IRSTI 27.39.21 DOI: https://doi.org/10.26577/JMMCS2024-122-02-b2

M.K. Dauylbayev[1], M. Akhmet[2], N. Aviltay[1*]
[1]Al-Farabi Kazakh national university, Kazakhstan, Almaty
[2]Middle East Technical University, Turkey, Ankara
[*]e-mail: avyltay.nauryzbay@gmail.com

ASYMPTOTIC EXPANSION OF THE SOLUTION FOR SINGULAR PERTURBED LINEAR IMPULSIVE SYSTEMS

In this study, a singularly perturbed linear impulsive system with singularly perturbed impulses is considered. Many books discuss different types of singular perturbation problems. In the present work, an impulse system is considered in which a small parameter is introduced into the impulse equation. This is the main novelty of our study, since other works [25] have only considered a small parameter in the differential equation. A necessary condition is also established to prevent the impulse function from bloating as the parameter approaches zero. As a result, the notion of singularity for discontinuous dynamics is greatly extended. An asymptotic expansion of the solution of a singularly perturbed initial problem with an arbitrary degree of accuracy for a small parameter is constructed. A theorem for estimating the residual term of the asymptotic expansion is formulated, which estimates the difference between the exact solution and its approximation. The results extend those of [32], which formulates an analogue of Tikhonov's limit transition theorem. The theoretical results are confirmed by a modelling example.
Key words: singular perturbation, differential equations with singular impulses, small parameter.

М.Қ. Дауылбаев[1], М. Ахмет[2], Н. Авилтай[1*]
[1]Әл-Фараби атындағы Қазақ ұлттық университеті, Қазақстан, Алматы қ.
[2]Орта-Шығыс техникалық университеті, Түркия, Анкара қ.
[*]e-mail: avyltay.nauryzbay@gmail.com

Сингулярлы ауытқыған сызықты импульсті жүйе шешімінің асимптотикалық жіктелуі

Бұл мақалада импульсті бөлігіде сингулярлы ауытқыған сызықты импульстік жүйе қарастырылады. Көптеген жұмыстарда сингулярлы ауытқыған әртүрлі типтегі есептер қаралды. [25] кітапта және кейбір басқа мақалаларда тек дифференциалдық теңдеуінде кіші параметрі бар импульстік жүйелер қарастырылды. Бұл жұмыста импульсті теңдеуіне кіші параметр енгізілді. Бұл осы зерттеудің басты жаңалығы. Сондай ақ, кіші параметр нөлге ұмтылған кезде импульстік функцияның шексіздікке кетуін болдырмау үшін қажетті қосымша шарт қойылды. Нәтижесінде үзіліссіз динамика теориясы үшін сингулярлық тұжырымдамасы айтарлықтай кеңейтілді. Бұл жұмыста сингулярлы ауытқыған бастапқы есеп шешімінің кез келген дәлдіктегі асимптотикалық жіктелуі құрылды. Асимптотикалық жіктелудің қалдық мүшесін бағалау теоремасы тұжырымдалды және ол нақты шешім мен оның жуықталған шешімінің айырымын бағалайды. Алынған нәтижелер Тихоновтың шектік көшу теоремасы аналогын тұжырымдайтын [32] жұмыстың нәтижелерін кеңейтеді. Теориялық нәтижені растайтын нақты мысал графикалық көрнекілікпен келтірілді.
Түйін сөздер: сингулярлы ауытқу, сингулярлы импульсті дифференциалдық теңдеулер, кіші параметр.

М.К. Дауылбаев[1], М. Ахмет[2], Н. Авилтай[1*]
[1]Казахский национальный университет им. аль-Фараби, Казахстан, г. Алматы
[2]Средне-Восточный технический университет, Турция, г. Анкара
[*]e-mail: avyltay.nauryzbay@gmail.com

Асимптотическое разложение решения задачи для сингулярно возмущенных линейных импульсных систем

В статье рассматривается сингулярно возмущенная линейная импульсная система, в которой импульсы также сингулярно возмущены. Во многих книгах обсуждались различные типы задач с сингулярными возмущениями. В [25] и в некоторых других статьях были изучены импульсные системы с малым параметром, присутствующим только в дифференциальных уравнениях. В настоящей работе был введен малый параметр также в уравнение импульса. Это представляет собой главную новизну данного исследования. Более того, было установлено необходимое условие, предотвращающее коллапс импульсной функции при уменьшении параметра до нуля. Этот результат значительно расширяет понятие сингулярности в разрывной динамике. В настоящей работе построено асимптотическое разложение решения сингулярно возмущенной начальной задачи с произвольной степенью точности по малому параметру. Сформулирована теорема об оценке остаточного члена асимптотического разложения, что показывает оценку разности между точным решением и его приближенным решением. Эти результаты расширяют результаты работы [32], в которой сформулирован аналог теоремы Тихонова о предельном переходе. Приведен пример с моделированием, подтверждающий теоретический результат.

Ключевые слова: сингулярное возмущение, дифференциальные уравнения с сингулярными импульсами, малый параметр.

1 Introduction

Singularly perturbed equations have found extensive application as mathematical representations for various phenomena in physics, chemical kinetics [1], mathematical biology [2], hydrodynamics [3], among others. Moreover, these equations are commonly encountered in the exploration of practical engineering and technological challenges [2–5]. A singular wave arises in case when a sudden force is added, for example, an earthquake might lead to a catastrophic tsunami wave [6], a sudden temperature shock might also lead to a thermal tsunami for a porous material [7], and a singular nonlinear oscillator behaves extremely miraculously [8]. Now, the singular wave travelling becomes a hot topic in mathematics [9], especially the quasi-periodic bifurcations [10], singular dissipations [11]. Due to the parameter dependence, the solutions to these problems exhibit non-uniform behavior over time as the parameters approach zero. Many authors are now actively studying singularly perturbed differential equations. There are effective asymptotic methods for singular perturbation problems that allow the construction of uniform approximations with any desired accuracy. The boundary function method is one of them [12]. This method can be used to solve a singularly perturbed problem when the Tikhonov theorem holds in part of the domain.

In this study, we will use the boundary function method to perform an analysis of an impulsive system. The suggested model with singular impulsive can be investigated by developing homotopy perturbation method [13]. Impulsive differential equations play a significant role in multiple scientific fields, including physics, biology, medicine [16], engineering [14] and chemistry [15]. They provide a more accurate representation of certain natural phenomena than ordinary differential equations. Impulsive equations are particularly important for controlling chaos and bifurcation in engineering systems, modelling epidemic scenarios with impulsive births [17], and managing complex dynamic systems [18]. Some of these systems are impulsive in nature and can be affected by small parameters, especially singular ones [19, 20]. However, solving an impulsive differential equation with a singular perturbation is a very complicated task, leading to a lack of research in this particular field. Impulse effects occur in various evolutionary processes that are characterized by abrupt changes of states [21]. They are also present in many systems along with singular perturbations [22–24].

Many papers have discussed different types of singular perturbation problems [12,25–27]. Consider the following singularly perturbed differential equation

$$\begin{aligned} \varepsilon z' &= f(z, y, t), \\ y' &= g(z, y, t), \end{aligned}$$

where ε represents a small positive real number. In the literature, the result that follows from this equation is known as the Tikhonov's theorem [26,28,29]. Bainov and Kovachev [25] were the first to extend the impulsive analogue of Tikhonov's theorem for the system in the form of

$$\begin{aligned} &\varepsilon z' = f(z, y, t), \quad \Delta z|_{t=t_i} = I_i(y(t_i)), \\ &y' = g(z, y, t), \qquad \Delta y|_{t=t_i} = J_i(y(t_i)), \end{aligned}$$

where $0 < t_1 < t_2 < ... < t_p < T$ and $i = 1, 2, ..., p$. It is important to note that only the differential equation in their problem has a perturbation singularity. Akhmet and Çağ [30–32] were the first to consider differential equations with singular impulses in addition to differential equations. They presented the following problem

$$\varepsilon z' = f(z, y, t), \qquad y' = g(z, y, t), \tag{1a}$$

$$\varepsilon \Delta z|_{t=\theta_i} = I(z, y, \varepsilon), \quad \Delta y|_{t=\eta_j} = J(z, y), \tag{1b}$$

where z, f and I are m -dimensional vector valued functions, y, g and J are n - dimensional vector valued functions. The impulse system consists of differential equations (1a) and impulse equations (1b). In addition, for the impulse function, the following condition

$$\lim_{(z,y,\varepsilon)\to(\varphi,\overline{y},0)} \frac{I(z, y, \varepsilon)}{\varepsilon} = I_0 \neq 0, \tag{$*$}$$

was used, which prevents the blow-up of the impulse function when the parameter approaches zero, where $\overline{y} = \overline{y}(\theta_i)$ representing the values for each impulse moment at $t = \theta_i, i = 1, 2, ..., p$. The main novelty of [32] is to extend Tikhonov's theorem in such a way that in the system (1) the impulse function has small parameter. The singularity of the impulse part of the system is analysed using perturbation theory methods. In [32], the behaviour of solutions in a singularly perturbed system is investigated, differentiating between single-layer and multi-layer dependence on condition $(*)$. The results show that the transition to the limit for $y(t, \varepsilon)$ is uniform over the entire interval $0 \le t \le T$. However, the transition to the limit for $z(t, \varepsilon)$ is not uniform over the entire interval $0 \le t \le T$, but only within the subintervals $\delta \le t \le \theta_i, i = 1, 2, ..., p$ for $\delta > 0$, excluding the boundary layers.

The theorems presented in the paper [32] do not provide the precise order of accuracy for the asymptotic approximation $\overline{y}(t)$ for the solution $y(t, \varepsilon)$ in the interval $0 \le t \le T$ and $\overline{z}(t)$ for $z(t, \varepsilon)$ outside the boundary layer. Our goal is to construct complete asymptotic expansions with higher degree of accuracy for solutions of systems with singularly perturbed impulses [33], [34]. In this study, we focus on singularly perturbed differential equations with singular impulses and construct a uniform asymptotic approximation of the solution that is valid over the entire interval $0 \le t \le T$, using the method of boundary functions.

2 Main Result

In this part of the paper, we consider the inhomogeneous linear differential system where impulses are singularly perturbed. Let us consider the following system

$$\begin{aligned} \varepsilon\frac{dz}{dt} &= A_1(t)z + B_1(t)y + \varepsilon f_1(t), \\ \frac{dy}{dt} &= A_2(t)z + B_2(t)y + f_2(t), \end{aligned} \tag{2}$$

and

$$\begin{aligned} \varepsilon\Delta z|_{t=\theta_i} &= C_1(\theta_i)z + C_2(\theta_i)y + \varepsilon I_1(\theta_i), \\ \Delta y|_{t=\theta_i} &= C_3(\theta_i)y + I_2(\theta_i) \end{aligned} \tag{3}$$

with initial condition

$$z(0,\varepsilon) = z^0, \quad y(0,\varepsilon) = y^0, \tag{4}$$

where $\varepsilon > 0$ is a small positive real number, z^0 and y^0 are assumed to be independent of ε, $0 < \theta_1 < \theta_2 < ... < \theta_p < T, \theta_i, i = 1,2,...,p$, are distinct discontinuity moments in $(0,T)$.

The solution of the problem (2)-(4) as $\varepsilon \to 0$ tends to solve the degenerate system

$$\begin{aligned} 0 &= A_1(t)\overline{z} + B_1(t)\overline{y}, & 0 &= C_1(\theta_i)\overline{z} + C_2(\theta_i)\overline{y}, \\ \frac{d\overline{y}}{dt} &= A_2(t)\overline{z} + B_2(t)\overline{y} + f_2(t), & \Delta\overline{y}|_{t=\theta_i} &= C_3(\theta_i)\overline{y} + I_2(\theta_i), \end{aligned}$$

with initial condition

$$\overline{y}(0) = y^0.$$

We need the following conditions:

$(C1)$ The functions $A_i(t), B_i(t), f_i(t), I_i(t), i = 1,2$, and $C_i(t), i = 1,2,3$, are differentiable infinitely many times on the segment $0 \le t \le T$.

$(C2)$ $A_1(t) < 0, 0 \le t \le T$.

$(C3)$ $1 + \dfrac{C_1(\theta_i)}{\varepsilon} \neq 0, 1 + C_3(\theta_i) \neq 0$.

$(C4)$ $\lim\limits_{(z,y,\varepsilon)\to(\varphi,\overline{y},0)} \dfrac{C_1(\theta_i)z + C_2(\theta_i)y + \varepsilon I_1(\theta_i)}{\varepsilon} = I_0 \neq 0$

where $\overline{y} = \overline{y}(\theta_i)-$ are the values for each impulse moment at the points $t = \theta_i, i = 1,2,...,p$.

We will look for the formal asymptotic expansion of the solution of (2)-(4) in the form

$$\begin{aligned} z(t,\varepsilon) &= \overline{z}(t,\varepsilon) + \omega^{(i)}(\tau_i,\varepsilon), \tau_i = \frac{t-\theta_i}{\varepsilon}, \theta_i < t \le \theta_{i+1}, \\ y(t,\varepsilon) &= \overline{y}(t,\varepsilon) + \varepsilon\nu^{(i)}(\tau_i,\varepsilon), \theta_0 = 0, \theta_{p+1} = T, i = \overline{0,p}, \end{aligned} \tag{5}$$

where

$$\begin{aligned} \overline{z}(t,\varepsilon) &= \sum_{k=0}^{\infty}\varepsilon^k\overline{z}_k(t), & \overline{y}(t,\varepsilon) &= \sum_{k=0}^{\infty}\varepsilon^k\overline{y}_k(t), \\ \omega^{(i)}(\tau_i,\varepsilon) &= \sum_{k=0}^{\infty}\varepsilon^k\omega_k^{(i)}(\tau_i), & \nu^{(i)}(\tau_i,\varepsilon) &= \sum_{k=0}^{\infty}\varepsilon^k\nu_k^{(i)}(\tau_i). \end{aligned} \tag{6}$$

The coefficients $\omega_k^{(i)}(\tau_i)$ and $\nu_k^{(i)}(\tau_i)$ in the expansions (6) are called boundary functions. The additional conditions are imposed on them:

$$\omega_k^{(i)}(\infty) = 0,\ \nu_k^{(i)}(\infty) = 0,\quad (i = 1, 2, ..., p.) \tag{7}$$

Substituting the series (5) into the system (2), we obtain

$$\varepsilon(\overline{z}'(t,\varepsilon) + \frac{1}{\varepsilon}\dot{\omega}^{(i)}(\tau_i,\varepsilon)) = A_1(t)(\overline{z}(t,\varepsilon) + \omega^{(i)}(\tau_i,\varepsilon)) + B_1(t)(\overline{y}(t,\varepsilon) + \varepsilon\nu^{(i)}(\tau_i,\varepsilon)) + \varepsilon f_1(t),$$
$$\overline{y}'(t,\varepsilon) + \dot{\nu}^{(i)}(\tau_i,\varepsilon) = A_2(t)(\overline{z}(t,\varepsilon) + \omega^{(i)}(\tau_i,\varepsilon)) + B_2(t)(\overline{y}(t,\varepsilon) + \varepsilon\nu^{(i)}(\tau_i,\varepsilon)) + f_2(t).$$

The following two equations (8), (9) are obtained to determine the coefficients of the regular and boundary layer parts of the series (6) from the last equations.

$$\begin{aligned} &\varepsilon\overline{z}'(t,\varepsilon) = A_1(t)\overline{z}(t,\varepsilon) + B_1(t)\overline{y}(t,\varepsilon) + \varepsilon f_1(t),\\ &\overline{y}'(t,\varepsilon) = A_2(t)\overline{z}(t,\varepsilon) + B_2(t)\overline{y}(t,\varepsilon) + f_2(t), \end{aligned} \tag{8}$$

and

$$\begin{aligned} &\dot{\omega}^{(i)}(\tau_i,\varepsilon) = A_1(\varepsilon\tau_i + \theta_i)\omega^{(i)}(\tau_i,\varepsilon) + \varepsilon B_1(\varepsilon\tau_i + \theta_i)\nu^{(i)}(\tau_i,\varepsilon),\\ &\dot{\nu}^{(i)}(\tau_i,\varepsilon) = A_2(\varepsilon\tau_i + \theta_i)\omega^{(i)}(\tau_i,\varepsilon) + \varepsilon B_2(\varepsilon\tau_i + \theta_i)\nu^{(i)}(\tau_i,\varepsilon). \end{aligned} \tag{9}$$

Now, represent $A_i(\varepsilon\tau_i + \theta_i), B_i(\varepsilon\tau_i + \theta_i), i = 1, 2,$ in the form of power series in ε,

$$A_i(\varepsilon\tau_i + \theta_i) = A_i(\theta_i) + A_i'(\theta_i)\varepsilon\tau_i + A_i''(\theta_i)\frac{(\varepsilon\tau_i)^2}{2!} + ...,$$
$$B_i(\varepsilon\tau_i + \theta_i) = B_i(\theta_i) + B_i'(\theta_i)\varepsilon\tau_i + B_i''(\theta_i)\frac{(\varepsilon\tau_i)^2}{2!} + ...$$

In both parts of equations (8), (9), the coefficients are equated according to the powers of ε, we obtain a sequence of ordinary differential equations for coefficients of the expansions in (6).

$$\begin{aligned} &\varepsilon^0 : 0 = A_1(t)\overline{z}_0(t) + B_1(t)\overline{y}_0(t),\\ &\overline{y}_0'(t) = A_2(t)\overline{z}_0(t) + B_2(t)\overline{y}_0(t) + f_2(t), \end{aligned} \tag{10}$$

$$\begin{aligned} &\varepsilon^1 : \overline{z}_0'(t) = A_1(t)\overline{z}_1(t) + B_1(t)\overline{y}_1(t) + f_1(t),\\ &\overline{y}_1'(t) = A_2(t)\overline{z}_1(t) + B_2(t)\overline{y}_1(t), \end{aligned} \tag{11}$$

$$\begin{aligned} &\varepsilon^k : \overline{z}_{k-1}'(t) = A_1(t)\overline{z}_k(t) + B_1(t)\overline{y}_k(t), k \geq 2,\\ &\overline{y}_k'(t) = A_2(t)\overline{z}_k(t) + B_2(t)\overline{y}_k(t), \end{aligned} \tag{12}$$

and

$$\begin{aligned} &\varepsilon^0 : \dot{\omega}_0^{(i)}(\tau_i) = A_1(\theta_i)\omega_0^{(i)}(\tau_i),\\ &\dot{\nu}_0^{(i)}(\tau_i) = A_2(\theta_i)\omega_0^{(i)}(\tau_i), \end{aligned} \tag{13}$$

$$\begin{aligned} &\varepsilon^k : \dot{\omega}_k^{(i)}(\tau_i) - A_1(\theta_i)\omega_k^{(i)}(\tau_i) = \Gamma_k(\tau_i), \\ &\dot{\nu}_k^{(i)}(\tau_i) - A_2(\theta_i)\omega_k^{(i)}(\tau_i) = \Theta_k(\tau_i), \end{aligned} \tag{14}$$

where functions $\Gamma_k(\tau_i)$ and $\Theta_k(\tau_i)$ are expressed recursively by $\omega_j^{(i)}(\tau_i)$ and $\nu_j^{(i)}(\tau_i)$ with $j < k$.

Consider the interval $t \in [0, \theta_1]$. For the determination of the expansion terms in (6) from equations (8), (12), it is necessary to have the initial conditions.

In order to determine the expansion terms in (6) from equations (8), (12), the initial conditions must be set. In the initial conditions (4) substitute the series (5).

$$\begin{aligned} &\overline{z}_0(0) + \varepsilon\overline{z}_1(0) + ... + \omega_0^{(0)}(0) + \varepsilon\omega_1^{(0)}(0) = z^0, \\ &\overline{y}_0(0) + \varepsilon\overline{y}_1(0) + ... + \varepsilon\nu_0^{(0)}(0) + \varepsilon^2\nu_1^{(0)}(0) = y^0. \end{aligned} \tag{15}$$

In both parts of the equations, equate the coefficients according to powers of ε,

$$\begin{aligned} &\varepsilon^0 : \overline{z}_0(0) + \omega_0^{(0)}(0) = z^0, \\ &\overline{y}_0(0) = y^0, \end{aligned} \tag{16}$$

$$\begin{aligned} &\varepsilon^k : \overline{z}_k(0) + \omega_k^{(0)}(0) = 0, \\ &\overline{y}_k(0) + \nu_{k-1}^{(0)}(0) = 0. \end{aligned} \tag{17}$$

For the leading term $\overline{z}_0(t), \overline{y}_0(t)$ of the regular part of the approximation obtain the systems

$$\begin{aligned} &0 = A_1(t)\overline{z}_0(t) + B_1(t)\overline{y}_0(t), \\ &\overline{y}_0'(t) = A_2(t)\overline{z}_0(t) + B_2(t)\overline{y}_0(t) + f_2(t), \quad \overline{y}_0(0) = y^0, \end{aligned}$$

which obviously coincide with the degenerate system. To find $\omega_0^{(0)}(\tau_0)$, solve the equation

$$\dot{\omega}_0^{(0)}(\tau_0) = A_1(0)\omega_0^{(0)}(\tau_0)$$

with initial condition

$$\omega_0^{(0)}(0) = z^0 - \overline{z}_0(0).$$

Using the second equation in (13) and formula (7), obtain

$$\nu_0^{(0)}(0) = \frac{A_2(0)}{A_1(0)}(z^0 - \overline{z}_0(0)). \tag{18}$$

It remains now to solve equation

$$\dot{\nu}_0^{(0)}(\tau_0) = A_2(0)\omega_0^{(0)}(\tau_0)$$

with initial condition (18).

In this way, all the terms of the approximation of order zero can be defined. Suppose we have already defined all terms up to order $k-1$. To determine the coefficients of ε^k for the approximation $\overline{z}_k(t)$ and $\overline{y}_k(t)$, apply the systems

$$\overline{z}'_{k-1}(t) = A_1(t)\overline{z}_k(t) + B_1(t)\overline{y}_k(t),$$
$$\overline{y}'_k(t) = A_2(t)\overline{z}_k(t) + B_2(t)\overline{y}_k(t), \quad \overline{y}_k(0) = -\nu^{(0)}_{k-1}(0).$$

To find $\omega^{(0)}_k(\tau_0)$ it is needed to solve the following system

$$\dot{\omega}^{(0)}_k(\tau_0) - A_1(0)\omega^{(0)}_k(\tau_0) = \Gamma_k(\tau_0),$$
$$\omega^{(0)}_k(0) = -\overline{z}_k(0).$$

Using the equation (14) and the condition (7), obtain the following initial condition

$$\nu^{(0)}_k(0) = \frac{A_2(0)}{A_1(0)}\omega^{(0)}_k(0) + \int_0^\infty (\frac{A_2(0)}{A_1(0)}\Gamma_k(s) - \Theta_k(s))ds. \tag{19}$$

Solving the second equation of (14)

$$\dot{\nu}^{(0)}_k(\tau_0) - A_2(0)\omega^{(0)}_k(\tau_0) = \Theta_k(\tau_0)$$

with the initial condition (19), find $\nu^{(0)}_k(\tau_0)$.

Now consider the following interval $t \in (\theta_i, \theta_{i+1}], i = 1, 2, ..., p$. $\overline{z}(\theta_i, \varepsilon)$ and $\overline{y}(\theta_i, \varepsilon)$ are the initial values for this interval. Substituting the given series (5) into the impulsive equation (3), obtain the equalities

$$\varepsilon(\overline{z}(\theta_i+, \varepsilon) + \omega^{(i)}(0, \varepsilon) - \overline{z}(\theta_i, \varepsilon) - \omega^{(i-1)}(\frac{\theta_i - \theta_{i-1}}{\varepsilon}, \varepsilon)) = C_1(\theta_i)(\overline{z}(\theta_i, \varepsilon) - \omega^{(i-1)}(\frac{\theta_i - \theta_{i-1}}{\varepsilon}, \varepsilon))+$$
$$+ C_2(\theta_i)(\overline{y}(\theta_i, \varepsilon) + \varepsilon\nu^{(i-1)}(\frac{\theta_i - \theta_{i-1}}{\varepsilon}, \varepsilon)) + \varepsilon I_1(\theta_i),$$
$$\overline{y}(\theta_i+, \varepsilon) + \varepsilon\nu^{(i)}(0, \varepsilon) - \overline{y}(\theta_i, \varepsilon) - \varepsilon\nu^{(i-1)}(\frac{\theta_i - \theta_{i-1}}{\varepsilon}, \varepsilon) = C_3(\theta_i)(\overline{y}(\theta_i, \varepsilon) + \varepsilon\nu^{(i-1)}(\frac{\theta_i - \theta_{i-1}}{\varepsilon}, \varepsilon))$$
$$+ I_2(\theta_i).$$

Taking into account (6), equate the coefficients according to the powers of ε

$$\begin{aligned} &\varepsilon^0 : 0 = C_1(\theta_i)\overline{z}_0(\theta_i) + C_2(\theta_i)\overline{y}_0(\theta_i), \\ &\Delta\overline{y}_0|_{t=\theta_i} = C_3(\theta_i)\overline{y}_0(\theta_i) + I_2(\theta_i), \end{aligned} \tag{20}$$

$$\varepsilon^1 : \omega^{(i)}_0(0) = C_1(\theta_i)\overline{z}_1(\theta_i) + C_2(\theta_i)\overline{y}_1(\theta_i) + I_1(\theta_i) - \Delta\overline{z}_0|_{t=\theta_i},$$
$$\Delta\overline{y}_1|_{t=\theta_i} = C_3(\theta_i)\overline{y}_1(\theta_i) - \nu^{(i)}_0(0),$$
$$\varepsilon^k : \omega^{(i)}_k(0) = C_1(\theta_i)\overline{z}_{k+1}(\theta_i) + C_2(\theta_i)\overline{y}_{k+1}(\theta_i) - \Delta\overline{z}_k|_{t=\theta_i},$$
$$\Delta\overline{y}_k|_{t=\theta_i} = C_3(\theta_i)\overline{y}_k(\theta_i) - \nu^{(i)}_{k-1}(0).$$

In order to determine the approximation of the zero order $\overline{z}_0(t)$ and $\overline{y}_0(t)$, consider the systems

$$0 = A_1(t)\overline{z}_0(t) + B_1(t)\overline{y}_0(t), \qquad 0 = C_1(\theta_i)\overline{z}_0(\theta_i) + C_2(\theta_i)\overline{y}_0(\theta_i),$$
$$\overline{y}'_0(t) = A_2(t)\overline{z}_0(t) + B_2(t)\overline{y}_0(t) + f_2(t), \quad \Delta\overline{y}_0|_{t=\theta_i} = C_3(\theta_i)\overline{y}_0(\theta_i) + I_2(\theta_i).$$

To find $\omega_0^{(i)}(\tau_i)$, we need to solve the equation

$$\dot{\omega}_0^{(i)}(\tau_i) = A_1(\theta_i)\omega_0^{(i)}(\tau_i), i = 1, 2, ..., p$$

with initial condition

$$\omega_0^{(i)}(0) = C_1(\theta_i)\overline{z}_1(\theta_i) + C_2(\theta_i)\overline{y}_1(\theta_i) + I_1(\theta_i) - \Delta\overline{z}_0|_{t=\theta_i},$$

where $\omega_0^{(i)}(0)$ may be modified as below. From the first equation (10) and (20), obtain

$$\overline{z}_0(t) = -\frac{B_1(t)}{A_1(t)}\overline{y}_0(t), \ \ \overline{z}_0(\theta_i) = -\frac{C_2(\theta_i)}{C_1(\theta_i)}\overline{y}_0(\theta_i),$$

Hence,

$$\frac{B_1(\theta_i)}{A_1(\theta_i)} = \frac{C_2(\theta_i)}{C_1(\theta_i)}. \tag{21}$$

The first equation (11) can be written in the form

$$\overline{z}_1(t) + \frac{B_1(t)}{A_1(t)}\overline{y}_1(t) = \frac{1}{A_1(t)}(\overline{z}_0'(t) - f_1(t)).$$

As a result, using the last equation and equality (21), the initial condition $\omega_0^{(i)}(0)$ is transformed into the form

$$\omega_0^{(i)}(0) = \frac{C_1(\theta_i)}{A_1(\theta_i)}(\overline{z}_0'(\theta_i) - f_1(\theta_i)) + I_1(\theta_i) - \Delta\overline{z}_0|_{t=\theta_i}.$$

From the second equation (13) and (7), we find initial condition

$$\nu_0^{(i)}(0) = \frac{A_2(\theta_i)}{A_1(\theta_i)}\omega_0^{(i)}(0), i = 1, 2, ..., p. \tag{22}$$

It is left to solve equation

$$\nu_0^{(i)}(\tau_i) = A_2(\theta_i)\omega_0^{(i)}(\tau_i), i = 1, 2, ..., p.$$

with initial conditions (22).

In order to determine the coefficients of ε^k for the approximation $\overline{z}_k(t)$ and $\overline{y}_k(t)$, have the systems

$$\overline{z}_{k-1}'(t) = A_1(t)\overline{z}_k(t) + B_1(t)\overline{y}_k(t),$$
$$\overline{y}_k'(t) = A_2(t)\overline{z}_k(t) + B_2(t)\overline{y}_k(t), \quad \Delta\overline{y}_k|_{t=\theta_i} = C_3(\theta_i)\overline{y}_k(\theta_i) - \nu_{k-1}^{(i)}(0).$$

To find $\omega_k^{(i)}(\tau_i)$ it is needed to solve the system

$$\dot{\omega}_k^{(i)}(\tau_i) - A_1(\theta_i)\omega_k^{(i)}(\tau_i) = \Gamma_k(\tau_i),$$
$$\omega_k^{(i)}(0) = C_1(\theta_i)\overline{z}_{k+1}(\theta_i) + C_2(\theta_i)\overline{y}_{k+1}(\theta_i) - \Delta\overline{z}_k|_{t=\theta_i},$$

where $\omega_k^{(i)}(0)$ can be changed as follows. The first equation (12) can be written in the form

$$\overline{z}_{k+1}(t)+\frac{B_1(t)}{A_1(t)}\overline{y}_{k+1}(t)=\frac{\overline{z}'_k(t)}{A_1(t)}.$$

Using the last equation and equality (21), the initial condition $\omega_k^{(i)}(0)$ is transformed as follows

$$\omega_k^{(i)}(0)=\frac{C_1(\theta_i)}{A_1(\theta_i)}\overline{z}'_k(\theta_i)-\Delta\overline{z}_k|_{t=\theta_i}.$$

By using the equation (14) and the condition (7), we get the following initial condition

$$\nu_k^{(i)}(0)=\frac{A_2(\theta_i)}{A_1(\theta_i)}\omega_k^{(i)}(0)+\int_0^{\infty}(\frac{A_2(\theta_i)}{A_1(\theta_i)}\Gamma_k(s)-\Theta_k(s))ds. \tag{23}$$

Then the boundary functions $\nu_k^{(i)}(\tau_i), i=1,2,...,p$ can be determined by the second equation of (14) and the initial condition (23).

Functions $\Gamma_k(\tau_i)$ and $\Theta_k(\tau_i)$ possess the exponential estimate. Therefore, it can be proved that the following inequalities hold,

$$|\omega_k^{(i)}(\tau_i)|\leq K\exp(-\gamma\tau_i), i=1,2,...,p,$$
$$|\nu_k^{(i)}(\tau_i)|\leq K\exp(-\gamma\tau_i), i=1,2,...,p,$$

where K and γ are positive numbers.

Thus, the coefficients of the expansions (6) are obtained at least up to order $k=n$. On the basis of above discussion one can conclude that the following assertion is correct.

Theorem 1 *Under conditions $(C1)-(C4)$, the series (5) is the asymptotic expansion as $\varepsilon\to 0$ for the solution $z(t,\varepsilon), y(t,\varepsilon)$ of the problem (2)-(4) in the interval $0\leq t\leq T$, i.e. the following estimate holds*

$$|z(t,\varepsilon)-Z_n(t,\varepsilon)|=O(\varepsilon^{n+1}),\ \ 0\leq t\leq T,$$
$$|y(t,\varepsilon)-Y_n(t,\varepsilon)|=O(\varepsilon^{n+1}),\ \ 0\leq t\leq T,$$

where

$$Z_n(t,\varepsilon)=Z_n^{(i)}(t,\varepsilon), Y_n(t,\varepsilon)=Y_n^{(i)}(t,\varepsilon), \theta_i<t\leq\theta_{i+1},$$
$$Z_n^{(i)}(t,\varepsilon)=\sum_{k=0}^{n}\varepsilon^k\overline{z}_k(t)+\sum_{k=0}^{n}\varepsilon^k\omega_k^{(i)}(\tau_i), \tau_i=\frac{t-\theta_i}{\varepsilon},$$
$$Y_n^{(i)}(t,\varepsilon)=\sum_{k=0}^{n}\varepsilon^k\overline{y}_k(t)+\varepsilon\sum_{k=0}^{n}\varepsilon^k\nu_k^{(i)}(\tau_i), i=1,2,...,p.$$

3 Example

Consider the following system with impulsive singularity

$$\begin{aligned} &\varepsilon\frac{dz}{dt} = -(t+2)z-(t+1)y-\varepsilon t, \quad \varepsilon\Delta z|_{t=\theta_i} = (\theta_i+2)z+(\theta_i+1)y+4\varepsilon\theta_i, \\ &\frac{dy}{dt} = -(t+2)z-12y, \qquad \Delta y|_{t=\theta_i} = 12y-2z \end{aligned} \tag{24}$$

with initial conditions

$$z(0,\varepsilon)=3, \quad y(0,\varepsilon)=2. \tag{25}$$

where $\theta_i = \dfrac{i}{3}, i=1,2,3,4.$

The solution of problem (24)-(25) as $\varepsilon \to 0$ tends to solve the degenerate system

$$\begin{aligned} &0 = -(t+2)\overline{z}-(t+1)\overline{y}, \quad 0 = (\theta_i+2)\overline{z}+(\theta_i+1)\overline{y}, \\ &\frac{d\overline{y}}{dt} = -(t+2)\overline{z}-12\overline{y}, \qquad \Delta\overline{y}|_{t=\theta_i} = 12\overline{y}-2\overline{z} \end{aligned}$$

with initial condition

$$\overline{y}(0)=2.$$

From the first line, we find the root $\overline{z} = \varphi = -\dfrac{(t+1)}{(t+2)}\overline{y}$. One can verify that condition $(C4)$ is valid

$$\lim_{(z,y,\varepsilon)\to(\varphi,\overline{y},0)} \frac{(\theta_i+2)z+(\theta_i+1)y+4\varepsilon\theta_i}{\varepsilon} = 4\theta_i \neq 0.$$

The solution $z(t,\varepsilon)$ of system (24) with initial value (25) has multi-layers near $t=0$ and $t=\theta_i+, i=1,2,3,4$. It is clear from Figure 1 that there are multi-layers.

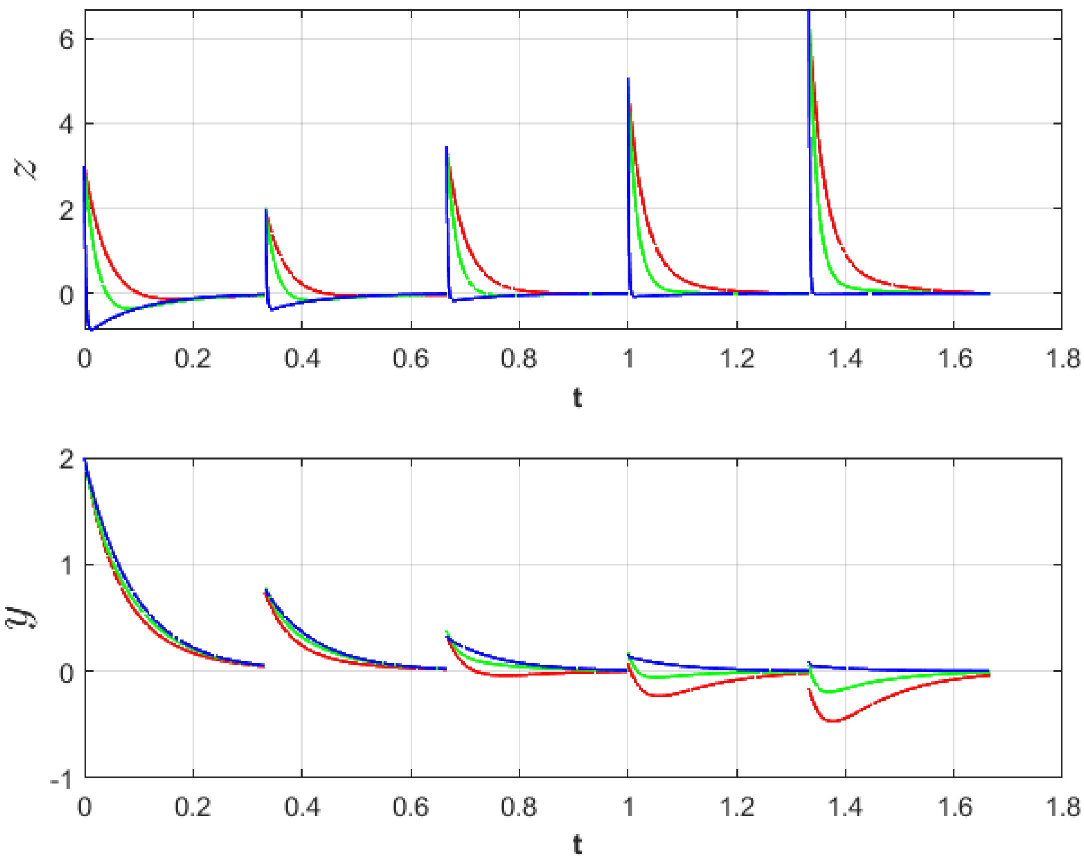

Figure 1: Red, green and blue represent the solution $z(t, \varepsilon), y(t, \varepsilon)$ of (24) with initial conditions $z(0, \varepsilon) = 3$ and $y(0, \varepsilon) = 2$, for $\varepsilon : 0.1, 0.05, 0.005$, respectively.

4 Conclusion

In this article, the singular linear impulsive system is considered. The boundary function method is used to construct the required asymptotic solutions. The asymptotic expansion of solutions with arbitrary degree of accuracy on a small parameter is constructed. To verify the theoretical results, an illustrative example is provided through simulation.

5 Acknowledgements

M. Akhmet is supported by 2247-A National Leading Researchers Program of TUBITAK, Turkey, N 120C138.

References

[1] Segel L. A., Slemrod M. The quasi-steady state assumption: A case study in perturbation. SIAM Review, 1989; 31: 446-477.

[2] Hek G. Geometric singular perturbation theory in biological practice. Journal of Mathematical Biology, 2010; 60: 347-386.

[3] Damiano E. R., Rabbitt R. D. A singular perturbation model of fluid dynamics in the vestibular semicircular canal and ampulla. Journal of Fluid Mechanics, 1996; 307: 333-372.

[4] Gondal I. On the application of singular perturbation techniques to nuclear engineering control problems. IEEE Transactions on Nuclear Science, 1988; 35(5): 1080-1085.

[5] Kokotovic P. V. Applications of singular perturbation techniques to control problems. SIAM Review, 1984; 26(4): 501-550.

[6] Wang Y., Deng, Q. G. Fractal derivative model for tsunami traveling. Fractals, 2019; 27(1):1950017.

[7] Liu F. J., Zhang T., He C. H., Tian D. Thermal oscillation arising in a heat shock of a porous hierarchy and its application. Facta Universitatis Series: Mechanical Engineering, 2022; 20(3): 633-645.

[8] El-Dib Y. O. Insightful and comprehensive formularization of frequency-amplitude formula for strong or singular nonlinear oscillators, Journal of Low Frequency Noise Vibration and Active Control, 2023. DOI:10.1177/14613484221118177.

[9] He C. H., Liu C. Variational principle for singular waves. Chaos, Solitons and Fractals, 2023; Volume 172, July:113566.

[10] Wilkening J., Zhao X. Spatially quasi-periodic bifurcations from periodic traveling water waves and a method for detecting bifurcations using signed singular values. Journal of Computational Physics, 2023; Volume 478, April:111954.

[11] Sebih M. E., Wirth J. On a wave equation with singular dissipation. Mathematische Nachrichten, July 2022; 295(9): DOI: 10.1002/mana.202000076.

[12] Vasil'eva A., Butuzov V., Kalachev L. The Boundary Function Method for Singular Perturbation Problems. Society for Industrial and Applied Mathematics, 1995.

[13] Anjum N., He J. H., Ain Q. T., Tian D. Li-HeЃfs modified homotopy perturbation method for doublyclamped electrically actuated microbeams-based microelectromechanical system. Facta Universitatis Series: Mechanical Engineering, 2021; Vol. 19, No 4: 601 . 612.

[14] Li X., Bohner M., Wang C. K. Impulsive differential equations: periodic solutions and applications. Automatica, 2015; vol. 52:173.178.

[15] Kang A., Xue Y., Jin Z. Dynamic behavior of an ecoepidemic system with impulsive birth. Journal of Mathematical Analysis and Applications, 2008, vol. 345, no. 2: 783.795.

[16] D'onofrio A. On pulse vaccination strategy in the SIR epidemic model with vertical transmission. Appl. Math. Lett., 2005; 18: 729-732.

[17] Dishliev A., Bainov D. D. Dependence upon initial conditions and parameters of solutions of impulsive differential equations with variable structure. International Journal of Theoretical Physics, 1990; 29: 655-676.

[18] Bainov D. D., Simenov P. S. Systems with Impulse Effect Stability Theory and Applications. Ellis Horwood Limited, Chichester, 1989.

[19] He D., Xu L. Integrodifferential inequality for stability of singularly perturbed impulsive delay integrodifferential equations. Journal of Inequalities and Applications, 2009; 11 pages: ID 369185.

[20] Witayakiattilerd W. PID Controller Singularly Perturbing Impulsive Differential Equations and Optimal Control Problem. Advances in Mathematical Physics 2017(2):1-11, DOI:10.1155/2017/1938513.

[21] Akhmet M. Principles of Discontinuous Dynamical Systems, Springer. New York, 2010.

[22] Chen W. H., Yuan G., Zheng W. X. Robust stability of singularly perturbed impulsive systems under nonlinear Perturbation. IEEE Transactions on Automatic Control, 2013; 58(1): 168-174.

[23] Chen W. H., Chen F., Lu X. Exponential stability of a class of singularly perturbed stochastic time-delay systems with impulse effect. Nonlinear Analysis: Real World Applications, 2010; 11(5): 3463-3478.

[24] Simeonov P., Bainov D. Stability of the solutions of singularly perturbed systems with impulse effect. Journal of Mathematical Analysis and Applications, 1988; 136(2): 575-588.

[25] Bainov D., Covachev V. Impulsive Differential Equations with a Small Parameter. World Scientific, 1994.

[26] O'Malley REJ. Singular Perturbation Methods for Ordinary Differential Equations. New York: Applied Mathematical Sciences, Springer, 1991.

[27] Vasil'eva A., Butuzov V.F. Asymptotic Expansion of Solutions of Singularly Perturbed Equations. Moscow: Nauka, 1973 (in russian).

[28] Tikhonov A. N. Systems of differential equations containing small parameters in the derivatives. Matematicheskii sbornik, 1952; 73(3): 575-586.

[29] Akhmet M., Dauylbayev M., Mirzakulova A. A singularly perturbed differential equation with piecewise constant argument of generalized type. Turkish Journal of Mathematics, 2018; 42(4): 1680-1685.

[30] Akhmet M., Çağ S. Chattering as a Singular Problem, Nonlinear Dynamics. 2017; 90(4): 2797.2812.

[31] Akhmet M., Çağ S. Bifurcation analysis of Wilson-Cowan model with singular impulses. Discontinuity, Nonlinearity, and Complexity, 2021; 10(1): 161-172.

[32] Akhmet M., Çağ S. Tikhonov theorem for differential equations with singular impulses. Discontinuity, Nonlinearity, and Complexity, 2018; 7(3): 291-303.

[33] M. Akhmet, N. Aviltay, M. Dauylbayev, R. Seilova. A case of impulsive singularity. Journal of Mathematics, Mechanics and Computer Science, 2023; 117(1): 3-14.

[34] N. Aviltay, M. Akhmet, A. Zhamanshin. Asymptotic solutions of differential equations with singular impulses. Carpathian Journal of Mathematics, 2024; 40(3): 581-598. DOI:10.37193/CJM.2024.03.02.

Information about authors:

Dauylbayev Muratkhan – Doctor of Physical and Mathematical Sciences, Professor, Professor of the Department of Mathematics of al-Farabi Kazakh National University (Almaty, Kazakhstan, email: mdauylbayev@gmail.com);

Akhmet Marat – Doctor of Physical and Mathematical Sciences, Professor, Professor of the Department of Mathematics of Middle East Technical University (Ankara, Turkey, email: marat@metu.edu.tr);

Aviltay Nauryzbay (corresponding author) – Lecture of the Department of Mathematics of al-Farabi Kazakh National University(Almaty, Kazakhstan, email: avyltay.nauryzbay@gmail.com).

Авторлар туралы мәлімет:

Дауылбаев Муратхан Кудайбергенович – Физика және математика ғылымдарының докторы, профессор, әл-Фараби атындағы ҚазҰУ Математика кафедрасының профессоры (Алматы қ., Қазақстан, электрондық пошта: mdauylbayev@gmail.com);

Ахмет Марат – Физика - математика ғылымдарының докторы, профессор, Орта шығыс техникалық университеті Математика кафедрасының профессоры (Анкара қ., Түркия, электрондық пошта: marat@metu.edu.tr);

Авилтай Наурызбай (корреспондент автор) – әл-Фараби атындағы ҚазҰУ математика кафедрасының оқытушысы, (Алматы қ., Қазақстан, электрондық пошта: avyltay.nauryzbay@gmail.com).

Received: April 04, 2024
Accepted: May 28, 2024