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ON THE INITIAL BOUNDARY PROBLEM FOR HYPERBOLIC EQUATIONS WITH EXPONENTIAL DEGENERATION $t^{12/7}$

Degenerate equations have been and are the object of numerous studies. They have not only theoretical but also practical significance. Let us only point out the fact that they arise when modeling subsonic and supersonic processes flows in a gaseous environment, filtration processes and movement of groundwater, in climate forecasts, etc. Mathematically, the degeneracy of a differential equation can be different. In this paper we consider a degenerate equation of the form $\partial_t \left(t^\beta \partial_t u(x,t)\right) - \Delta u(x,t) = f(x,t)$. In a bounded cylindrical domain, when the degree of degeneracy $\beta = 12/7$, we have established the unique solvability of the Cauchy-Dirichlet problem for the considered degenerate hyperbolic equation. Based on the solution of the spectral problem for the Laplace operator with Dirichlet conditions are introduced spectral decompositions of the right side of the differential equation and the desired solution to the Cauchy-Dirichlet problem. For the Fourier coefficients we obtain a family of Cauchy problems for a degenerate second-order ordinary differential equation, moreover, the second initial the condition must be met with weight. The latter is determined by the degree of degeneracy of the equation. The solutions to each of the Cauchy problems are represented by using Bessel functions. A priori estimates are established, on the basis of which is established the solvability of the initial-boundary problems for a degenerate hyperbolic equation.
Key words: Degenerate equations, degree of degeneracy, hyperbolic equations, a priori estimate.

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Жойылым дәрежесі $t^{12/7}$ гиперболалық теңдеу үшін бастапқы шекаралық есеп туралы

Жойылмалы теңдеулер көптеген зерттеулердің объектісі болды да және болып табылады. Олардың тек теориялық емес, практикалық маңызы бар. Атап кететін болсақ олар газды ортадағы дыбысқа дейінгі және супер дыбыстық ағындар, сүзу және жер асты суларының қозғалысының үдерістерін модельдеу кезінде, климаттық болжамдарда және т.б. пайда болады. Математикалық тұрғыдан дифференциалдық теңдеудің жойылымдығы әртүрлі болуы мүмкін. Бұл жұмыста $\partial_t \left(t^\beta \partial_t u(x,t)\right) - \Delta u(x,t) = f(x,t)$ түріндегі жойылмалы теңдеуін қарастырамыз. Шектелген цилиндрлік облысында, жойылым дәрежесі $\beta = 12/7$ болғанда, қарастырылып отырған жойылмалы гиперболалық теңдеу үшін Коши-Дирихле есебінің бірмәнді шешімділігін орнаттық. Дирихле шарттары бар Лаплас операторы үшін спектрлік есептің шешімі негізінде, берілген функция болып табылатын дифференциалдық теңдеудің оң жағы мен Коши-Дирихле есебінің ізделінді шешімінің спектрлік жіктелуі енгізіледі. Фурье коэффициенттері үшін біз, екінші бастапқы шарты салмақтықпен орындалуы қажет болатын, жойылмалы екінші ретті қарапайым дифференциалдық теңдеулердің Коши есептерінің тобын аламыз. Соңғысы теңдеудің жойылым дәрежесімен анықталады. Коши есептерінің әрқайсысының шешімдері Бессель функциялары арқылы анықталады. Жұмыста сонымен қатар жойылмалы гиперболалық теңдеу үшін бастапқы шекаралық есептің шешімділігіне негіз болатын априорлы бағалаулар алынды.
Түйін сөздер: Жойылмалы теңдеулер, жойылым дәрежесі, гиперболалық теңдеу, априорлы бағалаулар.

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О начально-граничной задаче для гиперболического уравнения со степенным вырождением $t^{12/7}$

Вырождающиеся уравнения являлись и являются объектом многочисленных исследований. Они имеют не только теоретическую, но и практическую значимость. Укажем лишь на тот факт, что они возникают при моделировании процессов до-звуковых и сверх-звуковых течений в газовой среде, процессов фильтрации и движения подземных вод, в прогнозе климата и т.д. Математически, вырождение дифференциального уравнения может быть различным. В настоящей работе рассматривается вырождающееся уравнение вида $\partial_t\left(t^{\beta}\partial_t u(x,t)\right) - \Delta u(x,t) = f(x,t)$. В ограниченной цилиндрической области, когда степень вырождения $\beta = 12/7$, нами установлена однозначная разрешимость задачи Коши-Дирихле для рассматриваемого вырождающегося гиперболического уравнения. На основе решения спектральной задачи для оператора Лапласа с условиями Дирихле вводятся спектральные разложения заданных функций – правой части дифференциального уравнения и искомого решения задачи Коши-Дирихле. Для коэффициентов Фурье мы получаем семейство задач Коши для вырождающегося обыкновенного дифференциального уравнения второго порядка, причем второе начальное условие должно выполняться с весом. Последнее определяется степенью вырождения уравнения. Решения каждой из задач Коши представляется с помощью функций Бесселя. Установлены априорные оценки, на основе которых установлена разрешимость начально-граничной задачи для вырождающегося гиперболического уравнения.
Ключевые слова: Вырождающиеся уравнения, степень вырождения, гиперболическое уравнение, априорные оценки.

1 Introduction

Would like to note that the authors study degenerate equations and investigate the solvability of various initial-boundary value problems in degenerate domains [1]– [4]. The presented work is a continuation of these studies.

The following initial-boundary value problem for a model degenerate hyperbolic equation

$$\partial_t(t^{\beta}\partial_t u) - \Delta u = f \quad \text{in} \quad Q = \Omega \times (0,T), \tag{1}$$

$$u = 0 \quad \text{on} \quad \Sigma = \partial\Omega \times (0,T), \tag{2}$$

$$u(x,0) = 0, \ \lim_{t\to+0} t^{\beta}\partial_t u(x,t) = 0 \quad \text{in} \quad \Omega, \tag{3}$$

was studied in the dissertation of N. Kaharman [5]. In particular, he established the following result:

Theorem 1 *Let $\beta \in [0,1)$, $f \in L^2(Q)$, $(-\Delta)^{1-\nu} f \in L^2(Q)$. Then problem (1)–(3) is uniquely solvable, and there is an a priori estimate*

$$\|u\|^2_{W^{2,2}(Q;\,t^{\beta})} \equiv \|u\|^2_{L^2(Q)} + \|t^{\beta}\partial_t u\|^2_{W_2^1(0,T;\,L^2(\Omega))} + \|\Delta\, u\|^2_{L^2(Q)}$$

$$\leq C\left[\|f\|^2_{L^2(Q)} + \|(-\Delta)^{1-\nu} f\|^2_{L^2(Q)}\right], \ \ \text{where } \nu = \frac{1-\beta}{2-\beta}, \tag{4}$$

that is, the parameter ν changes within the half-open segment: $\nu \in (0, 1/2]$.

As is known from [6], [7]– [12], if the degree of degeneracy $\beta \in [0,1)$, then this is a case of weak degeneracy of the equation. If $\beta \in [1,2]$, then this is a case of strong degeneracy. Thus, in the dissertation of N.Kaharman [5] the case of weak degeneracy is considered. The case of strong degeneracy is more difficult to study. Here, each value of the parameter β from the interval $[1,2]$ requires separate consideration. In this paper we study the case $\beta = 12/7$, parameter $\nu = 5/2$.

2 Statement of the problem. Main result

Let $0 < T < \infty$, $\Omega \subset R^n$ is a bounded domain with the boundary $\partial\Omega \in C^2$, $Q = \Omega \times (0,T)$, $\Sigma = \partial\Omega \times (0,T)$. Consider the following initial boundary value problem

$$\partial_t\big(t^{12/7}\,\partial_t u\big) - \Delta u = f \ \text{ in } Q, \tag{5}$$

$$u = 0 \text{ on } \Sigma, \tag{6}$$

$$u(x,0) = 0, \ \lim_{t\to+0} t^{12/7}\,\partial_t u(x,t) = 0 \text{ in } \Omega. \tag{7}$$

The following result is valid.

Theorem 2 *(Main result). Let the following conditions be satisfied:*

$$f(x,t) \in L^2(Q), \ \frac{f(x,t)}{t^\alpha} \in L^2(Q), \ \frac{\Delta f(x,t)}{t^\alpha} \in L^2(Q), \ \alpha > 23/14.$$

Then the problem (5)–(7) *is uniquely solvable, and there is an a priori estimate*

$$\|u\|^2_{W(Q;t^{12/7})} \equiv \|u\|^2_{L^2(Q)} + \|t^{12/7}\,\partial_t u\|^2_{W_2^1(0,T;L^2(\Omega))} + \|\Delta u\|^2_{L^2(Q)} \le$$

$$\le C\left[\|f(x,t)\|^2_{L^2(Q)} + \left\|\frac{f(x,t)}{t^\alpha}\right\|^2_{L^2(Q)} + \left\|\frac{\Delta f(x,t)}{t^\alpha}\right\|^2_{L^2(Q)}\right]. \tag{8}$$

3 Methods and materials. Proof of theorem2. A priori estimates

Applying the Fourier method to problem (5)–(7), namely $u(x,t) = \sum\limits_{j=1}^{\infty} c_j(t)z_j(x)$ we obtain the Cauchy problem for the Fourier coefficients $c_j(t)$

$$\big(t^{12/7}c_j'(t)\big)' + \lambda_j\,c_j(t) = f_j(t) \text{ in } (0,T), \tag{9}$$

$$c_j(0) = 0, \ \lim_{t\to+0} t^{12/7}\,c_j'(t) = 0, \tag{10}$$

where $\{z_j(x),\,\lambda_j\}$ is a solution to the spectral problem:

$$-\Delta z(x) = \lambda\,z(x), \ x \in \Omega, \ z|_{\partial\Omega} = 0, \tag{11}$$

and let the solution has the form

$$\{z_j(x),\ \ \lambda_j,\ \ j=1,2,..\},\ \ \text{moreover,}\ \ 0<\lambda_1<\lambda_2<..., \tag{12}$$

and the system of eigenfunctions $\{z_j(x),\ j=1,2,...\}$ is orthonormal.

Temporarily, for simplicity we will omit the index j. Let us rewrite equation (9) and the initial conditions (10) in the following form:

$$t^2c''(t)+\frac{12}{7}\,t\,c'(t)+\lambda\,t^{2/7}c(t)=t^{2/7}f(t)\ \text{in}\ (0,T). \tag{13}$$

$$c(0)=0,\ \ \lim_{t\to+0}t^{12/7}\,c'(t)=0. \tag{14}$$

Let us find a general solution to the inhomogeneous equation (13). According to ([13], chapter 2, § 2.1.2, formula 62 or 127), the general solution to the homogeneous version of equation (13) has the form:

$$c_{\text{hom.eq.}}(t)=t^{-5/14}\left[AJ_{5/2}\left(7\sqrt{\lambda}t^{1/7}\right)+BY_{5/2}\left(7\sqrt{\lambda}t^{1/7}\right)\right],\ \ t\in(0,T). \tag{15}$$

Since in (15) the Bessel functions have an index $\nu=5/2$, then according to ([14], 10.1.1, 10.1.11–10.1.12 at $\nu=\frac{5}{2}$) for the homogeneous equation (13) (for $f(t)\equiv 0$) the fundamental solutions can be written in the following form

$$\varphi_1(t)=\left(\frac{3}{[7\sqrt{\lambda}t^{1/7}]^3}-\frac{1}{7\sqrt{\lambda}t^{1/7}}\right)\sin(7\sqrt{\lambda}\,t^{1/7})-\frac{3}{[7\sqrt{\lambda}t^{1/7}]^2}\cos(7\sqrt{\lambda}\,t^{1/7}),$$

$$\varphi_2(t)=\left(-\frac{3}{[7\sqrt{\lambda}t^{1/7}]^3}+\frac{1}{7\sqrt{\lambda}t^{1/7}}\right)\cos(7\sqrt{\lambda}\,t^{1/7})-\frac{3}{[7\sqrt{\lambda}t^{1/7}]^2}\sin(7\sqrt{\lambda}\,t^{1/7}),$$

$$\bar{\varphi}_1(z)=\left(\frac{3}{z^2}-1\right)\sin z-\frac{3}{z}\cos z,\ \ \bar{\varphi}_2(z)=\left(-\frac{3}{z^2}+1\right)\cos z-\frac{3}{z}\sin z, \tag{16}$$

that is

$$\tilde{\varphi}_1(t)=\bar{\varphi}_1(z)\big|_{z=7\sqrt{\lambda}\,t^{1/7}},\ \ \tilde{\varphi}_2(t)=\bar{\varphi}_2(z)\big|_{z=7\sqrt{\lambda}\,t^{1/7}}, \tag{17}$$

therefore, the general solution (15) for the homogeneous equation (13) (for $f(t)\equiv 0$) is written as:

$$c_{\text{hom}}(t)=t^{-2/7}[A\,\varphi_1(t)+B\,\varphi_2(t)]. \tag{18}$$

Note, that the necessity for additional notations (16) will become evident later (formula(21)).

To find a general solution to the inhomogeneous equation (13) (where $f(t)\neq 0$) we use the method of variation of constants. We will have

$$c(t)=A(t)\,\varphi_1(t)+B(t)\,\varphi_2(t). \tag{19}$$

Let us write the system of algebraic equations in terms of the unknown functions $A'(t)$ and $B'(t)$:

$$\begin{cases}\varphi_1(t)\,A'(t)+\varphi_2(t)\,B'(t)=0,\\[2ex] \varphi_1'(t)\,A'(t)+\varphi_2'(t)\,B'(t)=\frac{f(t)}{t^{12/7}},\end{cases} \tag{20}$$

for which we calculate the Wronskian ([14], 10.1.6) and the corresponding determinants

$$W = W\{\varphi_1(t),\, \varphi_2(t)\} = \begin{Vmatrix} \varphi_1(t) & \varphi_2(t) \\ \varphi_1'(t) & \varphi_2'(t) \end{Vmatrix} = \frac{1}{[7\sqrt{\lambda}\, t^{1/7}]^2},$$

$$W_A = \begin{Vmatrix} 0 & \varphi_2(t) \\ \dfrac{f(t)}{t^{12/7}} & \varphi_2'(t) \end{Vmatrix} = -\frac{f(t)}{t^{12/7}}\,\varphi_2(t),\;\; W_B = \begin{Vmatrix} \varphi_1(t) & 0 \\ \varphi_1'(t) & \dfrac{f(t)}{t^{12/7}} \end{Vmatrix} = \frac{f(t)}{t^{12/7}}\,\varphi_1(t).$$

Hence, for the unknown coefficients of the general solution of equation (13) from (19), we, respectively, obtain

$$A'(t) = \frac{W_A}{W},\;\; B'(t) = \frac{W_B}{W},$$

that is,

$$A(t) = -7\sqrt{\lambda}\int_0^t \frac{f(\tau)}{\tau^{11/7}}\left[\left(-\frac{3}{[7\sqrt{\lambda}\,\tau^{1/7}]^2}+1\right)\cos(7\sqrt{\lambda}\,\tau^{1/7}) - \frac{3}{7\sqrt{\lambda}\,\tau^{1/7}}\sin(7\sqrt{\lambda}\,\tau^{1/7})\right]d\tau + a,$$

$$B(t) = 7\sqrt{\lambda}\int_0^t \frac{f(\tau)}{\tau^{11/7}}\left[\left(\frac{3}{[7\sqrt{\lambda}\,\tau^{1/7}]^2}-1\right)\sin(7\sqrt{\lambda}\,\tau^{1/7}) - \frac{3}{7\sqrt{\lambda}\,\tau^{1/7}}\cos(7\sqrt{\lambda}\,\tau^{1/7})\right]d\tau + b.$$

To satisfy the first initial condition from (14), it is necessary that $a = 0,\; b = 0$. Then for the general solution of equation (13) from (19) we obtain

$$c(t) = -\int_0^t \frac{f(\tau)}{t^{1/7}\tau^{11/7}}\left[\tilde{\varphi}_1(t)\tilde{\varphi}_2(\tau) - \tilde{\varphi}_1(\tau)\tilde{\varphi}_2(t)\right]d\tau, \tag{21}$$

where according to (16)–(17) the functions $\tilde{\varphi}_1(t),\; \tilde{\varphi}_2(t)$ are defined by the following formulas

$$\tilde{\varphi}_1(t) = \left(\frac{3}{[7\sqrt{\lambda}\,t^{1/7}]^2}-1\right)\sin(7\sqrt{\lambda}\,t^{1/7}) - \frac{3}{7\sqrt{\lambda}\,t^{1/7}}\cos(7\sqrt{\lambda}\,t^{1/7}),$$

$$\tilde{\varphi}_2(t) = \left(-\frac{3}{[7\sqrt{\lambda}\,t^{1/7}]^2}+1\right)\cos(7\sqrt{\lambda}\,t^{1/7}) - \frac{3}{7\sqrt{\lambda}\,t^{1/7}}\sin(7\sqrt{\lambda}\,t^{1/7}).$$

For convenience of calculations, we introduce the functions $\bar{c}(z),\; \bar{f}(z),\; \bar{\varphi}_1(z),\; \bar{\varphi}_2(z)$, so that the following equalities hold:

$$c(t) = \bar{c}(z)\big|_{z=7\sqrt{\lambda}t^{1/7}},\;\; f(t) = \bar{f}(z)\big|_{z=7\sqrt{\lambda}t^{1/7}},\;\; \tilde{\varphi}_1(t) = \bar{\varphi}_1(z)\big|_{z=7\sqrt{\lambda}t^{1/7}},\;\; \tilde{\varphi}_2(t) = \bar{\varphi}_2(z)\big|_{z=7\sqrt{\lambda}t^{1/7}}, \tag{22}$$

where

$$\bar{\varphi}_1(z) = \left(\frac{3}{z^2}-1\right)\sin z - \frac{3}{z}\cos z,\;\; \bar{\varphi}_2(z) = \left(-\frac{3}{z^2}+1\right)\cos z - \frac{3}{z}\sin z. \tag{23}$$

Then from (21)–(23) for the solution $\bar{c}(z)$ we obtain

$$\bar{c}(z) = -7^6\lambda^{5/2}\int\limits_0^z \frac{\bar{f}(\zeta)}{z\,\zeta^5}\left[\bar{\varphi}_1(z)\bar{\varphi}_2(\zeta) - \bar{\varphi}_1(\zeta)\bar{\varphi}_2(z)\right]d\zeta =$$

$$= -7^6\lambda^{5/2}\int\limits_0^z \frac{\bar{f}(\zeta)}{z\zeta^5}\left\{\left[\left(\frac{3}{z^2}-1\right)\sin z - \frac{3}{z}\cos z\right]\left[\left(-\frac{3}{\zeta^2}+1\right)\cos\zeta - \frac{3}{\zeta}\sin\zeta\right] - \right.$$
$$\left.\left[\left(\frac{3}{\zeta^2}-1\right)\sin\zeta - \frac{3}{\zeta}\cos\zeta\right]\left[\left(-\frac{3}{z^2}+1\right)\cos z - \frac{3}{z}\sin z\right]\right\} d\zeta, \tag{24}$$

or

$$\bar{c}(z) = -7^6\lambda^{5/2}\int\limits_0^z \frac{\bar{f}(\zeta)}{z\zeta^5}\cdot\frac{3(z-\zeta)}{\zeta^2}\left\{\left(1-3\frac{\zeta}{z}+\frac{\zeta^2}{z^2}\right)\frac{\sin(z-\zeta)}{z-\zeta} + \left(\frac{\zeta}{z}+\frac{3}{z^2}\right)\cos(z-\zeta)\right\}d\zeta =$$

$$= -3{\cdot}7^6\lambda^{5/2}\left[\int\limits_0^z \frac{\bar{f}(\zeta)}{z\zeta^7}\left(1-3\frac{\zeta}{z}+\frac{\zeta^2}{z^2}\right)\sin(z-\zeta)d\zeta - \int\limits_0^z \frac{\bar{f}(\zeta)}{z^2\zeta^7}(z\,\zeta+3)\left(1-\frac{\zeta}{z}\right)\cos(z-\zeta)d\zeta\right]. \tag{25}$$

From (25), using formulas (21)–(24), we obtain:

$$|c(t)| \le K_1\int\limits_0^t \frac{|f(\tau)|}{t^{1/7}\tau^{13/7}}d\tau + K_2\int\limits_0^t \frac{|f(\tau)|}{t^{2/7}\tau^{13/7}}d\tau \le$$

$$\le \left[\frac{K_1 t^{\alpha_1-3/2}}{\sqrt{2\alpha_1-3}} + \frac{K_2 t^{\alpha_2-23/14}}{\sqrt{2\alpha_2-23/7}}\right]\left\|\frac{f(\tau)}{\tau^\alpha}\right\|_{L^2(0,T)}, \quad \alpha = \max\{\alpha_1,\,\alpha_2\} > 23/14, \tag{26}$$

where the constants K_1 and K_2 do not depend on λ, and satisfy the inequalities

$$K_1 \ge \frac{3}{7^2\lambda},\quad K_2 \ge \frac{3}{7^3\lambda^{3/2}}. \tag{27}$$

From (26)–(27) ,we obtain the fulfillment of the first initial condition from (14).

Let us now verify the fulfillment of the second initial condition from (14). We will calculate the derivative with respect to z of the function $\bar{c}(z)$ (25):

$$\bar{c}'(z) = -3\cdot 7^6\lambda^{5/2}\int\limits_0^z \frac{\bar{f}(\zeta)}{z^2\zeta^7}\left\{\left(-z\zeta+\zeta^2-4+9\frac{\zeta}{z}-3\frac{\zeta^2}{z^2}\right)\sin(z-\zeta)+\right.$$
$$\left.+\frac{1}{z}\left(z^2-4z\zeta+3(\zeta^2-2)+9\frac{\zeta}{z}\right)\cos(z-\zeta)\right\}d\zeta. \tag{28}$$

Furthermore, by using the formula

$$t^{12/7}c'(t) = \frac{z^{12}}{7^{12}\lambda^6}\,\bar{c}'(z)\big|_{z=7\sqrt{\lambda}\,t^{1/7}}\frac{dz(t)}{dt} = \frac{z^6}{7^6\lambda^{5/2}}\,\bar{c}'(z)\big|_{z=7\sqrt{\lambda}\,t^{1/7}}, \tag{29}$$

we establish

$$\frac{z^6}{7^6\lambda^{5/2}}\bar{c}'(z) = -3z^4\int\limits_0^z \frac{\bar{f}(\zeta)}{\zeta^7}\left\{\left(-z\zeta+\zeta^2-4+9\frac{\zeta}{z}-3\frac{\zeta^2}{z^2}\right)\sin(z-\zeta)+\right.$$

$$\left.+\frac{1}{z}\left(z^2-4z\zeta+3(\zeta^2-2)+9\frac{\zeta}{z}\right)\cos(z-\zeta)\right\}d\zeta. \tag{30}$$

From relation (30), using formula (29), we, respectively, obtain

$$|t^{12/7}c'(t)| \le K_1^1\int\limits_0^t \frac{t^{10/7}|f(\tau)|}{\tau^{13/7}}d\tau + K_2^1\int\limits_0^t \frac{t^{9/7}|f(\tau)|}{\tau^{13/7}}d\tau \le$$

$$\le \left[\frac{K_1^1 t^{\alpha_1^1-9/7}}{\sqrt{2\alpha_1^1-19/7}} + \frac{K_2^1 t^{\alpha_2^1-10/7}}{\sqrt{2\alpha_2^2-19/7}}\right]\left\|\frac{f(\tau)}{\tau^{\alpha_1}}\right\|_{L^2(0,T)},\ \ \alpha_1>19/14,$$

where the constants K_1^1 and K_2^1 do not depend on λ, and satisfy the following inequalities

$$K_1^1 \ge \frac{3}{7^3\lambda^{3/2}},\ \ K_2^1 \ge \frac{3}{7^4\lambda^2}.$$

We have shown that the second initial condition from (14) also holds if there is the following requirement on the right-hand side of equation (13) $t^{-\alpha_1}f(t)\in L^2(0,T)$, $\alpha_1 > 19/14$.

So, we have shown that if the conditions of theorem 2 are met, function (21) satisfies Cauchy problem(13)–(14).

Let us proceed to establish the a priori estimate (8). For the solution $c(t)$ (21) according to (26) , we will have:

$$|c(t)| \le \left[\frac{K_1 T^{\alpha_1-3/2}}{\sqrt{2\alpha_1-3}} + \frac{K_2 T^{\alpha_2-23/14}}{\sqrt{2\alpha_2-23/7}}\right]\left\|\frac{f(\tau)}{\tau^{\alpha}}\right\|_{L^2(0,T)},\ \ \alpha=\max\{\alpha_1,\,\alpha_2\}>23/14. \tag{31}$$

From (31) we have

$$\|c(t)\|_{L^2(0,T)} \le T^{1/2}\left[\frac{K_1 T^{\alpha_1-3/2}}{\sqrt{2\alpha_1-3}} + \frac{K_2 T^{\alpha_2-23/14}}{\sqrt{2\alpha_2-23/7}}\right]\left\|\frac{f(\tau)}{\tau^{\alpha}}\right\|_{L^2(0,T)},\ \ \alpha=\max\{\alpha_1,\,\alpha_2\}>23/14. \tag{32}$$

Furthermore, from the equation (13), we obtain:

$$\left|\left(t^{12/7}c'(t)\right)'\right| \le |f(t)| + \lambda|c(t)|,$$

Using the last inequality and (32), we will have the following estimate

$$\left\| \left(t^{12/7} c'(t)\right)' \right\|^2_{L^2(0,T)} \le K_3 \left[\|f(t)\|^2_{L^2(0,T)} + \left\| \frac{\lambda f(t)}{t^\alpha} \right\|^2_{L^2(0,T)} \right], \quad \alpha > 23/14. \tag{33}$$

Finally, using the relations

$$t^{12/7} c'(t) = \int\limits_0^t \left(\tau^{12/7} c'(\tau)\right)' d\tau, \quad \left|t^{12/7} c'(t)\right|^2 \le T \int\limits_0^T \left| \left(\tau^{12/7} c'(\tau)\right)' \right|^2 d\tau$$

and the estimate (33), we obtain the following estimate

$$\left\| t^{12/7} c'(t) \right\|^2_{L^2(0,T)} \le K_4 \left[\|f(t)\|^2_{L^2(0,T)} + \left\| \frac{\lambda f(t)}{t^\alpha} \right\|^2_{L^2(0,T)} \right], \quad \alpha > 23/14. \tag{34}$$

Now, returning the indices j to the functions $c(t)$ and $f(t)$ we note, that these are the Fourier coefficients of the functions $u(x,t)$ and $f(x,t)$ in the expansions:

$$u(x,t) = \sum_{j=1}^{\infty} c_j(t) z_j(x), \quad f(x,t) = \sum_{j=1}^{\infty} f_j(t) z_j(x), \tag{35}$$

where $\{z_j(x),\ \lambda_j,\ j = 1, 2, ...\}$ are solutions to the spectral problem (11)–(12).

As a result, using formulas (21) and (35), due to the Parseval-Stecklov equality and the estimate (32), we have the first a priori estimate:

$$\|u(x,t)\|^2_{L^2(Q)} = \sum_{j=1}^{\infty} \|c_j(t)\|^2_{L^2(0,T)} \le K_5 \left\| \frac{f(x,t)}{t^\alpha} \right\|^2_{L^2(Q)}, \quad \alpha > 23/14. \tag{36}$$

Next, for $-\Delta u(x,t)$ according to (36), we obtain the second a priori estimate:

$$\|\Delta u(x,t)\|^2_{L^2(Q)} = \sum_{j=1}^{\infty} \|\lambda_j\, c_j(t)\|^2_{L^2(0,T)} \le K_6 \left\| \frac{\Delta f(x,t)}{t^\alpha} \right\|^2_{L^2(Q)}, \quad \alpha > 23/14. \tag{37}$$

Now let us establish an estimate for the expression $\partial_t\big(t^{\frac{12}{7}}\, \partial_t u(x,t)\big)$. First of all, from equation (5), we obtain equalities

$$\partial_t\big(t^{12/7}\, \partial_t u(x,t)\big) = f(x,t) + \Delta u(x,t), \tag{38}$$

$$t^{12/7}\, \partial_t u(x,t) = \int\limits_0^t \left[f(x,\tau) + \Delta u(x,\tau)\right] d\tau. \tag{39}$$

From (38)–(39) and estimate (37) we have the third and fourth a priori estimates

$$\left\| \partial_t\big(t^{12/7}\, \partial_t u(x,t)\big) \right\|^2_{L^2(Q)} \le K_7 \left[\|f\|^2_{L^2(Q)} + \left\| \frac{\Delta f(x,t)}{t^\alpha} \right\|^2_{L^2(Q)} \right], \quad \alpha > 23/14, \tag{40}$$

$$\left\| t^{12/7}\, \partial_t u(x,t) \right\|^2_{L^2(Q)} \le K_8 \left[\|f\|^2_{L^2(Q)} + \left\| \frac{\Delta f(x,t)}{t^\alpha} \right\|^2_{L^2(Q)} \right], \quad \alpha > 23/14. \tag{41}$$

The set of inequalities (36), (37), (40)–(41) is equivalent to the a priori estimate (8). Theorem 2 is completely proven.

4 Conclusion

In the work, sufficient conditions are found to be imposed on the right-hand side of the differential equation, which ensure in the Sobolev space the unique solvability of the homogeneous Cauchy-Dirichlet problem for one inhomogeneous degenerate hyperbolic equation, the degree of degeneracy β of which is determined by the relation: $\beta = 12/7$.

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