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ON THE OPTIMAL DISCRETIZATION OF THE SOLUTION POISSON'S EQUATION

The paper studies the problem of discretizing the solution of the Poisson equation with the right-hand side f belonging to the multidimensional periodic Sobolev class. The research methodology is based on considering the problem of discretizing the solution of the Poisson equation as one of the concretizations of the general problem of optimal recovery of the operator Tf and using well-known statements of approximation theory. Within the framework of this general optimal recovery problem, we first estimate from above the smallest discretization error δ_N of the solution of the Poisson equation in the Hilbert metric using the discretization operator $\left(\tilde{l}^{(N)}, \widetilde{\varphi}_N\right)$ constructed from a finite set of Fourier coefficients of the function f. A lower estimate, coinciding in order with the upper estimate, for the smallest error δ_N was obtained by involving all linear functionals defined on the multidimensional Sobolev class. It should be noted that the optimal discretization operator $\left(\tilde{l}^{(N)}, \widetilde{\varphi}_N\right)$ better approximates the solution under consideration in the Hilbert metric than any discretization operator constructed from values f at given points. Poisson's equation is an elliptic partial differential equation and describes many physical phenomena such as electrostatic field, stationary temperature field, pressure field and velocity potential field in hydrodynamics. Therefore, the relevance of the research conducted here is beyond doubt.

Key words: Poisson's equation, discretization operator, optimal discretization, Fourier coefficients, discretization error, linear functionals, Sobolev class

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Пуассон теңдеуі шешімін оптималды дискреттеу туралы

Жұмыста оң жағындағы f функциясы көпөлшемді периодты Соболев класына тиесілі Пуассон теңдеуінің шешімін дискреттеу есебі зерттелген. Зерттеу әдіснамасы Пуассон теңдеуінің шешімін дискреттеу есебін Tf операторын оптималды қалыптастырудың жалпы есебінің көп нақтыланууларының бірі ретінде қарастыруға негізделген және жуықтаулар теориясының белгілі тұжырымдарын пайдалануға бағытталған. Осы оптималды қалыптастырудың жалпы есебі аясында алдымен Пуассон теңдеуінің шешімін f функциясының Фурье коэффициенттерінің ақырлы жиынтығы бойынша құрылған $\left(\tilde{l}^{(N)}, \widetilde{\varphi}_N\right)$ дискреттеу операторымен гильберттік метрикада дискреттеуде пайда болатын ең аз δ_N қателігі жоғарыдан бағаланған. Одан әрі ең аз δ_N қателігінің, реті бойынша жоғарғы бағамен беттесетін, төменгі бағасы көпөлшемді периодты Соболев класында анықталған барлық сызықтық функцияларды қарастыру нәтижесінде алынған. $\left(\tilde{l}^{(N)}, \widetilde{\varphi}_N\right)$ оптималды дискреттеу операторы қарастырылып отырылған шешімді гильберттік метрикада f функциясының берілген нүктелердегі мәндері бойынша құрылған кез келген оператордан жақсы жуықтайтынын атап өткен жөн. Пуассон теңдеуі дербес туындылы эллипстік дифференциалдық теңдеулер қатарына жатады және электростатикалық өріс, температураның стационарлық өрісі, қысым өрісі, сондай – ақ, гидродинамикадағы жылдамдық потенциалының өрісі сияқты біраз физикалық құбылыстарды сипаттайды. Сондықтан, осы жұмыста жүргізілген зерттеудің өзектілігі ешқандай күмән туғызбайтыны сөзсіз.

Түйін сөздер: Пуассон теңдеуі, дискреттеу операторы, оптималды дискреттеу, Фурье коэффициенттері, дискреттеу қателігі, сызықтық функционалдар, Соболев класы

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Об оптимальной дискретизации решения уравнения Пуассона

В работе изучена задача дискретизации решения уравнения Пуассона с правой частью f принадлежащей многомерному периодическому классу Соболева. Методология исследования основана на рассмотрении задачи дискретизации решения уравнения Пуассона как одной из конкретизации общей задачи оптимального восстановления оператора Tf и в использовании известных утверждений теории приближений. В рамках этой общей задачи оптимального восстановления сначала оценена сверху наименьшая погрешность δ_N дискретизации решения уравнения Пуассона в гильбертовой метрике с помощью оператора дискретизации $\left(\tilde{l}^{(N)}, \widetilde{\varphi}_N\right)$, построенного по конечному набору коэффициентов Фурье функции f. Оценка снизу, совпадающая по порядку с оценкой сверху, наименьшей погрешности δ_N получена в результате привлечения всех линейных функционалов, определенных на многомерном периодическом классе Соболева. Следует отметить, что оптимальный оператор дискретизации $\left(\tilde{l}^{(N)}, \widetilde{\varphi}_N\right)$ лучше приближает в гильбертовой метрике рассматриваемое решение, чем любой оператор дискретизации, построенный по значениям f в заданных точках. Уравнение Пуассона является эллиптическим дифференциальным уравнением в частных производных и описывает многие физические явления такие, как электростатическое поле, стационарное поле температуры, поле давления и поле потенциала скорости в гидродинамике. Поэтому актуальность проведенного здесь исследования не вызывает сомнений.

Ключевые слова: уравнение Пуассона, оператор дискретизации, оптимальная дискретизация, коэффициенты Фурье, погрешность дискретизации,линейные функционалы, класс Соболева

1 Introduction

The paper considers the Poisson equation

$$\triangle u \equiv \frac{\partial^2 u}{\partial x_1^2} + \ldots + \frac{\partial^2 u}{\partial x_s^2} = f \tag{1}$$

with the right -hand side f from the multidimensional periodic Sobolev class $W_2^r \equiv W_2^r[0,1]^s$ with parameters $r > 0$ and $s \in \mathbb{N} \backslash \{1\}$, where $\triangle$ is the Laplace operator, $s \in \{2, 3, \ldots\}, x = (x_1, \ldots, x_s), u = u(x), f = f(x)$.

It is easy to check that if $r > s/2$ and $\hat{f}(0) \neq 0$ are true, then for any boundary condition there is a function $\omega = \omega(x) \in C[0,1]^s$ with $\triangle \omega = 1$ on $[0,1]^s$ such that the solution to equation (1) has the form

$$u_\omega(x; f) = \omega(x)\hat{f}(0) - \frac{1}{4\pi^2} \sum_{m \in Z^s}^{*} \frac{\hat{f}(m)}{(m,m)} \exp\{-2\pi i(m,x)\}, \tag{2}$$

here and everywhere below, the $*$ sign above the sum sign means that the vector $m = (0, \ldots, 0) \in Z^s$ does not participate in the summation. Conversely, any function of the form (2) satisfies equation (1). Since the multiple functional series from (2) is an infinite object, the problem arises of discretizing (approximating) the solution with a finite object and establishing the accuracy of the discretization error.

The first result on discretization of the solution $u_\omega(x;f)$ was obtained in [1] under the condition that the right-hand side of (1) is odd and a one-periodic function $f(x) = f(x_1,\ldots,x_s)$ on each variable $x_1,\ldots,x_s$ belonging to the Korobov class. There a discretization operator is proposed, constructed on the value of the function at the points

$$(\{a_1k/N\},\ldots,\{a_sk/N\}),k\in\{1,\ldots,N\} \tag{3}$$

and approximating solution

$$u(x;f) = -\frac{1}{4\pi^2}\sum_{m\in Z^s}^{*}\frac{\hat{f}(m)}{(m,m)}\exp\{-2\pi i(m,x)\}$$

with accuracy

$$\mathcal{O}\left(\frac{(\ln N)^{r\beta/2+s}}{N^{(r-1)/2+1/s}}\right), \tag{4}$$

where $\{d\}$−the fractional part of the number d, $a_1,\ldots,a_s$ is the optimal coefficients on modulus N and index β.

In [2], the authors, using nodes of a modified Korobov grid, constructed a discretization operator $\Lambda_N(x;f)$, that approximates solution (2) in the metric of space $L^p(2\le p\le\infty)$ with accuracy

$$\mathcal{O}\left(\frac{(\ln N)^{(r+2/s)(s-1)}}{N^{r-(1-1/p-2/s)}}\right) \quad \text{and} \quad \mathcal{O}\left(\frac{(\ln N)^{r(\beta+s)+s}}{N^r}\right) \tag{5}$$

in case $1-\frac{1}{p}-\frac{2}{s}>0$ and $1-\frac{1}{p}-\frac{2}{s}\le 0$ accordingly. Comparing (4) and (5), we conclude that the estimates from (5), corresponding to the case $p=\infty$, are almost “square times” better than the estimate (4).

Further in [3], using the nodes of the Smolyak grid (see, for example, [4], [5]) and the results of the article [6], a discretization operator $(Jf)_N(x)$ was constructed such that

$$\sup_{f\in E_s^r}\|u_\omega(x;f)-(Jf)_N(x)\|_p \underset{\omega,s,r}{\ll} \begin{cases} \frac{(\ln N)^{(r+2/s)(s-1)}}{N^{r-(1-1/p-2/s)}}, 1-\frac{1}{p}-\frac{2}{s}>0,\\ \frac{(\ln N)^{(r+2/s)(s-1)}}{N^r}, 1-\frac{1}{p}-\frac{2}{s}<0,\\ \frac{(\ln N)^{(r(s-1)+2s-1-s/p}}{N^r}, 1-\frac{1}{p}-\frac{2}{s}=0.\end{cases} \tag{6}$$

Thus, in the case $1-\frac{1}{p}-\frac{2}{s}>0$ of the estimate (5) coincides with the estimate (6), and in the other two cases the differences are only in the exponents of logarithms.

In [7], to discretize solutions $u_\omega(x;f)$ with the right side $f\in E_s^r$ discretization operators were used, constructed from a finite set of Fourier coefficients of functions f, and the following results were obtained: firstly, a specific discretization operator $\Psi_N(x;f)$ was proposed, which is optimal in order in the power scale in the metric of space L^2 ; secondly, the error in calculating the Fourier coefficients was found, preserving the optimality of the discretization operator; thirdly, $\Psi_N(x;f)$ has a simpler form than the discretization operators $\Lambda_N(x;f)$ and $(Jf)_N(x;f)$. It should be noted that the order of estimating the error of the discretization operator $\Psi_N(x;f)$ in one case is better than the estimates of the errors of discretization operators $\Lambda_N(x; f)$ and $(Jf)_N(x; f)$, in the other case it coincides with them. Finally, in [8] the error in calculating the values of the function at points (3) was found, preserving the order of estimating the error of the discretization operator from [9] with the algorithm for finding optimal coefficients. The solution $u_\omega(x; f), f \in E_s^r$ discretization operators proposed in [2], [3] and [7] are optimal in order in the power scale. For the first time, the optimal discretization operator for the solution $u_\omega(x; f)$ was constructed in [9], in the case when f belongs to the multidimensional periodic Nikol'skii – Besov class $B^r_{q,\theta}(0,1)^s$ with parameters $s \in \mathbb{N}\backslash\{1\}, r > s/2, 1 \leq \theta \leq \infty, 1 \leq q \leq 2$. Here we note that to construct the optimal discretization operator from [9], the nodes of the uniform grid of a unit $s-$ dimensional cube were used.

In this work, using a finite set of Fourier coefficients of a function $f \in W_2^r$, an optimal discretization operator for the solution $u_\omega(x; f)$ is constructed.

2 Research methodology

Achieving the aim of the research is carried out as a result of considering the problem of discretization of the solution $u_\omega(x; f)$ as one of the concretizations of the problem of optimal recovery of the operator, formulated in [10] based on work [11]. Now we present from [10] the formulation of the problem of recovering the operator with some clarifications. Let be given normed spaces X and Y, consisting of functions $f : \Omega_X \mapsto R$ and $g : \Omega_Y \mapsto R$, respectively, a functional class $F \subset X$, operator $T : F \mapsto Y$ and also a function

$$\varphi_N \equiv \varphi_N(z_1, \ldots, z_N; y) : \mathbb{C}^N \times \Omega_Y \mapsto \mathbb{C}(N = 1, 2, \ldots),$$

which for every fixed $(z_1, \ldots, z_N)$ as a function of a variable y belongs to the space Y. Next, denoting by the symbol $\{(l^{(N)}, \varphi_N)\}$ the set of all possible pairs $(l^{(N)}, \varphi_N)$ formed from a $N-$ dimensional vector $l^{(N)} = \left(l_N^{(1)}, \ldots, l_N^{(N)}\right)$ with components $l_N^{(1)} : F \mapsto \mathbb{C}, \ldots, l_N^{(N)} : F \mapsto \mathbb{C}$ and function φ_N, for a given class F, space Y, operator $T : F \mapsto Y$, set $D_N \subset \{(l^{(N)}, \varphi_N)\}$, we determine the quantity

$$\delta_N(D_N, T, F)_Y = \inf_{(l^{(N)}, \varphi_N) \in D_N} \delta_N\left((l^{(N)}, \varphi_N), T, F\right)_Y, \tag{7}$$

where $\delta_N\left((l^{(N)}, \varphi_N), T, F\right)_Y = \sup\limits_{f \in F} \left\|(Tf)(\cdot) - \varphi_N(l_N^{(1)}(f), \ldots, l_N^{(N)}(f); \cdot)\right\|_Y$.

In the following presentation, we will call a numerical function $\varphi_N(l_N^{(1)}(f), \ldots, l_N^{(N)}(f); y)$ of a variable y a computing unit. For sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ with positive members we will use a notation $a_n \underset{\alpha,\beta,\ldots}{\ll} b_n$, that means the existence of a some quantity $C_k(\alpha, \beta, \ldots) > 0, k = 1, 2, \ldots,$ depending only on the parameters indicated in brackets, such that $a_n \leq C_k(\alpha, \beta, \ldots)b_n$ for all $n \in \mathbb{N}$. And the simultaneous fulfillment of the inequalities $a_n \underset{\alpha,\beta,\ldots}{\ll} b_n$ and $b_n \underset{\alpha,\beta,\ldots}{\ll} a_n$ is written in the form $a_n \underset{\alpha,\beta,\ldots}{\succ\prec} b_n$. According to the above notations and definitions, the problem of optimal recovery of the operator $T : F \mapsto Y$ by computing units $(l^{(N)}, \varphi_N) \equiv \varphi_N(l_N^{(1)}(f), \ldots, l_N^{(N)}(f); \cdot) \in D_N$ in the metric of space Y is to find the exact order of quantity (7) (i.e., to determine a sequence $\{\Psi_N\}_{N\geq 1}$ of positive members that satisfies the relation

$$\delta_N(D_N, T, F)_Y \underset{\alpha,\beta,\ldots}{\succ\prec} \psi_N,$$

here $\alpha, \beta, \ldots$ are parameters of class F and space Y) and in the indication of the computing unit releasing the established exact order, i.e. such that

$$\delta_N((\overline{l}^{(N)}, \overline{\varphi}_N), T, F)_Y \underset{\alpha,\beta,\ldots}{\succ\prec} \psi_N.$$

A computing unit that releases exact order is called an optimal computing unit. The concretization in (7) of the class F, space Y, operator $T : F \mapsto Y$ and of the set D_N gives rise to various optimal recovery problems (see, for example, [10]- [14]). Further, following the works [2], [3] and [7] in this case $(Tf)(\cdot) = u_\omega(\cdot; f)$ instead of the terms "recovery" and "computing unit" we will use the terms "discretization" and "discretization operator" respectively.

In this paper, the discretization problem $u_\omega(x; f)$ is considered as concretization

$$(Tf)(\cdot) = u_\omega(\cdot; f), F = W_2^r[0,1]^s, Y = L^2[0,1]^s, D_N = L_N,$$

where L_N is the set of all pairs $\{(l^{(N)}, \varphi_N)\}$ with linear functionals

$$l_N^{(1)} : W_2^r \mapsto \mathbb{C}, \ldots, l_N^{(N)} : W_2^r \mapsto \mathbb{C},$$

and a discretization operator $(\tilde{l}^{(N)}, \widetilde{\varphi}_N)$ with the following properties is constructed:

I. The discretization operator $(\tilde{l}^{(N)}, \widetilde{\varphi}_N)$ approximates in the metric of space $L^2[0,1]^s$ the solution $u_\omega(x; f)$ of equation (1) with the right-hand side $f \in W_2^r$ with accuracy $\frac{C_1(s,r)}{N^{(r+2)/s}}$, i.e., the inequality

$$\sup_{f \in W_2^r} \left\| u_\omega(\cdot; f) - \widetilde{\varphi}_N(\tilde{l}_N^{(1)}(f), \ldots, \tilde{l}_N^{(N)}(f); \cdot) \right\|_{L^2} \underset{s,r}{\ll} \frac{1}{N^{(r+2)/s}}; \tag{8}$$

holds for $(\tilde{l}^{(N)}, \widetilde{\varphi}_N)$;

II. Estimate (8) cannot be improved in order;

III. The discretization operator $(\tilde{l}^{(N)}, \widetilde{\varphi}_N)$ is optimal, i.e. an arbitrarily chosen discretization operator $(l^{(N)}, \varphi_N) \in L_N$ does not improve estimate (8) in order.

3 Main Result

First, we give the definition of the class W_2^r. The multidimensional periodic Sobolev class $W_2^r \equiv W_2^r[0,1]^s$ is the set of all one-periodic functions $f(x) = f(x_1, \ldots, x_s)$ by each variable $x_1, \ldots, x_s$ and summable on $[0,1]^s$, whose trigonometric Fourier coefficients satisfy the condition

$$\sum_{m \in Z^s} |\hat{f}(m)|^2 (\overline{m}_1^{2r} + \ldots + \overline{m}_s^{2r}) \leq 1, \tag{9}$$

where $(m, x) = m_1 x_1 + \ldots + m_s x_s, \overline{m}_j = \max\{1; |m_j|\} (j = 1, \ldots, s)$.

Now, using properties I – III, we formulate the main result of this work.

Theorem. Let numbers $s \in \mathbb{N}\backslash\{1\}$ and $r > s/2$ be given. Then for each $N = (2n+1)^s, n \in \mathbb{N}$ one has the relations

$$\delta_N(L_N, (Tf)(\cdot) = u_\omega(\cdot; f), W_2^r)_{L^2} \underset{s,r}{\succ\prec}$$
$$\underset{s,r}{\succ\prec} \delta_N\left((\tilde{l}^{(N)}, \widetilde{\varphi}_N), (Tf)(\cdot) = u_\omega(\cdot; f), W_2^r\right)_{L^2} \underset{s,r}{\succ\prec} \frac{1}{N^{(r+2)/s}}, \quad (10)$$

here $l^{(N)}$ consists of components

$$\tilde{l}_N^{(1)}(f) = \hat{f}(\widetilde{m}^{(1)}) = \hat{f}(0), \tilde{l}_N^{(2)}(f) = \hat{f}(\widetilde{m}^{(2)}), \ldots, \tilde{l}_N^{(N)}(f) = \hat{f}(\widetilde{m}^{(N)})$$

and the function $\varphi_N \equiv \varphi_N(z_1, \ldots, z_N; \cdot)$ is defined by the equality

$$\varphi_N(z_1, \ldots, z_N; x) = -\frac{1}{4\pi^2} \sum_{k=1}^{N} z_k d_k(x) \exp\{2\pi i(\widetilde{m}^{(k)}, x)\},$$

where $\left\{\widetilde{m}^{(1)} = 0, \widetilde{m}^{(2)}, \ldots, \widetilde{m}^{(N)}\right\}$ is some ordering of the set

$$A_n = \{m \in Z^s : |m_1| \le n, \ldots, |m_s| \le n\},$$

$$d_k(x) = \begin{cases} \left(\widetilde{m}^{(k)}, \widetilde{m}^{(k)}\right)^{-1}, k \in \{2, \ldots, N\}, \\ -4\pi^2\omega(x), k = 1. \end{cases} .$$

4 Proof

Everywhere below, for each integer vector $m = (m_1, \ldots, m_s)$ we put $\|m\| = \max\{|m_1|, \ldots, |m_s|\}$. The symbol $|B|$ will denote the number of elements of a finite set B.

When estimating the quantity $\delta_N(L_N, (Tf)(\cdot) = u_\omega(\cdot; f), W_2^r)_{L^2}$ from below, we use the following lemma from [13].

Lemma. Let $s \ge 1$ be a given integer. Then, for each integer $N \ge 1$, for any set

$$G \equiv \{m^{(1)}, \ldots m^{(N')}\} \subset Z^s$$

such that $N' = |G| \ge 2N$ and $|G| \underset{s}{\succ\prec} N$, and for arbitrary linear functionals $l_1, \ldots, l_N$, defined at lest on the set of all trigonometric polynomials with spectrum in G, there exists complex numbers $\{c_k\}_{k=1}^{N'}$, satisfying conditions

$$\sum_{k=1}^{N'} |c_k| \ge N, \sum_{k=1}^{N'} |c_k|^2 = N;$$

further, if $\chi(x) = \sum_{k=1}^{N'} e^{2\pi i(m^{(k)}, x)}$, then $l_1(\chi) = 0, \ldots, l_N(\chi) = 0$ and $\|\chi\|_\infty \ge N, \|\chi\|_2 = \sqrt{N}$.

First, let us estimate from above the quantity $\delta_N((\tilde{l}^{(N)}, \widetilde{\varphi}_N), (Tf)(\cdot) = u_\omega(\cdot; f), W_2^r)_{L^2}$. For any function $f \in W_2^r$ the equality

$$\widetilde{\varphi}_N(\tilde{l}_N^{(1)}(f), \dots, \tilde{l}_N^{(N)}(f); x) = \omega(x)\hat{f}(0) - \frac{1}{4\pi^2} \sum_{m \in A_n}^{*} \frac{\hat{f}(m)}{(m,m)} \exp\{2\pi i(m,x)\}$$

holds. Therefore, according to (2) and Parseval's equality we obtain

$$\left\| u_\omega(x; f) - \widetilde{\varphi}_N(\tilde{l}_N^{(1)}(f), \dots, \tilde{l}_N^{(N)}(f); \cdot) \right\|_{L^2} = \frac{1}{4\pi^2} \left(\sum_{m \in Z^s \setminus A_n} \frac{|\hat{f}(m)|^2}{(m,m)^2} \right)^{1/2}. \tag{11}$$

Further,

$$\sum_{m \in Z^s \setminus A_n} \frac{|\hat{f}(m)|^2}{(m,m)^2} = \sum_{m \in Z^s \setminus A_n} \frac{|\hat{f}(m)|^2 (\overline{m}_1^{2r} + \ldots + \overline{m}_s^{2r})}{(m_1^2 + \ldots + m_s^2)^2 (\overline{m}_1^{2r} + \ldots + \overline{m}_s^{2r})} \leq$$
$$\leq \sum_{m \in Z^s \setminus A_n} \frac{|\hat{f}(m)|^2 (\overline{m}_1^{2r} + \ldots + \overline{m}_s^{2r}))}{\overline{m}_1^{2r+4} + \ldots + \overline{m}_s^{2r+4}}.$$

For each $m \in Z^s \setminus A_n$ there is an index $\theta \in \{1, \dots, s\}$ such that $|m_\theta| > n$. Therefore, continuing the expression written above taking into account the inequalities

$$\frac{1}{\overline{m}_1^{2r+4} + \ldots + \overline{m}_s^{2r+4}} \leq \frac{1}{\overline{m}_\theta^{2r+4}} \leq \frac{1}{n^{2r+4}} \underset{s}{\ll} \frac{1}{N^{(2r+4)/s}}$$

and (9) we arrive at the inequality $\sum\limits_{m \in Z^s \setminus A_n} \frac{|\hat{f}(m)|^2}{(m,m)^2} \underset{s}{\ll} \frac{1}{N^{(2r+4)/s}}$.

Therefore, according to (11), the inequality

$$\left\| u_\omega(x; f) - \widetilde{\varphi}_N(\tilde{l}_N^{(1)}(f), \dots, \tilde{l}_N^{(N)}(f); \cdot) \right\|_{L^2} \underset{s}{\ll} \frac{1}{N^{(r+2)/s}}. \tag{12}$$

is true, from where, taking into account the arbitrariness of the function $f \in W_2^r$, we arrive at the estimate

$$\delta_N \left((\tilde{l}^{(N)}, \widetilde{\varphi}_N), (Tf)(\cdot) = u_\omega(\cdot; f), W_2^r \right)_{L^2} \underset{s}{\ll} \frac{1}{N^{(r+2)/s}}. \tag{13}$$

Let linear functionals

$$l_N^{(1)} : W_2^r \mapsto \mathbb{C}, \dots, l_N^{(N)} : W_2^r \mapsto \mathbb{C} \tag{14}$$

and function $\varphi_N(z_1, \dots, z_N; y) : \mathbb{C}^N \times [0,1]^s \mapsto \mathbb{C}$ be given. For some $C_2(s,r) > 0$ the conditions $|U_N| > 2N$ and $|U_N| \underset{s}{\succ\prec} N$ are satisfied for the set

$$U_N = \{m \in Z^s : 1 \leq \|m\| \leq C_2(s,r) N^{1/s}\}.$$

Therefore, due to the above lemma for linear functionals (14), there are complex numbers such that

$$\sum_{m\in U_N} |c_m|^2 = N \tag{15}$$

and if $g_N(x) = \sum\limits_{m\in U_N} c_m \exp\{2\pi i(m,x)\}$, then

$$l_N^{(1)}(g_N) = 0, \ldots, l_N^{(N)}(g_N) = 0. \tag{16}$$

The function

$$f_N(x) = \frac{C_3(s,r)}{N^{r/s}\sqrt{N}} g_N(x)$$

belongs to the class W_2^r. Indeed, taking into account (15) and the relation $|U_N| \underset{s}{\succ\prec} N$, we obtain

$$\sum_{m\in U_N} |\hat{f}(m)|^2 \left(\overline{m}_1^{2r} + \ldots + \overline{m}_s^{2r}\right) \underset{s,r}{\ll}$$

$$\underset{s,r}{\ll} \sum_{m\in U_N} \frac{|c_m|^2}{N^{2r/s}N} \left(\overline{m}_1^{2r} + \ldots + \overline{m}_s^{2r}\right) \underset{s,r}{\ll}$$

$$\underset{s,r}{\ll} \frac{1}{N^{2r/s}N} \sum_{m\in U_N} |c_m|^2 \left(\overline{m}_1^{2r} + \ldots + \overline{m}_s^{2r}\right) \underset{s,r}{\ll} \frac{1}{N} \sum_{m\in U_N} |c_m|^2 \le 1.$$

Since $0 \notin U_N$, then $\hat{f}_N(0) = 0$. Therefore, from equality (2) follows

$$u_\omega(x; f_N) = -\frac{C_3(s,r)}{4\pi^2 N^{s/r}\sqrt{N}} \sum_{m\in U_N} \frac{c_m \exp\{-2\pi i(m,x)\}}{(m,m)}$$

Because

$$\|u_\omega(\cdot; f_N)\|_{L^2} \underset{s,r}{\gg} \frac{1}{N^{r/s}\sqrt{N}} \left(\sum_{m\in U_N} \frac{|c_m|^2}{(m,m)}\right)^{1/2}, \tag{17}$$

then due to equalities $(m,m) = m_1^2 + \ldots + m_s^2 \underset{s}{\ll} \|m\|^2$ and (15)

$$\|u_\omega(\cdot; f_N)\|_{L^2} \underset{s,r}{\gg} \frac{1}{N^{(r+2)/s}}. \tag{18}$$

According to (16) the equalities $l_N^{(1)}(f_N) = 0, \ldots, l_N^{(N)}(f_N) = 0$ are true. Hence, taking into account the inclusion $f \in W_2^r$ we have

$$\sup_{f\in W_2^r} \left\| u_\omega(\cdot; f_N) - \varphi_N(l_N^{(1)}(f_N), \ldots, l_N^{(N)}(f_N); \cdot) \right\|_{L^2} \ge$$

$$\ge \frac{1}{2}\Big(\|u_\omega(\cdot; f_N) - \varphi_N(l_N^{(1)}(f_N), \ldots, l_N^{(N)}(f_N); \cdot)\|_{L^2} +$$

$$+\|u_\omega(\cdot;-f_N)-\varphi_N(l_N^{(1)}(-f_N),\ldots,l_N^{(N)}(-f_N);\cdot)\|_{L^2}\Big)=$$
$$=\frac{1}{2}\Big(\|u_\omega(\cdot;f_N)-\varphi_N(0,\ldots,0;\cdot)\|_{L^2}+$$
$$+\|u_\omega(\cdot;-f_N)-\varphi_N(0,\ldots,0;\cdot)\|_{L^2}\Big)\geq\|u_\omega(\cdot;f_N)\|_{L^2},$$

whence, due to (18), we obtain

$$\delta_N(L_N,(Tf)(\cdot)=u_\omega(\cdot;f),W_2^r)_{L^2}\underset{s,r}{\gg}\frac{1}{N^{(r+2)/s}}. \tag{19}$$

Therefore, according to (13) and

$$\delta_N(L_N,(Tf)(\cdot)=u_\omega(\cdot;f),W_2^r)_{L^2}\leq\delta_N((\tilde{l}^{(N)},\widetilde{\varphi}_N),(Tf)(\cdot)=u_\omega(\cdot;f),W_2^r)_{L^2}$$

there are relations (10). The theorem is proven.

5 Conclusions

1) If we take as a set D_N the set P_N of all pairs $(l^{(N)},\varphi_N)$ with functionals

$$l_N^{(1)}(f)=f(\xi^{(1)}),\ldots,l_N^{(N)}(f)=f(\xi^{(N)}),$$

where $\xi^{(i)}\in[0,1]^s$ for each $i\in\{1,\ldots,s\}$ then, due to the fact that the Sobolev class W_2^r coincides with the Nikolskii–Besov class $B_{2,2}^r$ from Theorem 3.2, formulated and proven in [9], we obtain the following inequality

$$\delta_N(P_N,(Tf)(\cdot)=u_\omega(\cdot;f),W_2^r)_{L^2}\underset{s,r}{\gg}\frac{1}{N^{r/s}}.$$

This inequality allows us to assert that any discretization operator $(l^{(N)},\varphi_N)$ constructed from the values of the function at given points , including the optimal discretization operator

$$\overline{\varphi}_N\left(f(\xi^{(1)}),\ldots,f(\xi^{(N)}),x\right)=\frac{1}{N}\sum_{\xi^{(n)}\in S_N}f(\xi^{(n)})\times$$
$$\times\left(\omega(x)-\frac{1}{4\pi^2}\sum_{\|k\|<t/2}^{*}\frac{\exp(2\pi i(k,x-\xi^{(n)}))}{(k,k)}\right)$$

from [9], where

$$S_N=\left\{\xi^{(n)}=\left(\frac{n_1}{t},\ldots,\frac{n_s}{t}\right),n\in Z^s,0\leq n_j<t(j=1,\ldots,s)\right\}$$

for each $N=t^s(t=1,2,\ldots)$, approximates the solution $u_\omega(x;f)$ in the metric of space L^2 worse than the discretization operator $\left(\tilde{l}^{(N)},\widetilde{\varphi}_N\right)$ from the theorem above.

2) The theorem we formulated is a new result in approximation theory, numerical analysis and computational mathematics. Due to the optimality of the discretization operator we constructed, this research can be continued by considering the problem of finding the limit error of the optimal discretization operator, the formulation of which is presented in [10].

3) Another direction of development of the research carried out here is the consideration of other periodic functional classes ensuring the absolute convergence of the multiple functional series (2) for each $f\in F$, and normed spaces Y.

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