infty,\infty}(M)}\right)$$

$$\le \inf_{b\in D_\tau(a)}\Big(\sup_{1\le k<\infty} k\overline{(a-b)}_k(M) + t\sup_{1\le k<\infty}\bar{b}_k^0(M)\Big) \le \sup_{1\le k<\infty} k\overline{(a-b^0)}_k(M) + t\bar{a}_\tau(M).$$

Consider the first term

$$\sup_{1\le k<\infty} k\overline{(a-b^0)}_k(M) \asymp \sup_{1\le k<\tau} k\overline{(a-b^0)}_k(M) + \sup_{\tau\le k<\infty} k\overline{(a-b^0)}_k(M).$$

Let I be an arbitrary segment from M such that $|I| \ge \tau$. Hence

$$\left|\sum_{n\in I}(a_n - b_n^0)\right| = \left|\sum_{I_k\subset I}\sum_{n\in I_k}(a_n - b_n^0) + \sum_{n\in I\cap I_{k_0}}(a_n - b_n^0) + \sum_{n\in I\cap I_{k_1}}(a_n - b_n^0)\right|$$

$$\le \left|\sum_{n\in I\cap I_{k_0}} a_n\right| + \left|\sum_{n\in I\cap I_{k_1}} a_n\right| + \left(|I\cap I_{k_0}| + |I\cap I_{k_1}|\right)\bar{a}_\tau(M),$$

i.e., the following estimate holds

$$\left|\sum_{n\in I}(a_n - b_n^0)\right| \le s_1\bar{a}_{s_1} + s_2\bar{a}_{s_2} + 2\tau\bar{a}_\tau(M),$$

where $s_1 = |I\cap I_{k_0}| \le \tau$, $s_2 = |I\cap I_{k_1}| \le \tau$. Hence we have

$$\left|\sum_{n\in I}(a_n - b_n^0)\right| \le 4\sup_{\tau\ge s>1} s\,\bar{a}_s(M).$$

Hence,

$$\sup_{\tau\le k<\infty} k\overline{(a-b^0)}_k = \sup_{\tau\le k<\infty} k\sup_{|I|\ge k}\frac{1}{|I|}\left|\sum_{n\in\mathbb{Z}}(a_n - b_n^0)\right| \le 4\sup_{\tau\ge s\ge 1} s\bar{a}_s(M).$$

For the first term we have

$$\sup_{1\le k<\infty} k\overline{(a-b^0)}_k \le c \sup_{\tau\ge k\ge 1} k\bar{a}_k(M).$$

Thus,

$$K(t,a;n_{1,\infty}(M),n_{\infty,\infty}(M)) \lesssim \sup_{\tau\ge k>1} k\bar{a}_k(M)+t\bar{a}_\tau(M).$$

Then we have

$$\|a\|_{(n_{1,\infty}(M),n_{\infty,\infty}(M))_{\theta,q}} \asymp \left(\sum_{m\in\mathbb{Z}}\left(2^{-\theta m}K(2^m,a)\right)^q\right)^{\frac{1}{q}}$$

$$= \left(\sum_{m=-\infty}^{-1}\left(2^{-\theta m}K(2^m,a)\right)^q\right)^{\frac{1}{q}} + \left(\sum_{m=0}^{\infty}\left(2^{-\theta m}K(2^m,a)\right)^q\right)^{\frac{1}{q}}.$$

For the first term, we have

$$\left(\sum_{m=-\infty}^{-1}\left(2^{-\theta m}K(2^m,a)\right)^q\right)^{\frac{1}{q}} = \left(\sum_{m=-\infty}^{-1}\left(2^{-\theta m}\inf_{a=a_0+a_1}(\|a_0\|_{n_{1,\infty}(M)}+2^m\|a_1\|_{n_{\infty,\infty}(M)})\right)^q\right)^{\frac{1}{q}}$$

$$\le \left(\sum_{m=-\infty}^{-1}\left(2^{(1-\theta)m}\|a\|_{n_{\infty,\infty}(M)}\right)^q\right)^{\frac{1}{q}} = c\|a\|_{n_{\infty,\infty}(M)} \lesssim \|a\|_{n_{p,q}(M)}.$$

For the second term, taking into account that $\tau=2^m$ and applying the above estimate for the Petre functional and Minkowski's inequality, we obtain

$$\left(\sum_{m=0}^{\infty}\left(2^{-\theta m}K(2^m,a)\right)^q\right)^{\frac{1}{q}} \le \left(\sum_{m=0}^{\infty}\left(2^{-\theta m}\sup_{\tau\ge k>1} k\bar{a}_k(M)+2^m\bar{a}_\tau(M)\right)^q\right)^{\frac{1}{q}}$$

$$\le \left(\sum_{m=0}^{\infty}\left(2^{-\theta m}\sup_{\tau\ge k>1} k\bar{a}_k(M)\right)^q\right)^{\frac{1}{q}} + \left(\sum_{m=0}^{\infty}\left(2^{(1-\theta)m}\bar{a}_{2^m}(M)\right)^q\right)^{\frac{1}{q}}.$$

Considering that $k \asymp \left(\sum_{r=1}^{2^m} r^{q-1}\right)^{\frac{1}{q}}$ and changing the order of summation we get

$$\left(\sum_{m=0}^{\infty}\left(2^{-\theta m}K(2^m,a)\right)^q\right)^{\frac{1}{q}} \le \left(\sum_{m=0}^{\infty}\left(2^{-\theta m}\left(\sum_{r=1}^{2^m} r^{q-1}\bar{a}_r^q(M)\right)^{\frac{1}{q}}\right)^q\right)^{\frac{1}{q}} + \|a\|_{n_{p,q}}$$

$$= \left(\sum_{r=1}^{\infty} r^{q-1}\bar{a}_r^q(M)\sum_{m=\log_2 r}^{\infty} 2^{-\theta mq}\right)^{\frac{1}{q}} + \|a\|_{n_{p,q}(M)}$$

$$\asymp \left(\sum_{r=1}^{\infty} r^{(1-\theta)q-1}\bar{a}_r^q(M)\right)^{\frac{1}{q}} + \|a\|_{n_{p,q}(M)} = c\|a\|_{n_{p,q}(M)}.$$

Hence,

$$\|a\|_{(n_{1,\infty}(M),n_{\infty,\infty}(M))_{\theta,q}} \lesssim \|a\|_{n_{p,q}}.$$

So we get the embedding

$$n_{p,q}(M) \hookrightarrow (n_{1,\infty}(M), n_{\infty,\infty}(M))_{\theta,q},$$

where $\frac{1}{p} = 1-\theta,\ \theta \in (0,1)$.

Let us now prove the reverse embedding. Let $k \in \mathbb{N}$, $a \in (n_{1,\infty}, n_{\infty,\infty})_{\theta,q}$ and $a = a_0 + a_1$ be an arbitrary representation, where $a_0 \in n_{1,\infty}(M)$ and $a_1 \in n_{\infty,\infty}(M)$. Obviously, $\bar{a}_k(M) \leq \bar{a}_k^0(M) + \bar{a}_k^1(M)$. Then, if we denote $v(t) = t$, where $t \in (1,\infty)$, then

$$\sup_{v\geq k} k\bar{a}_k(M) \leq \sup_{k>0} k\bar{a}_k^0(M) + \sup_{v(t)\geq k} k\bar{a}_k^1(M) \leq \sup_{k\geq 1} k\bar{a}_k^0(M) + t\sup_{k\geq 1}\bar{a}_k^1(M).$$

Taking into account the arbitrariness of the representation $a = a_0 + a_1$ we have

$$\sup_{v\geq k>0} k\bar{a}_k(M) \leq K(t, a; n_{1,\infty}(M), n_{\infty,\infty}(M)).$$

Therefore, for $0 < q \leq \infty$ we have

$$\int_0^\infty \left(t^{-\theta}K(t,a;n_{1,\infty}(M),n_{\infty,\infty}(M))\right)^q \frac{dt}{t} \geq \int_1^\infty \left(t^{-\theta}\sup_{v\geq k>0} k\bar{a}_k(M)\right)^q \frac{dt}{t}$$

$$\geq c_1 \int_1^\infty \left(t^{-\theta}\sup_{t\geq k>0} k\bar{a}_k(M)\right)^q \frac{dt}{t} \geq c_2 \sum_{r=0}^{\infty}\left(2^{-\theta r}\sup_{2^r\geq k>0} k\bar{a}_k(M)\right)^q$$

$$\geq c_2 \sum_{r=1}^{\infty}\left(2^{r/p}\bar{a}_{2^r}(M)\right)^q.$$

Thus we get the embedding

$$(n_{1,\infty}(M), n_{\infty,\infty}(M))_{\theta,q} \hookrightarrow n_{p,q}(M), \tag{3}$$

where $\frac{1}{p} = 1-\theta,\ \theta \in (0,1)$.

Hence the relation (2) holds. To prove the general case, we use the reiteration theorem [4, Theorem 3.5.3].

Let $1 < p_0 < p_1 < \infty$. From (2) it follows that there are $\theta_0, \theta_1 \in (0,1)$ such that

$$\begin{aligned} (n_{1,\infty}(M), n_{\infty,\infty}(M))_{\theta_0,q_0} &= n_{p_0,q_0}(M) \\ (n_{1,\infty}(M), n_{\infty,\infty}(M))_{\theta_1,q_1} &= n_{p_1,q_1}(M), \end{aligned} \tag{4}$$

then by the reiteration theorem it follows that

$$(n_{p_0,q_0}(M), n_{p_1,q_1}(M))_{\theta,q} = (n_{1,\infty}(M), n_{\infty,\infty}(M))_{\eta,q} = n_{p,q}(M).$$

In the last equality, we took into account that $\eta = (1-\theta)\theta_0 + \theta\theta_1$.

This proves the theorem.

3 Corollary

As a corollary, an interpolation theorem of Marcinkevich type is presented.

Corollary 1 *Let* $2 \leq p_0 < p_1 < \infty$, $1 \leq q_0, q_1 < \infty$, $q_0 \neq q_1$, $0 < \tau, \sigma < \infty$, M *is the set of all segments from* $\mathbb{Z}$, $G = \{a = \{a_k\}_{k\in\mathbb{Z}},\ a_k \geq 0\}$. *If the following inequalities hold for a quasilinear operator*

$$\|Ta\|_{n_{q_0,\infty}(M)} \leq F_0 \|a\|_{n_{p_0,\sigma}(M)},\ a \in n_{p_0,\sigma}(M), \tag{5}$$

$$\|Ta\|_{n_{q_1,\infty}(M)} \leq F_1 \|a\|_{n_{p_1,\sigma}(M)},\ a \in n_{p_1,\sigma}(M), \tag{6}$$

then for any $a \in G \cap n_{p,\tau}$ *we have*

$$\|Ta\|_{n_{q,\tau}(M)} \leq cF_0^{1-\theta} F_1^{\theta} \|a\|_{n_{p,\tau}(M)}, \tag{7}$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\theta \in (0,1)$ *and the corresponding constant* c *depends only on* $p_i, q_i, \sigma, i = 0, 1$.

Proof. According to the real interpolation method [4, Theorem 3.1.2] and the inequalities (5) and (6) it follows

$$\|Ta\|_{(n_{q_0,\infty}(M), n_{q_1,\infty}(M))_{\theta,\tau}} \leq F_0^{1-\theta} F_1^{\theta} \|a\|_{(n_{p_0,\sigma}(M), n_{p_1,\sigma}(M))_{\theta,\tau}}.$$

From the relation (3) we have that

$$\|Ta\|_{n_{q,\tau}(M)} \leq c\|Ta\|_{(n_{q_0,\infty}(M), n_{q_1,\infty}(M))_{\theta,\tau}}.$$

From Theorem 1, taking into account that $a = \{a_k\}_{k\in\mathbb{Z}},\ a_k \geq 0$, we get

$$\|a\|_{n_{p,\tau}(M)} \asymp \|a\|_{(n_{p_0,\sigma}(M), n_{p_1,\sigma}(M))_{\theta,q}}.$$

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