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SOME LOCAL WELL POSEDNESS RESULTS IN WEIGHTED SOBOLEV SPACE $H^{1/3}$ FOR THE 3-KDV EQUATION

The paper analyses the local well posedness of the initial value problem for the k-generalized Korteweg-de Vries equation for $k = 3$ with irregular initial data. k-generalized Korteweg-de Vries equations serve as a model of magnetoacoustic waves in plasma physics, of the nonlinear propagation of pulses in optical fibers. The solvability of many dispersive nonlinear equations has been studied in weighted Sobolev spaces in order to manage the decay at infinity of the solutions. We aim to extend these researches to the k-generalized KdV with $k = 3$. For initial data in classical Sobolev spaces there are many results in the literature for several nonlinear partial differential equations. However, our main interest is to investigate the situation for initial data in Sobolev weighted spaces, which is less understood. The low regularity Sobolev results for initial value problems for this dispersive equation was established in unweighted Sobolev spaces with $s \geq 1/12$ and later further improved for $s \geq \frac{-1}{6}$. The paper improves these results for 3-KdV equation with initial data from weighted Sobolev spaces.
Key words: Nonlinear equations, dispersive equations, contraction, semigroup, nonlinear propagation.

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Салмақтық Соболев кеңістігінде3-КдФ теңдеуі үшін $H^{1/3}$ жергілікті қисындылық жайлы кейбір нәтижелер

Бұл жұмыста бастапқы деректері регулярлы емес $k = 3$ болған жағдайдағы k-жалпыланған Кортевег-де Фриз теңдеуі үшін бастапқы есептің локалдықисындылығына талдау жасалады. k-жалпыланған Кортевег-де Фриз теңдеулері плазма физикасындағы магнитоакустикалық толқындардың, және сонымен қатар оптикалық талшықтардағы импульстердің сызықты емес таралуының моделі ретінде қызмет етеді. Шешімдердің шексіздікте ыдырауын жақсырақ бақылау үшін, көптеген сызықтық емес дисперсиялық теңдеулердің шешілімділігі салмақтық Соболев кеңістігінде зерттеледі. Біздің мақсатымыз $k = 3$ болатын k-жалпыланған Кортевег-де Фриз теңдеуі үшін осы зерттеулерді жалғастыру болып табылады. Әдебиетте бастапқы деректері классикалық Соболев кеңістітерінде жататын бірқатар сызықты емес дербес дифференциалдық теңдеулер үшін көптеген нәтижелер бар. Дегенмен, біздің басты мүддеміз бастапқы деректері салмақтық Соболев кеңістігінде жатқан жағдайды зерттеу болып табылады, бұл жағдай аса түсініксіз болып табылады. Қарастырылып отырған дисперсия теңдеулері үшін бастапқы есептерүшін төменгі регулярлық Соболевтік нәтижелер салмақсыз Соболев кеңістігінде $s \geq 1/12$ мәндері үшін орнатылған және кейінірек $s \geq \frac{-1}{6}$ мәндері үшін де жақсартылған. Біз бұл нәтижелерді алынған бастапқы деректері салмақтық Соболев кеңістігінде жататын $k = 3$ болған жағдайдағы k-жалпыланған Кортевег-де Фризтеңдеуі үшін жақсартамыз.
Түйін сөздер: Сызықты емес теңдеулер, дисперсиялық теңдеулер, сығу, жартылай топ, сызықтық емес таралу.

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Некоторые результаты о локальной корректности в весовом пространстве Соболева $H^{1/3}$ для уравнения 3-КдФ

В данной работе анализируется локальная корректность начальной задачи для нелинейного k-обобщенного уравнения Кортевега-де Фриза для $k = 3$ с нерегулярными начальными данными. k-обобщенные уравнения Кортевега-де Фриза служат моделью магнитоакустических волн в физике плазмы, а также нелинейного распространения импульсов в оптических волокнах. Разрешимость многих дисперсионных нелинейных уравнений изучены в весовых пространствах Соболева с целью лучшего управления распадом решений на бесконечности. Нашей целью является продолжить эти расследования для k-обобщенного уравнения Кортевега-де Фриза с $k = 3$. В литературе имеются множество результатов для ряда нелинейных уравнений в частных производных с начальными данными в классических пространствах Соболева. Однако наш основной интерес представляет исследование ситуации с начальными данными в весовых пространствах Соболева, которая остается менее понятной. Низко регулярные Соболевские результаты для рассматриваемых нелинейных дисперсионных уравнений были установлены в невесовых пространствах Соболева для значений $s \geq 1/12$ и позднее были улучшены для $s \geq \frac{-1}{6}$. В данной статье эти результаты были улучшены для уравнения 3-КдВ с начальными данными из весовых пространств Соболева.
Ключевые слова: Нелинейные уравнения, дисперсионные уравнения, сжатие, полугруппа, нелинейное распространение.

1 Introduction

We investigate the Cauchy problem for the k-generalized Korteweg-de Vries equation with $k = 3$ (or briefly gKdV-3)

$$v_t + v_{xxx} + (v^4)_x = 0, \; x \in \mathbb{R}, \, t > 0 \tag{1}$$

with initial data $v(x, 0) = v_0(x)$, $x \in \mathbb{R}$, from weighted Sobolev spaces $H^s(\mathbb{R}) \cap L^2(|x|^{2m}dx)$. Equation (1) serves as a model of magnetoacoustic waves in plasma physics [13], of the nonlinear propagation of pulses in optical fibers [18].

The well-posedness of the initial value problem for the gKdV-3 equation was firstly established in the work of C. Kenig, G. Ponce and L. Vega [16] in classical Sobolev spaces with regularity $s \geq 1/12$ and later optimally improved by A. Grünrock [10] and T. Tao [23] for $s \geq -1/6$, using Bourgain's spaces techniques.

Inspired by T. Kato [12], in order for manage the decay of the solutions as $x \to \infty$, the several nonlinear dispersive equations has been investigated in weighted Sobolev spaces $H^s(\mathbb{R}) \cap L^2(|x|^{2m}dx)$ ([3–9, 21]). We aim to extend these researches to 3-KdV as we detail below.

We claim the Banach fixed point theorem to the integral equation version of the initial value problem (1), i.e.

$$v(x,t) = W(t)v_0(x) - \int_0^t W(t-\tau)(v^4)_x(x,\tau)\, d\tau, \tag{2}$$

where $W(t)v_0(x)$ is the solution of the initial value problem for the associated linear partial differential equation, that introduced in (8) below.

2 Materials and methods

Our main result is the following theorem.

Theorem 1. *Suppose that* $m \in [0, 1/6]$. *For initial value* v_0 *from weighted Sobolev space* $H^{1/3}(\mathbb{R}) \cap L^2(|x|^{2m}dx)$ *there exist a unique solution* v *of the integral equation* (2) *that belongs to the weighted Sobolev space* $v(\cdot, t) \in H^{1/3}(\mathbb{R}) \cap L^2(|x|^{2m}dx), \quad t \in (0, T]$ *for* $T > 0$.

We mentioned above the sharp Sobolev results (for $s \geq -1/6$). So, it is natural to improve the regularity s on the weighted Sobolev resuls for $0 < s < 1/3$. Indeed, in [6] we considered the situation for $s = 1/12 + \varepsilon$, employing a more delicate analysis.

Now we introduce the notations. For a constant $c > 0$ satisfying inequality $a \leq cb$, we write $a \lesssim b$. And, if $a \lesssim b$ and $b \lesssim a$, then we write $a \sim b$.

We denote by

$$\mathcal{F}(h)(\xi) := \int_{-\infty}^{\infty} \exp(-ix\xi) h(x)\, dx, \quad \xi \in \mathbb{R}$$

the Fourier transform of $h \in L^2(\mathbb{R})$ and by

$$\mathcal{F}^{-1}(h)(x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(ix\xi) h(\xi)\, d\xi, \quad x \in \mathbb{R}$$

its inverse Fourier transform.

Let $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$. The Sobolev space $H^s(\mathbb{R})$ can be defined by the norm

$$\|h\|_{H^s} := \Big(\int_{-\infty}^{\infty} \langle \xi \rangle^{2s} |\mathcal{F}(h)(\xi)|^2\, d\xi \Big)^{1/2},$$

where $s \in \mathbb{R}$ is the order of the Sobolev space. The inclusion $H^{s'}(\mathbb{R}) \subset H^s(\mathbb{R})$ holds for $s \leq s'$, that is,

$$\|h\|_{H^s} \lesssim \|h\|_{H^{s'}}. \tag{3}$$

In order to treat functions defined in a space-time domain we introduce mixed norm spaces. Let $1 \leq p, q \leq \infty$. We say that $h \in L^p_x L^q_T$ if

$$\|h\|_{L^p_x L^q_T} := \Big\{ \int_{-\infty}^{\infty} \Big(\int_0^T |h(x,t)|^q\, dt \Big)^{p/q} dx \Big\}^{1/p}$$

and $h \in L^q_T L^p_x$, if

$$\|h\|_{L^q_T L^p_x} := \Big\{ \int_0^T \Big(\int_{-\infty}^{\infty} |h(x,t)|^p\, dx \Big)^{q/p} dt \Big\}^{1/q}.$$

For $p = \infty$ or $q = \infty$, we have the definition involving the essential supremum.

The fractional derivative D_x^λ for $\lambda \in \mathbb{C}$ can be defined as the Fourier multiplier given by

$$\mathcal{F}(D_x^\lambda h)(\xi) := |\xi|^\lambda \mathcal{F}(h)(\xi).$$

Similarly, the operator $(1+D_x^2)^\lambda$ is defined as follows

$$\mathcal{F}\big((1+D_x^2)^\lambda h\big)(\xi) := (1+|\xi|^2)^\lambda \mathcal{F}(h)(\xi).$$

Consequently, by Plancherel theorem we have

$$\|h\|_{H^s} \sim \|(1+D_x^2)^{s/2} h\|_{L^2} \lesssim \|h\|_{L^2} + \|D_x^s h\|_{L^2}.$$

We exploit the Hilbert transform H introduced as

$$\mathcal{F}(Hh)(\xi) := -i\mathrm{sgn}(\xi)\mathcal{F}(h)(\xi).$$

Hence, D_x can be expressed via $\frac{\partial}{\partial x}$ in the following way $D_x = H\frac{\partial}{\partial x}$.

We recall the the fractional version of Leibniz rule ([16, Theorem A.8]). Let $\lambda \in (0,1)$, $\lambda_1, \lambda_2 \in [0,\lambda]$ such that $\lambda = \lambda_1 + \lambda_2$. And let $p, p_1, p_2, q, q_1, q_2 \in (1,\infty)$ with

$$1/p = 1/p_1 + 1/p_2, \quad 1/q = 1/q_1 + 1/q_2, \tag{4}$$

then

$$\|D_x^\lambda(gh) - gD_x^\lambda h - hD_x^\lambda g\|_{L_x^p L_T^q} \lesssim \|D_x^{\lambda_1} g\|_{L_x^{p_1} L_T^{q_1}} \|D_x^{\lambda_2} h\|_{L_x^{p_2} L_T^{q_2}}. \tag{5}$$

Also, $q_1 = \infty$ for $\lambda_1 = 0$.

We evoke the derivative chains rules in fractional calculus ([16, Theorem A.6])

$$\|D_x^\lambda F(h)\|_{L_x^p L_T^q} \lesssim \|F'(h)\|_{L_x^{p_1} L_T^{q_1}} \|D_x^\lambda h\|_{L_x^{p_2} L_T^{q_2}} \tag{6}$$

with $0 < \lambda < 1$, $1 < p, p_1, p_2, q, q_2 < \infty$ and $1 < q_1 \leq \infty$ such that (4).

The solution of IVP for Airy equation

$$\begin{cases} v_t + v_{xxx} = 0, & x \in \mathbb{R},\ t > 0, \\ v(x,0) = v_0(x), & x \in \mathbb{R} \end{cases} \tag{7}$$

can be represented as $v(x,t) = W(t)v_0(x)$, where we denote by $W(t)$ the Fourier multiplier defined as

$$\mathcal{F}(W(t)v_0)(\xi) := \exp(it\xi^3)\mathcal{F}(v_0)(\xi). \tag{8}$$

Plancherel theorem implies that

$$\|W(t)v_0\|_{L_x^2} \sim \|v_0\|_{L^2}. \tag{9}$$

Some properties of the semigroup $\{W(t)\}_{t>0}$ can be applied to prove the theorem 2. We recall the estimates for semigroup from [14, Theorem 2.4]

$$\|W(t)v_0\|_{L_x^8 L_T^8} \lesssim \|v_0\|_{L^2}, \tag{10}$$

and from [22, Theorem 2]

$$\|W(t)v_0\|_{L_x^6 L_T^\infty} \lesssim \|v_0\|_{H^{1/3}} \tag{11}$$

and from [16, Theorem 3.5]

$$\|\frac{\partial}{\partial x}W(t)v_0\|_{L_x^\infty L_T^2} \lesssim \|v_0\|_{L^2}. \tag{12}$$

We exploit in Section 3 bounds of the Airy semigroup, that we present in the following lemma.

Lemma 1. *Suppose that* $v_0 \in H^{1/3}(\mathbb{R})$. *Then,*

$$\|\frac{\partial}{\partial x}W(t)v_0\|_{L_x^{24} L_T^{8/3}} \lesssim \|v_0\|_{H^{1/3}}. \tag{13}$$

Proof. First we construct operators

$$A_z := D_x^{4z/3}(1+D_x^2)^{-1/6}W(t)$$

which are analytic for $z \in \mathbb{C}$, $0 \le \operatorname{Re} z \le 1$. The estimates (11) above implies

$$\|A_{iy}v_0\|_{L_x^6 L_T^\infty} = \|W(t)(1+D_x^2)^{-1/6}D_x^{4iy/3}v_0\|_{L_x^6 L_T^\infty} \lesssim \|D_x^{4iy/3}v_0\|_{L^2} = \|v_0\|_{L^2},$$

for any $y \in \mathbb{R}$ and also (12) implies that

$$\begin{aligned}\|A_{1+iy}v_0\|_{L_x^\infty L_T^2} &= \|\frac{\partial}{\partial x}W(t)D_x^{(1+4iy)/3}(1+D_x^2)^{-1/6}Hv_0\|_{L_x^\infty L_T^2} \\ &\lesssim \|D_x^{1/3}(1+D_x^2)^{-1/6}Hv_0\|_{L^2} \le \|Hv_0\|_{L^2} \sim \|v_0\|_{L^2}.\end{aligned}$$

Consequently, by Stein's theorem [1] for any $\theta \in (0,1)$ and $p,q \in [1,\infty]$ such that

$$\frac{1}{p} = \frac{1-\theta}{6} + \frac{\theta}{\infty}, \quad \frac{1}{q} = \frac{1-\theta}{\infty} + \frac{\theta}{2}$$

we obtain

$$\|A_\theta v_0\|_{L_x^p L_T^q} \lesssim \|v_0\|_{L^2}.$$

Thus, we have

$$\|A_{3/4}v_0\|_{L_x^{24} L_T^{8/3}} \lesssim \|v_0\|_{L^2}$$

for $\theta = 3/4$. It follows that (13).

We present the following bounds [16] that

$$\left\|\frac{\partial}{\partial x}\int_0^t W(t-\tau)h(\cdot,\tau)\,d\tau\right\|_{L_T^\infty L_x^2} \lesssim \|h\|_{L_x^1 L_T^2} \tag{14}$$

and [9]

$$\left\|\int_0^t W(t-\tau)h(\cdot,t\tau)\,d\tau\right\|_{L_T^\infty L_x^2} \le \|h\|_{L_T^{q'} L_x^{p'}}, \tag{15}$$

for $p \geq 2$ and
$$\frac{1}{q} = \frac{1}{6} - \frac{1}{3p}, \quad \frac{1}{p} + \frac{1}{p'} = 1 = \frac{1}{q} + \frac{1}{q'}.$$

Now we recall Fonseca-Linares-Ponce pointwise formula established in [8] which allows to commute fractional powers $|x|^m$ and the Airy semigroup $W(t)$, with the proper adjustments. Namely, the following identity

$$|x|^m W(t)v_0(x) = W(t)(|x|^m v_0)(x) + W(t)\mathcal{F}^{-1}[\Phi_{t,m}(\mathcal{F}(v_0)(\xi)](x) \tag{16}$$

holds for all $t > 0$, $v_0 \in H^s(\mathbb{R}) \cap L^2(|x|^{2m}dx)$, with $0 < s < 2$ and $0 < m \leq s/2$, and almost any $x \in \mathbb{R}$. Also, the L^2–norm of the last term can be bound as followa

$$\|\mathcal{F}^{-1}[\Phi_{t,m}(\mathcal{F}(v_0)(\xi)]\|_{L^2_x} \lesssim (1+t)\big(\|v_0\|_{L^2} + \|D_x^{2m}v_0\|_{L^2}\big). \tag{17}$$

We note that only the particular case of $s = 2\lambda$ and $m = \lambda$, for $0 < \lambda < 1$ is considered in [8].

3 Proof of Theorem 2

In Section 3.1 we treat the the initial value problem for the 3-KdV in the Sobolev space. Previously we noted that the well posedness of the initial value problem for the 3-KdV in the Sobolev space $H^{1/3}(\mathbb{R})$ is already known. The local well posedness results for 3-KdV was proved in classical Sobolev spaces with $s \geq 1/12$ in [16, Theorem 2.6]. Then this result was extended up to $s \geq -1/6$ in [10, 23]. Nevertheless, the local well posedness of the IVP for 3-KdV in weighted Sobolev spaces with regularity $s \leq 1/3$ is interesting open question. Inspired by Kenig-Ponce-Vega ([16, pp. 583–585]) and Fonseca-Linares-Ponce ([8, pp. 5364–5366]) works, we prove our new local well posedness result in weighted Sobolev spaces (Section 3.2).

By using the Banach fixed-point theorem to the mapping

$$\Psi(v) := W(t)v_0 - \int_0^t W(t-\tau)(v^4)_x(\cdot,\tau)\,d\tau,$$

our goal is to establish that this mapping is a contraction on a conveniently chosen subspace of $L_T^\infty H_x^{1/3} \cap L_T^\infty L_x^2(|x|^{2m}dx)$.

3.1 Unweighted case ($m = 0$)

Let
$$Y_T^\delta := \{v \,:\, \|v\|_{Y_T} \leq \delta\},$$
will be the complete metric space (with δ, $T > 0$ that are fixed) with the norm

$$\|v\|_{Y_T} := \sum_{j=1}^{6} \sigma_j^T(v), \tag{18}$$

where
$$\begin{aligned} \sigma_1^T(v) &:= \|v\|_{L_T^\infty H_x^{1/3}}, & \sigma_4^T(v) &:= \|D_x^{1/3}v_x\|_{L_x^\infty L_T^2}, \\ \sigma_2^T(v) &:= \|v_x\|_{L_x^{24} L_T^{8/3}}, & \sigma_5^T(v) &:= \|v\|_{L_x^6 L_T^\infty}, \\ \sigma_3^T(v) &:= \|D_x^{1/3}v\|_{L_x^8 L_T^8}, & \sigma_6^T(v) &:= \|v_x\|_{L_x^\infty L_T^2}. \end{aligned}$$

Step 1. First, we will prove that Ψ is well defined on Y_T^δ. Now show that $\Psi(v) \in Y_T^\delta$ for any $v \in Y_T^\delta$, that is,

$$\|\Psi(v)\|_{Y_T} = \sum_{j=1}^{6} \sigma_j^T(\Psi(v)) \leq \delta. \tag{19}$$

We note that the treatment of the terms on the left hand side of (19) can be converted to $L_T^2 L_x^2$–norm of $D_x^{1/3}(v^4)_x$ and $(v^4)_x$. Leibniz rule (5), the fractional derivatives chain rule (6) and Hölder integral inequality imply that

$$\begin{aligned}
\|D_x^{1/3}(v^4)_x\|_{L_T^2 L_x^2} &\sim \|D_x^{1/3}(v^3 v_x)\|_{L_x^2 L_T^2} \\
&\leq \|D_x^{1/3}(v^3 v_x) - v^3 D_x^{1/3} v_x - v_x D_x^{1/3}(v^3)\|_{L_x^2 L_T^2} \\
&\quad + \|v^3 D_x^{1/3} v_x\|_{L_x^2 L_T^2} + \|v_x D_x^{1/3}(v^3)\|_{L_x^2 L_T^2} \\
&\lesssim \|D_x^{1/3}(v^3)\|_{L_x^{24/11} L_T^8} \|v_x\|_{L_x^{24} L_T^{8/3}} + \|v^3\|_{L_x^2 L_T^\infty} \|D_x^{1/3} v_x\|_{L_x^\infty L_T^2} \\
&\lesssim \|v^2\|_{L_x^3 L_T^\infty} \|D_x^{1/3} v\|_{L_x^8 L_T^8} \|v_x\|_{L_x^{24} L_T^{8/3}} + \|v\|_{L_x^6 L_T^\infty}^3 \|D_x^{1/3} v_x\|_{L_x^\infty L_T^2} \\
&= \|v\|_{L_x^6 L_T^\infty}^2 \|D_x^{1/3} v\|_{L_x^8 L_T^8} \|v_x\|_{L_x^{24} L_T^{8/3}} + \|v\|_{L_x^6 L_T^\infty}^3 \|D_x^{1/3} v_x\|_{L_x^\infty L_T^2} \\
&= \big(\sigma_5^T(v)\big)^2 \sigma_3^T(v) \sigma_2^T(v) + \big(\sigma_5^T(v)\big)^3 \sigma_4^T(v) \lesssim \|v\|_{Y_T}^4.
\end{aligned} \tag{20}$$

We observe that (20) motivates the choice of the norms σ_2^T, σ_3^T, σ_4^T and σ_5^T.

Otherwise, the necessity of the norm σ_6^T can be justified as below

$$\begin{aligned}
\|(v^4)_x\|_{L_T^2 L_x^2} &\sim \|v^3 v_x\|_{L_x^2 L_T^2} \leq \|v^3\|_{L_x^2 L_T^\infty} \|v_x\|_{L_x^\infty L_T^2} = \|v\|_{L_x^6 L_T^\infty}^3 \|v_x\|_{L_x^\infty L_T^2} \\
&= \big(\sigma_5^T(v)\big)^3 \sigma_6^T(v) \leq \|v\|_{Y_T}^4.
\end{aligned} \tag{21}$$

Now we will analyse the norms $\sigma_j^T(\Psi(v))$, $j = 1, \ldots, 6$, which rely on the Airy semigroup estimates and the estimates (20) and (21) that we deduced.

Plancherel formula, Minkowski inequality, (9) and Hölder integral inequality give us

$$\begin{aligned}
\sigma_1^T(\Psi(v)) &\lesssim \|W(t)v_0\|_{L_T^\infty L_x^2} + \int_0^T \|W(t-\tau)(v^4)_x(\cdot,\tau)\|_{L_T^\infty L_x^2}\, d\tau \\
&\qquad + \|W(t) D_x^{1/3} v_0\|_{L_T^\infty L_x^2} + \int_0^T \|W(t-\tau) D_x^{1/3}(v^4)_x(\cdot,\tau)\|_{L_T^\infty L_x^2}\, d\tau \\
&\lesssim \|v_0\|_{H^{1/3}} + \|(v^4)_x\|_{L_T^1 L_x^2} + \|D_x^{1/3}(v^4)_x\|_{L_T^1 L_x^2} \\
&\leq \|v_0\|_{H^{1/3}} + T^{1/2} \|(v^4)_x\|_{L_T^2 L_x^2} + T^{1/2} \|D_x^{1/3}(v^4)_x\|_{L_T^2 L_x^2} \\
&\lesssim \|v_0\|_{H^{1/3}} + T^{1/2} \|v\|_{Y_T}^4.
\end{aligned} \tag{22}$$

Here we are allowed to permuted $D_x^{1/3}$ and $W(t)$ since both are Fourier multipliers.

In the same way, by exploiting Lemma 2, the estimates (10), (12), (11) and the Sobolev embedding theorem(3) we deduce

$$\sum_{j=2}^{6}\sigma_j^T(\Psi(v)) \lesssim \|v_0\|_{H^{1/3}} + \|(v^4)_x\|_{L^1_TL^2_x} + \|D_x^{1/3}(v^4)_x\|_{L^1_TL^2_x}$$
$$\lesssim \|v_0\|_{H^{1/3}} + T^{1/2}\|v\|^4_{Y_T}. \tag{23}$$

Therefore, if $v \in Y_T^\delta$, collecting (22) and (23) one gives

$$\|\Psi(v)\|_{Y_T} \leq C\|v_0\|_{H^{1/3}} + CT^{1/2}\delta^4$$

for some constant $C > 0$. Consequently, taking

$$\delta := 2C\|v_0\|_{H^{1/3}} \tag{24}$$

and choosing $T > 0$ such that

$$\frac{\delta}{2} + CT^{1/2}\delta^4 \leq \delta, \tag{25}$$

we get (19).

Step 2. Secondly, we will show that Ψ is a contraction on Y_T^δ. Let $v, w \in Y_T^\delta$, for δ defined in (24). Our goal is to show that

$$\|\Psi(v) - \Psi(w)\|_{Y_T} \leq K\|v - w\|_{Y_T} \tag{26}$$

for some $0 < K < 1$ and T sufficiently small to specify below. We have

$$\Psi(v) - \Psi(w) = \int_0^t W(t-\tau)(v^4 - w^4)_x\,dt', \tag{27}$$

then we need to prove

$$\|(v^4 - w^4)_x\|_{L^2_TL^2_x} \lesssim \delta^3\|v - w\|_{Y_T} \tag{28}$$

and

$$\|D_x^{1/3}(v^4 - w^4)_x\|_{L^2_TL^2_x} \lesssim \delta^3\|v - w\|_{Y_T}. \tag{29}$$

Really, using the same argument as in the Step 1 and invoking (28) and (29), instead of (20) and (21), for some $C > 0$ we obtain

$$\|\Psi(v) - \Psi(w)\|_{Y_T} \leq CT^{1/2}\delta^3\|v - w\|_{Y_T}. \tag{30}$$

Consequently, by taking $T > 0$ such that $CT^{1/2}\delta^3 < 1$ and (25), we conclude that (26).

Notice that

$$v^4 - w^4 = (v - w)(v^3 + v^2w + vw^2 + w^3) \tag{31}$$

and differentiating,

$$(v^4 - w^4)_x = (v^3 + v^2 w + v w^2 + w^3)\,(v-w)_x \\ + (v-w)\big(3v^2 v_x + 2vwv_x + v^2 w_x + w^2 v_x + 2vww_x + 3w^2 w_x\big).$$

Therefore, (28) and (29) can be converted to

$$\|u_1 u_2 u_3 (u_4)_x\|_{L^2_x L^2_T} \lesssim \delta^3 \|v-w\|_{Y_T} \tag{32}$$

and

$$\|D_x^{1/3}(u_1 u_2 u_3 (u_4)_x)\|_{L^2_x L^2_T} \lesssim \delta^3 \|v-w\|_{Y_T} \tag{33}$$

for $u_1, u_2, u_3, u_4 \in \{v, w, v-w\}$ and one, and only one, of the u_j's being equal to $v-w$.

Inequality (32) can be proved by Hölder integral inequality

$$\|u_1 u_2 u_3 (u_4)_x\|_{L^2_x L^2_T} \leq \|u_1 u_2 u_3\|_{L^2_x L^\infty_T} \|(u_4)_x\|_{L^\infty_x L^2_T} \leq \prod_{j=1}^{3} \|u_j\|_{L^6_x L^\infty_T} \|(u_4)_x\|_{L^\infty_x L^2_T} \\ \leq \prod_{j=1}^{3} \sigma_5^T(u_j)\sigma_6^T(u_4) \leq \prod_{j=1}^{4} \|u_j\|_{Y_T} \leq \delta^3 \|v-w\|_{Y_T}.$$

We split the proof of (33) a in a few parts. By using the same argument as in (20), we obtain

$$\|D_x^{1/3}(u_1 u_2 u_3 (u_4)_x)\|_{L^2_x L^2_T} \lesssim \|D_x^{1/3}(u_1 u_2 u_3)\|_{L_x^{24/11} L_T^8} \|(u_4)_x\|_{L_x^{24} L_T^{8/3}} \\ + \|u_1 u_2 u_3\|_{L^2_x L^\infty_T} \|D_x^{1/3}(u_4)_x\|_{L^\infty_x L^2_T} \\ = \|D_x^{1/3}(u_1 u_2 u_3)\|_{L_x^{24/11} L_T^8}\, \sigma_2^T(u_4) + \prod_{j=1}^{3} \sigma_5^T(u_j)\sigma_4^T(u_4). \tag{34}$$

Further, the Leibniz rule (5) and Hölder integral inequality give us

$$\|D_x^{1/3}(u_1 u_2 u_3)\|_{L_x^{24/11} L_T^8} \lesssim \|u_1 u_2 D_x^{1/3} u_3\|_{L_x^{24/11} L_T^8} + \|u_3 D_x^{1/3}(u_1 u_2)\|_{L_x^{24/11} L_T^8} \\ + \|u_1 u_2\|_{L^3_x L^\infty_T} \|D_x^{1/3} u_3\|_{L^8_x L^8_T} \\ \lesssim \sigma_5^T(u_3)\|D_x^{1/3}(u_1 u_2)\|_{L_x^{24/7} L_T^8} + \prod_{j=1}^{2} \sigma_5^T(u_j)\sigma_3^T(u_3) \tag{35}$$

and

$$\|D_x^{1/3}(u_1 u_2)\|_{L_x^{24/7} L_T^8} \lesssim \|u_1 D_x^{1/3} u_2\|_{L_x^{24/7} L_T^8} + \|u_2 D_x^{1/3} u_1\|_{L_x^{24/7} L_T^8} \\ + \|u_1\|_{L^6_x L^\infty_T} \|D_x^{1/3} u_2\|_{L^8_x L^8_T} \\ \lesssim \sigma_5^T(u_1)\sigma_3^T(u_2) + \sigma_5^T(u_2)\sigma_3^T(u_1). \tag{36}$$

Putting together (34)–(36) we deduce (33).

3.2 Weighted case ($0 < m \le 1/6$)

Let us define the space
$$Z_T^\delta := \{v \,:\, \|v\|_{Z_T} < \delta\}$$
for some suitably taken $\delta, T > 0$, with
$$\|v\|_{Z_T} := \|v\|_{Y_T} + \||x|^m v\|_{L_T^\infty L_x^2} \tag{37}$$
and $\|v\|_{Y_T}$ introduced in (18).

Step 1a. First, we establish that Ψ is well defined on Z_T^δ. Above we examined the Y_T–norm of $\Psi(v)$. In this section we focus on $L_T^\infty L_x^2$–norm of $|x|^m\Psi(v)$. We can write
$$\begin{aligned}\||x|^m\Psi(v)\|_{L_T^\infty L_x^2} &\le \||x|^m W(t)v_0\|_{L_T^\infty L_x^2} + \Big\||x|^m \int_0^t W(t-\tau)(v^4)_x\,d\tau\Big\|_{L_T^\infty L_x^2}\\ &=: I + II.\end{aligned}$$

By (16), (17), Plancherel theorem (9) and (3), we control the linear term
$$\begin{aligned} I &\le \|W(t)(|x|^m v_0)\|_{L_T^\infty L_x^2} + \|W(t)(\mathcal{F}^{-1}[\Phi_{t,m}(\mathcal{F}(v_0)(\xi)])\|_{L_T^\infty L_x^2}\\ &\lesssim \||x|^m v_0\|_{L^2} + (1+T)\big(\|v_0\|_{L^2} + \|D_x^{2m}v_0\|_{L^2}\big)\\ &\lesssim \||x|^m v_0\|_{L^2} + (1+T)\|v_0\|_{H^{1/3}}. \end{aligned} \tag{38}$$

Let φ be a compact support, such that that $0 \le \varphi \le 1$ and $\varphi \equiv 1$ on $(-1,1)$. Using the pointwise formula (16) and Minkowski integral inequality, we split the nonlinear term II as follows
$$\begin{aligned} II \le& \int_0^T \Big\|W(t-\tau)\Big(|x|^m\varphi(x)(v^4)_x\Big)\Big\|_{L_T^\infty L_x^2}\,d\tau\\ &+ \Big\|\int_0^t W(t-\tau)\frac{\partial}{\partial x}[|x|^m(1-\varphi(x))v^4]\,d\tau\Big\|_{L_T^\infty L_x^2}\\ &+ \Big\|\int_0^t W(t-\tau)\Big(\frac{\partial}{\partial x}\big\{|x|^m(1-\varphi(x))\big\}v^4\Big)\,d\tau\Big\|_{L_T^\infty L_x^2}\\ &+ \int_0^T \Big\|W(t-\tau)\mathcal{F}^{-1}[\Phi_{t,m}(\mathcal{F}((v^4)_x))]\Big\|_{L_T^\infty L_x^2}\,d\tau\\ =:& II_1 + II_2 + II_3 + II_4. \end{aligned} \tag{39}$$

The estimates (9) and (21), the compact support of φ, Hölder integral inequality imply that
$$II_1 \lesssim \int_0^T \||x|^m\varphi(x)(v^4)_x\|_{L_x^2}\,d\tau \lesssim T^{1/2}\|(v^4)_x\|_{L_T^2L_x^2} \lesssim T^{1/2}\|v\|_{Z_T}^4. \tag{40}$$

By (14) we bound II_2

$$\begin{aligned} II_2 &\lesssim \||x|^m(1-\varphi(x))v^4\|_{L^1_xL^2_T} \lesssim \||x|^m v^4\|_{L^1_xL^2_T} \le \||x|^m v\|_{L^2_xL^2_T}\|v^3\|_{L^2_xL^\infty_T} \\ &= \||x|^m v\|_{L^2_TL^2_x}\|v\|^3_{L^6_xL^\infty_T} \le T^{1/2}\||x|^m v\|_{L^\infty_TL^2_x}\big(\sigma^T_5(v)\big)^3 \le T^{1/2}\|v\|^4_{Z_T}. \end{aligned} \tag{41}$$

By invoking the semigroup property (15) with $q = 18$ and $p = 3$ ($q' = 18/17$ and $p' = 3/2$) and Minkowski's integral inequality, we bound the following term

$$\begin{aligned} II_3 &\lesssim \|\frac{\partial}{\partial x}[|x|^m(1-\varphi(x)]v^4\|_{L^{18/17}_TL^{3/2}_x} \lesssim \|v^4\|_{L^{18/17}_TL^{3/2}_x} \\ &\lesssim T^\theta\|v\|^4_{L^\infty_TL^6_x} \le T^\theta\|v\|^4_{L^6_xL^\infty_T} = T^\theta\big(\sigma^T_5(v)\big)^4 \le T^\theta\|v\|^4_{Z_T} \end{aligned} \tag{42}$$

for some $\theta > 0$. Finally, formulas (9), (17), (20) and (21) allow to deduce

$$\begin{aligned} II_4 &\sim \int\limits_0^T \Big\|\mathcal{F}^{-1}[\Phi_{t,m}(\mathcal{F}((v^4)_x))]\Big\|_{L^\infty_TL^2_x}d\tau \\ &\lesssim (1+T)\int\limits_0^T \big(\|(v^4)_x\|_{L^2_x} + \|D^{2m}_x(v^4)_x\|_{L^2_x}\big)\,d\tau \\ &\lesssim (1+T)T^{1/2}\big(\|(v^4)_x\|_{L^2_TL^2_x} + \|D^{1/3}_x(v^4)_x\|_{L^2_TL^2_x}\big) \\ &\lesssim (1+T)T^{1/2}\|v\|^4_{Z_T}. \end{aligned} \tag{43}$$

Finally, the bounds (23), (38)–(43) give

$$\|\Psi(v)\|_{Z_T} \le C(1+T^\theta)\|v_0\|_{H^{1/3}} + C\||x|^m v_0\|_{L^2} + CT^\theta\delta^4$$

for $v \in Z^\delta_T$, $C, \theta > 0$. Therefore, if we take

$$\delta := 2C\big(\|v_0\|_{H^{1/3}} + \||x|^m v_0\|_{L^2}\big) \tag{44}$$

and T such that

$$\frac{\delta}{2} + CT^\theta\big(\|v_0\|_{H^{1/3}} + \delta^4\big) \le \delta, \tag{45}$$

then the following inequality holds

$$\|\Psi(v)\|_{Z_T} \le \delta.$$

Consequently, Ψ maps Z^δ_T into itself.

Step 2a. Now we need to prove that Ψ is a contraction on Z^δ_T. Suppose that $v, w \in Z^\delta_T$, where δ from (44) and T to determine in a moment. For some $\theta > 0$ to establish the following estimate

$$\big\||x|^m\big(\Psi(v) - \Psi(w)\big)\big\|_{L^\infty_TL^2_x} \lesssim T^\theta\delta^3\|v-w\|_{Z_T} \tag{46}$$

we is the main goal of the part Step 2a. Analogously, using the same argument as in estimating the nonlinear term II in Step 1a, we bound the left hand side norm of (46).

Applying the expression (27) we change v^4 by $v^4 - w^4$. We apply (28) and (29), instead of (20) and (21), for the new factors related to II_1 and II_4 . As for II_2 and II_3 it suffices to insert

$$\||x|^m(v^4 - w^4)\|_{L^1_x L^2_T} \lesssim T^\theta \delta^3 \|v - w\|_{Z_T}$$

and

$$\|v^4 - w^4\|_{L_T^{18/17} L_x^{3/2}} \lesssim T^\theta \delta^3 \|v - w\|_{Z_T}$$

in (41) and (42), respectively.

Furthermore, by using expressions (31) the last inequalities can be rewritten as follows

$$\||x|^m(v - w)u_1u_2u_3\|_{L^1_x L^2_T} \lesssim T^\theta \delta^3 \|v - w\|_{Z_T}$$

and

$$\|(v - w)u_1u_2u_3\|_{L_T^{18/17} L_x^{3/2}} \lesssim T^\theta \delta^3 \|v - w\|_{Z_T},$$

where u_1, u_2, u_3 represent the functions v or w. Really, these inequalities can be obtained by Hölder's inequality,

$$\begin{aligned}\||x|^m(v - w)u_1u_2u_3\|_{L^1_x L^2_T} &\leq \||x|^m(v - w)\|_{L^2_x L^2_T} \|u_1u_2u_3\|_{L^2_x L^\infty_T} \\ &\leq T^{1/2} \||x|^m(v - w)\|_{L^\infty_T L^2_x} \prod_{j=1}^{3} \|u_j\|_{L^6_x L^\infty_T} \leq T^{1/2} \delta^3 \|v - w\|_{Z_T}\end{aligned}$$

and

$$\begin{aligned}\|(v - w)u_1u_2u_3\|_{L_T^{18/17} L_x^{3/2}} &\leq T^{17/18} \|(v - w)u_1u_2u_3\|_{L_x^{3/2} L_T^\infty} \\ &\leq T^{17/18} \|v - w\|_{L^6_x L^\infty_T} \prod_{j=1}^{3} \|u_j\|_{L^6_x L^\infty_T} \leq T^{17/18} \delta^3 \|v - w\|_{Z_T}.\end{aligned}$$

In summary, collecting (30) and (46) we obtain

$$\|\Psi(v) - \Psi(w)\|_{Z_T} \leq CT^\theta \delta^3 \|v - w\|_{Z_T}$$

for some $C, \theta > 0$.

Finally, we prove that Ψ is a contraction on Z_T^δ for $T > 0$ such that $CT^\theta \delta^3 < 1$ and (45).

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