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INTEGRO-INTERPOLATION METHOD OF CONSTRUCTING A DIFFERENCE SCHEME IN A PROBLEM WITH A MOVING BOUNDARY

Working with systems that involve moving boundaries can be a very difficult task. Not only do we have to solve the equations describing the system, but we also have to find the region the system occupies at each step. One of the common moving-boundary classes, Stefan problems are systems of diffusion or heat-conduction where the boundaries between the different phases in the system change over time [1, 2]. Unfortunately, since Stefan problems can be so complex that an analytical solution of the system is often impossible. Therefore, approximate analytical methods or numerical methods, which are the most practical for working with these problems, are often used. This work is devoted to numerical investigation of nonlinear fluid filtration. Hydrodynamic study of non-Newtonian fluid filtration requires solving nonlinear differential equations with partial derivatives. The integration of these equations is associated with serious mathematical difficulties caused by moving boundaries, the dependence of the physical properties on the coordinates and time, the specifics of the boundary conditions. Therefore, in the works devoted to the study of nonlinear effects of filtering liquid and gas, approximate methods are used (quasistationary approximation, the integral relations and numerical). Among them, we can note the simplicity and versatility of finite difference method, which, however, requires the solution of a complex system of algebraic equations with simple computational algorithms. In our problem, in order to close the mathematical system, another equation is required is a type of Stefan's condition. This is the law of conservation of momentum balance, which determines the position of the moving interface. Note that this moving boundary is an unknown surface. Consequently, the problem we are considering is an example of a free boundary problem [3].

Key words: nonlinear fluid filtration, non-newtonian fluid, movable boundary, region of the grids, numerical solution, finite difference method, approximate analytical solution.

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Шекарасы жылжымалы есептің айырымдық схемасын құрудың интегро-интерполяция әдісі

Жүйе жылжымалы шекералар бойынша берілсе күрделі есепке жатады. Мұнда жүйені сипаттайтын теңдеуді шешіп қоймай, жүйенің өзгеріс аймағын білу қажет. Осындай көп тараған жылжымалы шекрасымен берілген есептердің диффузиялық немесе жылуөткізгіштік процестерде кездесуі Стефан типтес есептерге жатады. Бұндай жүйеде әртүрлі фазалардың шекарасы уақытқа байланысты өзгеріп отырады [1, 2]. Өкінішке орай, Стефан есептері күрделі болғандықтан аналитикалық шешімдерін анықтау мүмкіндігі қиын. Сондықтан мұндай есептерде жуық аналитикалық шешімі және тәжірибеде ыңғайлы сандық жуық шешімдері қолданылады. Бұл есеп сызықты емес флюидтің филтрленуінің сандық зерттеу жұмысына жатады. Ньютондық емес сұйықтардың фильтрленуінің гидродинамикасын зерттеу сызықты емес дербес туындылы дифференциалдық теңдеудің шешіміне байланысты күрделі болады. Мұндай есептердің интегралын анықтау келесі мататматикалық қиындықтар тудырады: процесті сипаттайтын физикалық шамалардың кеңістік координаттарына және уақытқа байланысты өзгерісі, жылжымалы шекаралардың және шекаралық шарттардың ерекшелігіне байланысты болады. Сондықтан сұйықтардың фильтрленуінің сызықты емес эффектілігін зерттеуде жуық шығару әдістері (квазисызықты, интегралдық қатынас немесе сандық) қолданылады. Бұлардың ішінде қарапайымдылығымен және жетімділігімен ақырлы-айырымдық әдіс ерекше орын алады. Бірақ есептеу алгоритмінің жеңілдігіне қарамастан күрделі алгебралық теңдеулер жүйесін құрып шығару керек. Біздің есепті тұйықталған математикалық жүйеге келтіру үшін Стефан шартында кездесетін теңдеу қажет. Бұл жерде сұйықтық қозғалысының шекарасына байланысты қозғалыс мөлшерінің сақталу импульс заңын беретін теңдеу болу керек. Мұндай есеп белгісіз шекараға байланысты жылжымалы бетті береді. Сондықтан еркін жылжыйтын шекараға байланысты шартпен берілген есепке мысал бола алады [3].

Түйін сөздер: сызықты емес сұйықтықты фильтрациялау, Ньютондық емес сұйықтық, жылжымалы шекара, тор аймағы, сандық шешім, шекті айырмдық әдіс, жуықталған аналитикалық шешім.

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Интегро-интерполяционный метод построения разностной схемы в задаче с подвижной границей

Работа с системами, которые имеют подвижные границы, может оказаться очень сложной задачей. Нужно не только решить уравнения, описывающие систему, но и найти область, которую система занимает на каждом шаге. Один из распространенных классов задач с движущимися границами, задачи Стефана – это системы диффузии или теплопроводности, в которых границы между различными фазами в системе меняются с течением времени [1,2]. К сожалению, поскольку задачи Стефана могут быть настолько сложными, что аналитическое решение системы часто оказывается невозможным. Поэтому часто используются приближенные аналитические методы или численные методы, которые наиболее практичны для работы с такими задачами. Данная работа посвящена численному исследованию нелинейной фильтрации флюида. Гидродинамическое исследование фильтрации неньютоновской жидкости ставит перед необходимостью решения нелинейных дифференциальных уравнений с частными производными. Интегрирование таких уравнений связано с серьезными математическими трудностями, обусловленными подвижными границами, зависимостью физических свойств от координат и времени, спецификой краевых условий. Поэтому в работах, посвященных исследованию нелинейных эффектов фильтрации флюида, применяются приближенные методы (квазистационарное приближение, интегральные соотношения и численные). Среди них простотой и универсальностью отличается метод конечных разностей, который, однако, требует решения громоздкой системы алгебраических уравнений при простате вычислительных алгоритмов. В нашей задаче, чтобы замкнуть математическую систему, требуется еще одно уравнение – типа условия Стефана. Это закон сохранения баланса импульса движения, который определяет положение движущейся границы раздела. Заметим, что эта движущаяся граница является неизвестной поверхностью. Следовательно, рассматриваемая нами задача является примером задачи со свободной границей [3].

Ключевые слова: нелинейная фильтрация жидкости, неньютоновская жидкость, подвижная граница, область сеток, численное решение, метод конечных разностей, приближенное аналитическое решение.

1 Introduction

Oils of Western Kazakhstan, containing a relatively large amount of paraffin-asphaltene-resinous substances, belong to non-Newtonian fluids. The study of the structural and mechanical properties of such oils is of great interest for solving various issues of oil production. The study of the rheological characteristics of non-Newtonian oils on a capillary viscometer (copper capillary tubes with a diameter of 2, 3 and 4 mm and a length of 200, 300 and 400 mm) was carried out according to a well-known technique, [4–6]. For oil with a 25% resin content at temperatures of 16, 18 and 210 C, the characteristic dependence $\vartheta = \vartheta(|\nabla P|)$ Figure 1 shows the curve, [7, 8]. For the test oil, the plot of dependence is nonlinear and passes through the origin, and it is convexed downwards at small pressure gradients, and at large it has a linear shape. Models of viscous and viscous-plastic media and various types of dependence were used to describe the structural behavior of the fluid. So, for example, the flow curve shown in Figure 1 can be approximated by two straight lines, in particular, one straight line OA, passing through the origin, and another AC, cutting off on the abscissa axis segment OB, corresponding to the limiting shear gradient g_*. If we restrict research to $|\nabla P| > g_*$, then we obtain a model similar to the Shvedov–Bingham model, and for $|\nabla P| < g_*$ is viscous fluid flow model.

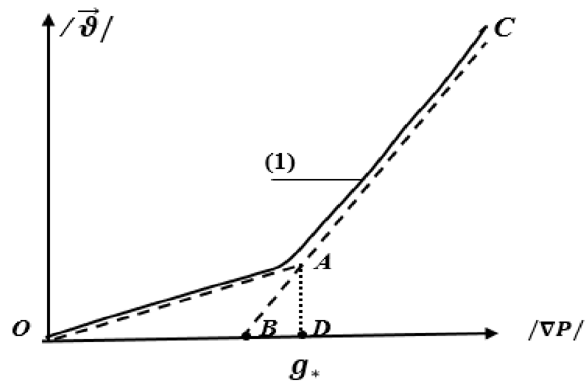

Figure 1

Figure 1. Shows the experimental dependence of the $\vartheta = \vartheta(|\nabla P|)$ fluid filtration rate (solid curve 1) and its approximation (dashed lines) BC or OAC. If the rate of fluid filtration at low pressure gradients cannot be neglected, then it is necessary to use filtration models that take into account the fluid flow at such gradients. These models include those based on polygonal or other approximations of the indicator curve

$$\vartheta = \vartheta(|\nabla P|).$$

When using the experimental curve (1), the velocity of fluid movement in a porous medium can be described by the nonlinear equation, [5, 8]:

$$\overrightarrow{\vartheta} = -\frac{k}{\mu} F(|\nabla P|) \frac{\nabla P}{|\nabla P|}. \quad (1)$$

In this case $F(|\nabla P|)$ is a continuous positive, monotonically increasing function ($F'(|\nabla P|) > 0$), derivative of which can have a finite number of discontinuities of the first kind.

Figure 1 (curve 1) shows a model of a viscous medium with an apparent viscosity [7] depending on the pressure gradient. In this case, the polygon fits into the indicator curve in such a way that its first link passes through the origin and characterizes filtration at low pressure gradients, and the second link coincides with the asymptote of the graph $\vartheta = \vartheta(|\nabla P|)$ and characterized the flow of fluids at large pressure gradients.

To calculate the filtration of a Newtonian fluid, along with mathematical methods, simulators (analog computers) are widely used, however, electric simulation of the flow of a non-Newtonian fluid in a porous medium using conventional simulators is in most cases impossible. In work [9] similarity criteria are derived and an analog simulator based on the well-known electrohydrodynamic analogy is described. In the case when the movements of the liquid at small pressure gradients are not taken into account, i.e. the curve is approximated by a half-line BC, cutting off on the abscissa axis a segment of OB.

Thus, approximation by a two-link polygon will give the following model of nonlinear filtering [8].

$$\overrightarrow{\vartheta} = \begin{cases} -\frac{k}{\nu}\nabla P, & |\nabla P| < g_*, \\ -\frac{k}{\mu}(|\nabla P| - \mu_* g_*)\frac{\nabla P}{|\nabla P|}, & |\nabla P| > g_*. \end{cases} \tag{2}$$

Here $\mu_* = \left(1 - \frac{\mu}{\nu}\right)$ is the apparent viscosity, and μ and ν are dynamic viscosities at small and large pressure gradients.

In other works [8], approximate analytical solutions for this problem were obtained using various approximations: either the half-line BC or the polygon OAC. To obtain approximate solutions, usually use the method of integral relations proposed by G.I. Barenblatt [6] and the method based on applying the Laplace transform. With the help of the Laplace transform, the approximate solution of the problem is limited to the initial stage of the process (quasi-stationary approximation).

2 Methods and materials

2.1 System of equations describing isothermal fluid filtration

It is assumed that the terrestrial rock is elastic, and the fluid belongs to the class of weakly compressible liquids. Under these assumptions, the mathematical model can be represented in the form of the following system of equations: continuity equation $\frac{\partial m\rho}{\partial t} + \mathrm{div}(\rho\overrightarrow{\vartheta}) = 0$; equation of the porous medium state is $dm = \beta_r dP$; equation for the fluid state is $\rho = \rho_0 \exp[\beta_f(P - P_0)]$.

Then the equation of continuity, taking into account the equations of state of the porous medium and fluid becomes

$$\beta^* \frac{\partial P}{\partial t} + \mathrm{div}\,\overrightarrow{\vartheta} = 0. \tag{3}$$

In the above formulas, the following designations are adopted: k is permeability coefficient, $\beta^* = \beta_r + \beta_f$ is reservoir coefficient of elasticity, β_r and β_f are rock and fluid compressibility factors.

2.2 Mathematical model of nonlinear fluid filtration

Let's now consider the problem of fluid filtration, i.e. with polygonal approximation of the experimental flow rate is depression curve (Figure 1). Let an isotropic layer of unit thickness and width be filled with a homogeneous liquid. Under the long-term influence of the temperature field, the liquid acquired structural and mechanical properties that were unequal along the length of the formation. In this case, the value of the gradient at the boundary of the viscosity discontinuity can depend on the x coordinate. If the structural and mechanical properties of liquid particles are distributed during their transfer, we will assume that the magnitude of the gradient at the viscosity discontinuity boundary $g(x,t)$ will change in proportion to the speed of the liquid. Then, taking into account the filtration law (2), the continuity equation (3) in a one-dimensional formulation for the case $g(x,0) = g_* = \text{const}$ is reduced to solving the equations [7, 8]. Thus, it is required to find the function $P_1(x,t)$, $P_2(x,t)$, $\xi(t)$ from the conditions:

$$c_1\frac{\partial P_1}{\partial t} = \frac{\partial}{\partial x}\left[k_1(x,t)\left(\frac{\partial P_1}{\partial x} - \mu_* g_*\right)\right], \quad x_0 < x \le \xi(t), \quad t > 0, \tag{4}$$

$$c_2\frac{\partial P_2}{\partial t} = \frac{\partial}{\partial x}\left[k_2(x,t)\frac{\partial P_2}{\partial x}\right], \quad \xi(t) \le x \le L, \quad t > 0. \tag{5}$$

where $c_1 = \mu\beta^*$, $c_2 = \nu\beta^*$.

Under the initial condition

$$P_2(x,0) = \varphi(x), \quad x_0 < x < L, \quad \xi(0) = x_0, \tag{6}$$

and the condition of matching the initial values P_1 and $P_2 : \lim\limits_{t\to 0} P_1(x,t) = \varphi(x_0)$.

Under the following conditions on the unknown boundary $\xi = \xi(t)$:

$$\lim_{x\to\xi-0} P_1(x,t) = \lim_{x\to\xi+0} P_2(x,t), \quad t > 0, \tag{7}$$

$$\lim_{x\to\xi-0}\frac{\partial}{\partial x}P_1(x,t) = \lim_{x\to\xi+0}\frac{\partial}{\partial x}P_2(x,t) = g_*, \quad t > 0. \tag{8}$$

and the corresponding condition on the gallery

$$\alpha_1\frac{k}{\mu}\left(\frac{\partial P_1(x_0,t)}{\partial x} - \mu_* g_*\right) + \beta_1 P_1\left(x_0,t\right) = q_1(t), \quad t > 0, \tag{9}$$

$$P_2(L,t) = \varphi(L), \quad t > 0. \tag{10}$$

where $\alpha_1 \cdot \beta_1 = 0$, $\alpha_1 + \beta_1 = 1$.

Unlike problems [10–12], here there is no explicit equation for determining the free boundary, however, the known value of the gradient at the viscosity discontinuity boundary allows us to construct a difference scheme that allows us to determine the position of the boundary.

It should be noted that in a layer of finite length L, filtration is divided into two periods. The first period is at $0 \le t \le T$, where T is the moment in time when the boundary reaches the right end of the formation ($\xi(T) = L$), the second period at $t \ge T$ is characterized by the solution of equation (4). An approximate solution of the problem by the method of integral relations was considered, for example, in [8].

In the case of rectilinear-parallel motion of the medium and $g_* = \text{const}$, taking into account the following dimensionless parameters:

$$\overline{x} = \frac{x}{L}, \quad \overline{\xi} = \frac{\xi}{L}, \quad \overline{u} = \frac{P_i\left(x,t\right)}{P_0}, \quad k_1(\overline{x},\overline{t}) = \frac{k_1(x,t)}{k_0},$$

$$k_2(\overline{x},\overline{t}) = \frac{\mu_0 k_2(x,t)}{k_0}, \quad \overline{t} = \frac{t}{t_0}, \quad \overline{g}_* = \frac{g_* L}{P_0}, \quad \overline{q}_1 = \left(\frac{\mu q_1}{k_1} + \mu_* g_*\right) \cdot \frac{L}{P_0},$$

$$t_0 = \frac{\mu \beta^* L^2}{k_0}, \quad \mu_0 = \frac{\nu}{\mu}, \quad c_1 = 1, \quad c_2 = 1, \quad i = 1, 2.$$

For P_1, P_2 in dimensionless form, one notation $\overline{u}(\overline{x},\overline{t})$ is adopted, since they are defined in non-intersecting areas, and the continuity conditions are satisfied at the interface. It is necessary to define the functions $u(x,t)$ so that they satisfy the filtration equation. Let's write the equations by omitting the dashes above the variables

$$c\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right), \quad t > 0. \tag{11}$$

Moreover, the functions u, c and k are defined in the intervals $0 < x < \xi(t)$ and $\xi(t) < x < 1$, i.e.

$$\begin{aligned} k &= \begin{cases} k_1(x,t), & 0 < x < \xi(t), \\ k_2(x,t), & \xi(t) < x < 1, \end{cases} \qquad 0 < k_i \le k_{i0}; \\ c &= \begin{cases} c_1(x,t), & 0 < x < \xi(t), \\ c_2(x,t), & \xi(t) < x < 1, \end{cases} \qquad 0 < c_i \le c_{i0}. \end{aligned} \tag{12}$$

Note that the functions c and k may have a discontinuity at $x = \xi(t)$. In addition to equation (11), the functions $u(x,t)$ and $\xi(t)$ at the interface $x = \xi(t)$ must satisfy the conditions of continuity of the desired function

$$u(\xi - 0, t) = u(\xi + 0, t),$$

and matching gradients at the viscosity discontinuity boundary:

$$\frac{\partial}{\partial x}u(\xi - 0, t) = \frac{\partial}{\partial x}u(\xi + 0, t) = g_*. \tag{13}$$

Initial conditions

$$u(x, 0) = \varphi(x), \quad \xi(0) = x_0, \tag{14}$$

and boundary conditions

$$l_1 u(x_0, t) = q_1(t), \quad u(1, t) = 1. \tag{15}$$

Here the operator corresponds to plane-parallel filtration

$$l_1 = \alpha_1 \left(k_1 \frac{\partial u}{\partial x}\right) + \beta_1 u.$$

Assuming that problem (11)–(15) is posed correctly and we assume that $\xi(t)$ is a monotonically increasing function $t \in (0, T]$, $\varphi'(x) \ge 0$ and $q_1(t) < 0$.

2.3 Difference scheme

For the numerical solution of the considered nonlinear problem, we construct an iterative implicit difference scheme based on the idea of the method of attaching a moving boundary $\xi = \xi(t)$ to grid node [11, 13, 14]. The domain for solving the problem is the half-grid $D = \{x, t | x_0 \leq x \leq 1,\ t \geq 0\}$. On the segment $[0, 1]$, we introduce a quasi-uniform grid of basic and flow nodes:

$$\left\{x_i = x_{i-1} + h,\, i = \overline{1, n-1};\ x_0 = 0;\ x_N = 1\right\},$$

$$\left\{x_{i-1/2} = x_{i-1} + 0.5h,\ i = \overline{1, n-1};\ x_0 = x_{-1/2};\ x_N = x_{N+1/2}\right\}.$$

The area $[0, 1]$ of streaming nodes is split into cells $i = \left[x_{i-1/2}, x_{i+1/2}\right],\ i = \overline{0, n}$. The line $x = \xi(t)$ in the solution area is the dividing one. Here we consider a uniform mesh in x and a non-uniform mesh in time $\widehat{\widehat{\omega}}_{h,\tau} = \{x_i, t_k | x_i = x_0 + ih,\ h > 0,\ i = \overline{0, n},\ t_k = t_{k-1} + \tau_k,\ \tau_k = \sum\limits_{j=0}^{k} \Delta\tau_j,\ \Delta\tau_j > 0,\ k \geq 1,\ n \geq 3\}$. In this case the time step τ_k we will take depending on k so that the end of the broken line approximating the movable boundaries $x = \xi(t)$ for any $\tau_k = \sum\limits_{j=0}^{k} \Delta\tau_j$ would hit the node of the difference grid.

The initial boundary value problem (11)–(15) corresponds to the following conservative, purely implicit two-layer difference scheme [7, 11]. Let us consider the case $c_1 = c_2 = 1$. Then we write (11) at all points except the point $x = \xi(t)$ in the form of

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right), \quad t > 0. \tag{16}$$

Integrate it within $\left(\xi_k - h_1/2\right), \left(\xi_k + h_1/2\right)$,

$$\int_{\xi_k - h_1/2}^{\xi_k + h_1/2} \frac{\partial u}{\partial t} dx = \int_{\xi_k - h_1/2}^{\xi_k} \frac{\partial}{\partial x}\left(k_1 \frac{\partial u}{\partial x}\right) dx + \int_{\xi_k}^{\xi_k + h_1/2} \frac{\partial}{\partial x}\left(k_2 \frac{\partial u}{\partial x}\right) dx. \tag{17}$$

Applying the mean value theorem to the integral on the left, we obtain

$$h_1 \frac{\partial u}{\partial t} = k_2 \frac{\partial u}{\partial x}\bigg|_{\xi_k + h_1/2} - k_1 \frac{\partial u}{\partial x}\bigg|_{\xi_k - h_1/2}. \tag{18}$$

Here the condition from the point of discontinuity is divided into two, which reduces the approximation error arising from the inaccurate determination of the interface. Moreover, if we divide (18) by h_1 and go to the difference derivatives [15], for the point $x = \xi(t)$ we get:

$$y_{\bar{t}} = \left(\overline{k} y_{\bar{x}}\right)_x, \tag{19}$$

where under $\overline{k}$ is considered

$$\overline{k} = \begin{cases} k_1\left(x - h_1/2, t\right) & \text{given that} \quad 0 < x < \xi(t), \\ k_2\left(x + h_1/2, t\right) & \text{given that} \quad \xi(t) < x < 1. \end{cases}$$

Passing to the difference derivatives in (16) and comparing them with (19), we obtain a homogeneous difference scheme over the entire interval $[0,1]$. Combining it with the initial and boundary conditions, we arrive at the difference problem in the entire domain:

$$\begin{array}{lll} y_{\bar t} = \left(\overline{k} y_{\bar x}\right)_x, & y_t^{(0)} = \varphi\left(x_i\right), & \\ \ell_1 y = q_1\left(t_k\right) & \text{given that} & x = 0, \\ y = 1 & \text{given that} & x = 1. \end{array} \tag{20}$$

The operator ℓ_1 is determined by the balance method by integrating (16) on the segments $(0, h/2)$ and using the boundary condition. As is known, in this case, the approximation error will be $o(h^2)$.

Difference problem (20) is supplemented by the condition within which $x = \xi(t)$. For this in the area D_ξ taking into account the condition on the fracture line of the fluid viscosity (13), at $t = t_k$, calculate the integral for a cell with a node of $i = i_k$. Then the equation (16) becomes:

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \frac{\partial u}{\partial t} dx = \left(1-\delta_i\right) \left.\frac{\partial u}{\partial t}\right|_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} + \delta_i \left[\left.\overline{k}\frac{\partial u}{\partial x}\right|_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} + \left.\overline{k}\frac{\partial u}{\partial x}\right|_{\xi_{k+0}}^{\xi_{k-0}} \right]$$

$$= k_{i+\frac{1}{2}} \left.\frac{\partial u}{\partial x}\right|_{x_{i+\frac{1}{2}}} - k_{i-\frac{1}{2}} \left.\frac{\partial u}{\partial x}\right|_{x_{i-\frac{1}{2}}} + \delta_i \left(k_{i-\frac{1}{2}} \left.\frac{\partial u}{\partial x}\right|_{\xi_{k-0}} - k_{i+\frac{1}{2}} \left.\frac{\partial u}{\partial x}\right|_{\xi_{k+0}} \right)$$

$$= k_{i+\frac{1}{2}} \left.\frac{\partial u}{\partial x}\right|_{x_{i+\frac{1}{2}}} - k_{i-\frac{1}{2}} \left.\frac{\partial u}{\partial x}\right|_{x_{i-\frac{1}{2}}} + \delta_i \left(k_{i-\frac{1}{2}} - k_{i+\frac{1}{2}} \right) g_*.$$

Here: given $\delta_i = 1$, $\xi \in \left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right]$ and $\delta_i = 0$, $\xi \notin \left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right]$.

We denote the grid function $y_{i_k} = u(x_i, t_k)$ and $\check{y}_{i_k} = u(x_i, t_{k-1})$.

Then we have:

$$\left(\frac{y_{i_k} - \check{y}_{i_k}}{\Delta\tau_k}\right) h_1 = k_{i_k+\frac{1}{2}} \frac{y_{i_k+1} - y_{i_k}}{h_1} - k_{i_k-\frac{1}{2}} \frac{y_{i_k} - y_{i_k-1}}{h_1} + \delta_i \left(k_{i_k-\frac{1}{2}} - k_{i_k+\frac{1}{2}}\right) g_*.$$

Hence, when the interface of two viscosities is displaced by one step, we find the corresponding time

$$\frac{h_1}{\Delta\tau_k^{(s+1)}} = \frac{\left[k_{i_k+\frac{1}{2}} \frac{y^{(s)}_{i_k+1} - y^{(s)}_{i_k}}{h_1} - k_{i_k-\frac{1}{2}} \frac{y^{(s)}_{i_k} - y^{(s)}_{i_k-1}}{h_1} + \left(k_{i_k-\frac{1}{2}} - k_{i_k+\frac{1}{2}}\right) g_* \right]}{\left(y^{(s)}_{i_k} - \check{y}_{i_k}\right)}, \tag{21}$$

where $\delta_i = 1$, s is t.

The computational algorithm is based on a counter sweep. Difference equation (20) over the entire grid region is then reduced to the form:

$$a_{z,i} y_{z,i-1} - b_{z,i} y_{z,i} + c_{z,i} y_{z,i+1} = d_{z,i}, \quad z = 1, 2. \tag{22}$$

Then, to determine the pressure value at the interface between two viscosities, we obtain a system of algebraic equations.

a) from left to right, using the right sweep formulas, we determine the sweeping coefficients

$$\alpha_i = \frac{c_{1,i}}{b_{1,i} - a_{1,i}\alpha_{i-1}}, \quad \beta_i = \frac{a_{1,i}\beta_{i-1} - d_{1,i}}{b_{1,i} - a_{1,i}\alpha_{i-1}}, \quad i = \overline{1, i_k - 1}, \quad i_k \geq 2; \tag{23}$$

b) from right to left, calculate the sweep coefficients of the left sweep

$$\psi_i = \frac{a_{2,i}}{b_{2,i} - c_{2,i}\psi_{i+1}}, \quad \eta_i = \frac{c_{2,i}\eta_{i+1} - d_i}{b_{2,i} - c_{2,i}\psi_{i+1}}, \quad i = \overline{n - 1, i_k}. \tag{24}$$

For the movable unit $\ll i = i_k \gg$, based on the formula of right and left runs

$$y_{1,i-1} = \alpha_{i-1}y_{1,i} + \beta_{i-1}, \quad i = \overline{i_k, -n_k}; \tag{25}$$

$$y_{2,i+1} = \psi_{i+1}y_{2,i} + \eta_{i+1}, \quad i = \overline{i_k, n_k}, \tag{26}$$

and taking into account (13), we find the required function $y_{1,i_k} = y_{2,i_k} = y^*_{i_k}$,

$$y^*_{i_k} = \frac{\beta_{i_k-1} + \eta_{i_k+1}}{2 - (\psi_{i_k+1} + \alpha_{i_k-1})}. \tag{27}$$

Wherein α_0, β_0 and ψ_n, η_n are determined depending on the setting of the boundary conditions.

Thus, when passing from the $(k-1)$ time layer, calculations are performed in the following order

$$\Delta\tau^{(0)}_{k-1} \Rightarrow (20) \Rightarrow (22) \Rightarrow (21) \Rightarrow (18) \Rightarrow$$
$$\Rightarrow \left|\Delta\tau^{(s)}_k - \Delta\tau^{(s-1)}_k\right| < \varepsilon_\tau \wedge \max_i \left|y^{(s)}_i - y^{(s-1)}_i\right| < \varepsilon_p.$$

If the convergence condition is satisfied, we assume that $\Delta\tau_{k-1} = \Delta\tau^{(s)}_{k-1}$ and $\xi_{k+1} = \xi_k + h_1$ and go to the next time layer, and if the inequalities are not satisfied, we repeat the iterative process.

2.4 Numerical solution results

For illustration, the numerical solution of the problem was carried out for a constant shear gradient $g_* = \text{const}$. This problem has an approximate analytical solution [8], and for a quasi-stationary approximation of a physical process, we execute the law of variation of the moving boundary.

When the gallery is set on the constant pressure of $\Delta p = p_0 - p_c$, then the solution has the form of:

$$\xi(t) = 2\sqrt{\varkappa_\mu t \ln\left(\frac{(1-\varepsilon)\Delta p}{(g_*\sqrt{\pi\varkappa_\mu t})}\right)}, \quad \varepsilon = \frac{(\sqrt{\varkappa_\mu} - \sqrt{\varkappa_\nu})}{(\sqrt{\varkappa_\mu} + \sqrt{\varkappa_\nu})}. \tag{28}$$

If the production gallery is set to a constant flow rate $D = g_* + \mu \cdot \dfrac{q_1}{k}$, then the solution has the following form:

$$\xi(t) = C\sqrt{\varkappa_\mu t},$$

$$\text{where } p_0 + D\sqrt{t} \cong p_0 + 2\sqrt{\frac{\varkappa_\mu t}{\pi}}\left[(1+\varepsilon)\,\varepsilon\left(xe^{-C^2} - C\sqrt{\pi}\,\mathrm{erfc}\,C\right) - 1\right]. \tag{29}$$

Here $\varkappa_\mu = \frac{k}{(\mu\cdot\beta^*)}$, $\varkappa_\nu = \frac{k}{(\nu\cdot\beta^*)}$, $g_* = 5\cdot 10^{-3}$atm/m, $p_0 = 150$atm, $p_c = 120$atm, $\mu = 2.5$cPs, $\nu = 3.6$cPs, $k = 0.4$D, $\beta^* = 16\cdot 10^{-6}\text{atm}^{-1}$, $q_1 = 6.077\cdot 10^2\text{cm}^3/\text{sec}$, $L = 10^3$m, $t_0 = 10^6$sec, $\varkappa_\mu$, $\varkappa_\nu$ are the piezoconductivity coefficients at high and low pressure gradients.

Position of the movable boundary versus time during operation with a given constant pressure on the gallery and at a constant gradient value $g_* = 5\cdot 10^{-3}$atm/m at the discontinuity of viscosities boundary is shown in Figure 2. Here, the absolute error, defined as the difference between the numerical and approximate analytical solutions depending on the operating time, varies from $1.73\cdot 10^{-2}$ to $7.22\cdot 10^{-2}$.

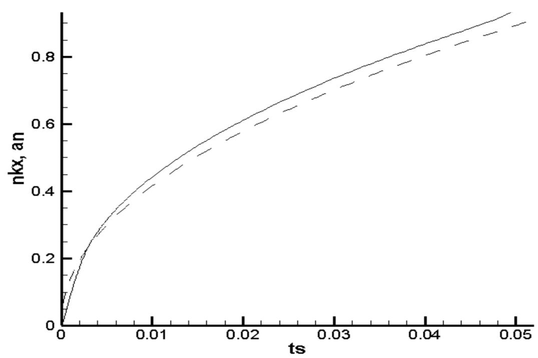

Figure 2

Figure 2. Graph of a moving border $\xi = \xi(t)$ depending on the time when the gallery is operated with constant pressure. The solid line corresponds to the numerical solution, and the dashed line to the approximate analytical solution (28).

The coordinate of the moving boundary depending on the time during operation with a given flow rate and at $g_* = 5\cdot 10^{-3}$ atm/m is shown in Figure 3. Here, the absolute error of the solution, depending on the operation time, varies from $1.92\cdot 10^{-2}$ to $8.15\cdot 10^{-2}$.

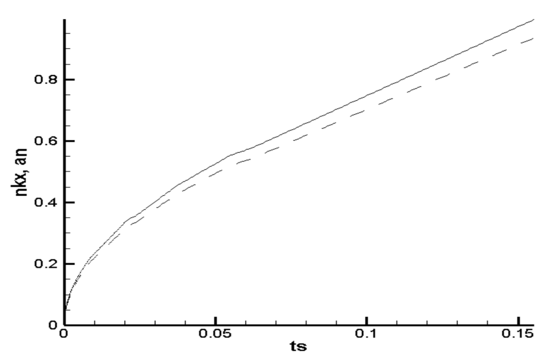

Figure 3

Figure 3. Graph of a moving border $\xi = \xi(t)$ depending on the time when the gallery is operating with a constant flow rate. The solid line corresponds to the numerical solution, and the dashed line to the approximate analytical solution (29).

For the value of the gradient at the boundary of the discontinuity of viscosities $g_* = 5 \cdot 10^{-3}$atm/m at $\Delta p = p_0 - p_c = 30$atm the moving front $\xi(t)$ reaches the right end of the formation $L = 10^3$m in 19.4 hours. If the value of the gradient at the boundary of the discontinuity of viscosities increases, then the time to reach the moving boundary $xi(t)$ the right end of the layer is correspondingly growing. Calculations were carried out for various constant values of the shear gradient: $7 \cdot 10^{-3}$atm/m; $8 \cdot 10^{-3}$atm/m and $9 \cdot 10^{-3}$atm/m. The time to reach the moving front $\xi(t)$ of the right end of the reservoir $L = 10^3$m is growing: 22.2 hours; 25 hours and 27.7 hours; respectively. In the case of the first stage of fluid filtration, the change in pressure as a function of time is shown in Figures 4 and 5 at $g_* = 5 \cdot 10^{-3}$atm/m.

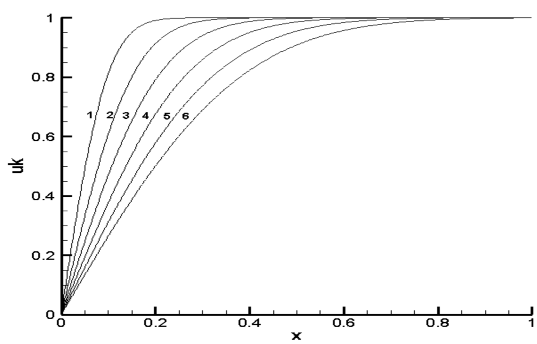

Figure 4

Figure 4. Pressure change graphs during gallery operation with constant pressure for various values of dimensionless time: 1 – 0.2924E – 02; 2 – 0.6866E – 02; 3 – 0.1275E – 01; 4 – 0.2078E – 01; 5 – 0.3118E – 01; 6 – 0.4404E – 01.

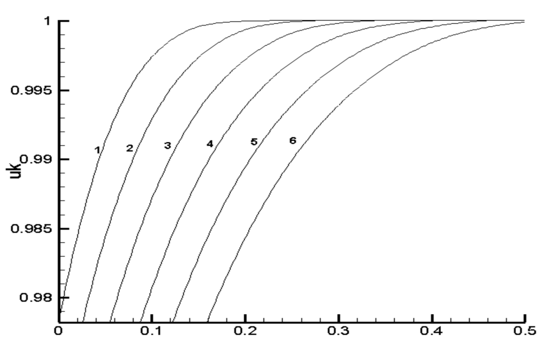

Figure 5

Figure 5. Graphs of pressure changes during gallery operation with constant flow rate for various values of dimensionless time: 1 – 0.3165E – 02; 2 – 0.5959E – 02; 3 – 0.9645E – 02; 4 – 0.1425E – 01; 5 – 0.1972E – 01; 6 – 0.2611E – 01.

3 Conclusion

Analysis of the results showed that the number of iterations depends on the step size of the grid region. In this case, the step along the spatial coordinate is selected depending on the value of the fracture gradient of the fluid viscosity. With a large shear gradient, one should take a smaller step along the spatial coordinate. In a numerical experiment, formula (21) was used to find the appropriate time for fishing in the node of the movable boundary. The results presented showed that the proposed method can be used to determine the free boundary in similar problems with conditions (7), (8), implicitly determining its position.

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