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STUDY ON THE INITIAL BOUNDARY VALUE PROBLEM FOR A FRACTIONAL DIFFERENTIAL EQUATION WITH A FRACTIONAL DERIVATIVE OF VARIABLE ORDER

This article studies the convergence of a numerical method for solving an initial-boundary value problem of a fractional differential equation with a variable order of the fractional derivative. In the generalized fractional differential filtration equation with a transitional filtration law in heterogeneous porous media, it is assumed that the order of the fractional derivative depends on the spatial variable. The main attention is paid to the development and theoretical justification of a method that provides high accuracy and efficiency of calculations with a variable order of the fractional derivative. For the numerical solution, an approximation was developed that combines the finite difference method for the time derivative and the finite element method for the spatial variable. The fractional derivative of variable order in the sense of Caputo is approximated by a formula of second order in time. The convergence of the constructed method is proven with order $O\left(\tau^2 + h^{k+1}\right)$ for the case $\alpha(x) \in (0,1)$. The results of computational experiments for various functions of the order of the fractional derivative are presented, confirming the reliability of the theoretical analysis. The conclusions drawn emphasize the importance and relevance of the further development of numerical methods for fractional differential equations of variable order in modern mathematics and applied sciences, including the modeling of complex processes.

Key words: Fractional differential problem, Filtration problem, Fractional derivative, Heterogeneous medium, Variable order of fractional derivative.

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Бөлшек туындысы айнымалы ретті бөлшек-дифференциалдық теңдеу үшін бастапқы-шеттік есепті зерттеу

Бұл мақалада бөлшек туындысы айнымалы ретті бөлшек-дифференциалдық теңдеу үшін бастапқы-шеттік есепті шешуге арналған сандық әдістің жинақтылығы зерттеледі. Гетерогенді кеуекті ортада өтпелі фильтрация заңы бар жалпыланған бөлшек-дифференциалды фильтрация теңдеуінде бөлшек туындының реті кеңістіктік айнымалыдан тәуелді деп есептеледі. Бөлшек туындының реті айнымалы болған жағдайда есептеулердің жоғары дәлдігі мен тиімділігін қамтамасыз ететін әдісті әзірлеуге және теориялық негіздеуге басты назар аударылады. Сандық шешім үшін уақыт туындысы бойынша ақырлы айырымдар әдісі мен кеңістіктік айнымалы үшін ақырлы элементтер әдісін біріктіретін жуықтау әзірленді. Капуто мағынасындағы айнымалы ретті бөлшек туынды уақыт бойынша екінші ретті формуламен жуықталады. Құрастырылған әдістің $\alpha(x) \in (0,1)$ жағдайы үшін $O\left(\tau^2 + h^{k+1}\right)$ ретімен жинақтылығы дәлелденді. Теориялық талдаудың дәлдігін растайтын бөлшек туынды ретінің әртүрлі функцияларына арналған есептеу тәжірибелерінің нәтижелері берілді. Жасалған қорытындылар қазіргі математика мен қолданбалы ғылымдарда, соның ішінде күрделі процестерді модельдеуде айнымалы ретті бөлшек-дифференциалдық теңдеулердің сандық әдістерін одан әрі дамытудың маңыздылығы мен өзектілігін атап көрсетеді.

Түйін сөздер: Бөлшек-дифференциалды есеп, Фильтрация есебі, Бөлшек туынды, Гетерогенді орта, Бөлшек туындының айнымалы реті.

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Исследование начально-краевой задачи для дробно-дифференциального уравнения с дробной производной переменного порядка

В этой статье исследуется сходимость численного метода для решения начально-краевой задачи дробно-дифференциального уравнения с переменным порядком дробной производной. В обобщенном дробно-дифференциальном уравнении фильтрации с переходным законом фильтрации в гетерогенных пористых средах предполагается, что порядок дробной производной зависит от пространственной переменной. Основное внимание уделено разработке и теоретическому обоснованию метода, обеспечивающего высокую точность и эффективность вычислений при переменном порядке дробной производной. Для численного решения была разработана аппроксимация, сочетающая метод конечных разностей для временной производной и метод конечных элементов для пространственной переменной. Дробная производная переменного порядка в смысле Капуто аппроксимирована формулой второго порядка по времени. Доказана сходимость построенного метода с порядком $O\left(\tau^2 + h^{k+1}\right)$ для случая $\alpha(x) \in (0,1)$. Представлены результаты вычислительных экспериментов для различных функций порядка дробной производной, подтверждающие достоверность теоретического анализа. Сделанные выводы подчеркивают важность и актуальность дальнейшего развития численных методов для дробно-дифференциальных уравнений переменного порядка в современной математике и прикладных науках, включая моделирование сложных процессов.

Ключевые слова: Дробно-дифференциальная задача, Задача фильтрации, Дробная производная, Гетерогенная среда, Переменный порядок дробной производной.

1 Introduction

Fractional differential equations with variable derivative order are an important tool for modeling various complex processes in science and technology. These equations allow more accurately describing the dynamics of systems with memory and heredity, which is especially relevant for applications in fields such as oil production, biophysics and fluid dynamics. In the oil industry, fractional differential models are used to describe fluid flow in porous media, taking into account their heterogeneity and time-varying parameters. One of the key tasks when working with fractional differential equations is the development of numerical methods that provide high accuracy and efficiency of calculations. Particular attention is paid to problems with a variable order of the fractional derivative, where the order of the derivative depends on the spatial variable and the desired solution. Such problems require the development of special approximation methods and analysis of their convergence.

Over the last two years there have been significant progress in the theory and methods for variable order fractional differential equations. Research during this period has substantially expanded the applicability of these methods in various scientific and engineering fields. The studies in [1] focus on developing of improved adaptive finite difference methods for variable order fractional equations. These works demonstrate that adaptive schemes can significantly reduce computational costs while maintaining high-accuracy solutions. In particular, they proposed new adaptive algorithms that take into account the variability of the equation order in each grid point, which improve the accuracy and efficiency of calculations. The papers [2,3] present new spectral element methods that have been adapted to solve fractional equations of variable order. These methods reduce computational costs and increase the stability of the solution. They have also developed hybrid approaches that combine spectral methods with finite difference methods to improve the performance and accuracy of numerical solutions. The authors of [4] show the prospects of using deep learning methods for solving fractional differential equations of variable order. Their approaches demonstrate significant improvements in accuracy and efficiency of the numerical scheme. They developed neural networks specifically adapted for modeling processes with variable order, which opens up new possibilities for the application of these methods in various scientific fields. In addition, [5,6] developed an efficient and accurate hybrid method based on shifted orthogonal Bernoulli polynomials and radial basis functions. Their research shows that such hybrid approaches can significantly improve the performance and accuracy of numerical solutions for complex problems involving variable-order fractional equations. Furthermore, the authors of [7, 8] proposed innovative methods based on the fractional Laplace transform for solving fractional differential equations. These methods have demonstrated high efficiency in a number of test problems and have opened up new prospects for the numerical solution of complex equations. They also showed that the use of such transformations can significantly simplify the solution process and increase the accuracy of the results. Research in [9, 10] shows the prospective application of machine learning methods for solving fractional differential equations. Their approaches demonstrate promising results in improving the accuracy and efficiency of numerical solutions, which opens new horizons for the application of these methods in various scientific fields.

The purpose of this research is the development and theoretical justification of a numerical method for solving an initial boundary value problem for a fractional differential equation with a variable order in the sense of Caputo. To achieve this goal, a numerical method was developed for solving the initial boundary value problem for a fractional differential filtration equation with a transitional filtration law. The convergence of the proposed numerical method has been investigated and its order of convergence has been determined.

The study and development of numerical methods for initial boundary value problems for variable-order fractional differential equations is an important and dynamically developing area. Progress in this area opens up new opportunities for modeling and analysis of complex systems, which has great practical importance in various fields of science and technology. Thus, we believe that this research is aimed at filling the existing gap in the field of numerical methods for variable-order fractional equations and will make a significant contribution to the development of this important and promising area of mathematical modeling.

2 Materials and methods

In [11], a system of partial differential equations is considered and it has been transformed into the following initial boundary value problem for a variable-order fractional differential equation describing a filtration process with the transitional filtration law in the domain

$\bar{Q}_T = \bar{\Omega} \times [0, T]$, where $\Omega = (0, 1)$:

$$\partial_t^{\alpha(x)} u(x,t) - \gamma \nabla^2 u(x,t) = f(x,t), \qquad \alpha(x) \in (0,1), \tag{1}$$

$$u(x,t)\,|_{t=0} = u_0(x), \qquad x \in \overline{\Omega}, \tag{2}$$

$$u(x,t)\,|_{x=0} = u(x,t)\,|_{x=1} = 0, \qquad t > 0. \tag{3}$$

The variable-order fractional derivative operator in the sense of Caputo is defined as follows [12]:

$$\partial_t^{\alpha(x)} u(t) = \frac{1}{\Gamma(1-\alpha(x))} \int_0^t \frac{u'(s)}{(t-s)^{\alpha(x)}} ds, \;\; 0 < \alpha(x) < 1. \tag{4}$$

Let us first present the variational formulation of the problem.

Problem 1 *Find $u \in H^1(0,T; H_0^1(\Omega))$, such that for any $v \in H_0^1(\Omega)$ the following identity holds:*

$$\left(\partial_t^{\alpha(x)} u, v\right) + (\gamma \nabla u, \nabla v) = (f, v), \tag{5}$$

$$u(x,t)\,|_{t=0} = u_0(x),\; x \in \overline{\Omega}, \tag{6}$$

$$u(x,t)\,|_{x=0} = u(x,t)\,|_{x=1} = 0,\; t > 0. \tag{7}$$

where $\alpha(x) \in (0,1)$.

To construct a semi-discrete formulation of the problem, we divide the time interval $[0,T]$ into segments using points $t_n = n\tau, \tau > 0$, $n = 0, 1, ..., N$, such that $N\tau = T$. Here u^n denotes the semi-discrete approximation of the function u at the point $t = t_n$. Let us introduce the notation

$$u^{n+1/2} = \frac{1}{2}\left(u^{n+1} + u^n\right), \quad \Delta_t u^{n+1/2} = \frac{1}{\tau}\left(u^{n+1} - u^n\right). \tag{8}$$

To discretize the variable-order fractional derivative in the sense of Caputo, we utilize the second-order approximation formula presented in [12]. As a result, we obtain the following semi-discrete formulation of Problem 1.

Problem 2 *Let the values* $u^n \in H_0^1(\Omega), u^0 = u_0(x)$ *be known. Find* $u^{n+1} \in H_0^1(\Omega), n = 1, 2, ..., N-1$ *such that for all* $v \in H_0^1(\Omega)$*:*

$$\left(\Delta_{0,t}^{\alpha_{n+\frac{1}{2}}} u^{n+\frac{1}{2}}, v\right) + \gamma\left(\nabla u^{n+\frac{1}{2}}, \nabla v\right) = (f, v), \tag{9}$$

or

$$\frac{\tau^{1-\alpha_{n+\frac{1}{2}}}}{\Gamma\left(2-\alpha_{n+\frac{1}{2}}\right)} \sum_{k=0}^{n} d_{n-k}^{\alpha_{n+\frac{1}{2}}} \left(\Delta_t u^{k+\frac{1}{2}}, v\right) + \gamma\left(\nabla u^{n+\frac{1}{2}}, \nabla v\right) = (f, v), \tag{10}$$

$$u(x,t)\,|_{t=0} = u_0(x),\ x \in \overline{\Omega}, \tag{11}$$

where $\alpha_{n+\frac{1}{2}} \in (0,1)$.

Let $\mathcal{K}_h$ be a uniform partition of the domain $\bar{\Omega}$. For $l \in \mathbb{N}$ we denote by $P_l(e)$ the space of polynomials of degree at most l on the element $e \in \mathcal{K}_h$.

Now we define the discrete space V_h, which is a subspace of the space $H_0^1(\Omega)$:

$$V_h = \left\{ v_h \in H_0^1(\Omega) \cap C^0\left(\overline{\Omega}\right) \;\middle|\; v_h\Big|_e \in P_1(e),\ \ \forall e \in \mathcal{K}_h \right\}.$$

Let us introduce the projection operator $Q_h : H_0^1(\Omega) \to V_h$, satisfying the condition

$$(\nabla(Q_h u - u), \nabla u_h) = 0 \quad \forall u \in H_0^1(\Omega),\ \ u_h \in V_h, \tag{12}$$

which has the following property:

$$\|u - Q_h u\| + h\,\|u - Q_h u\|_{H^1(\Omega)} \le C h^{k+1} \|u\|_{H^{k+1}(\Omega)} \tag{13}$$

for all $u \in H_0^1(\Omega) \cap H^{k+1}(\Omega)$.

Let

$$u(t_n) - u_h^n = (u(t_n) - Q_h u^n) + (Q_h u^n - u_h^n) = \psi^n + \xi^n. \tag{14}$$

Problem 3 *Let the values* $u_h^n \in H_0^1(\Omega)$*,* $u_h^0 = u_0(x)$ *be known. Find* $u_h^{n+1} \in V_h$*,* $n = 1, 2, ..., N-1$*, satisfying the following identities for any* $v_h \in V_h$*:*

$$\left(\Delta_{0,t}^{\alpha_{n+\frac{1}{2}}} u_h^{n+\frac{1}{2}}, v_h\right) + \gamma\left(\nabla u_h^{n+\frac{1}{2}}, \nabla v_h\right) = (f, v_h) \tag{15}$$

or

$$\frac{\tau^{1-\alpha_{n+\frac{1}{2}}}}{\Gamma\left(2-\alpha_{n+\frac{1}{2}}\right)} \sum_{k=0}^{n} d_{n-k}^{\alpha_{n+\frac{1}{2}}} \left(\Delta_t u_h^{k+\frac{1}{2}}, v_h\right) + \left(\gamma \nabla u_h^{n+\frac{1}{2}}, \nabla v_h\right) = (f, v_h),$$

where $\alpha_{n+\frac{1}{2}} \in (0,1)$.

2.1 Research results

The result of the main research is the convergence theorem for the constructed numerical scheme. To prove this theorem, the following assumptions are defined:

(AI) Problem 1 has a unique solution with the number of derivatives sufficient for the analysis.

(AII) There exists a finite positive real number γ_* such that for all $x \in \mathbb{R}$ the condition $0 < \gamma_* \leq \gamma(x)$ holds.

2.1.1 Convergence of a semi-discrete scheme

First, we prove an auxiliary lemma, that will be used to obtain estimates of terms containing fractional derivatives.

Lemma 1 *For any function $u \in L^2(\Omega)$, the following identity holds:*

$$\frac{1}{\tau^{\alpha_{n+\frac{1}{2}}}\Gamma\left(2-\alpha_{n+\frac{1}{2}}\right)}\sum_{k=0}^{n-1} d_{n-k}^{\alpha_{n+\frac{1}{2}}}\left(\left\|u^k\right\|-\left\|u^{k+1}\right\|\right)\left\|u^{n+1}\right\| =$$

$$= \frac{1}{\tau^{\alpha_{n+\frac{1}{2}}}\Gamma\left(2-\alpha_{n+\frac{1}{2}}\right)}\left\|u^{n+1}\right\|\left[d_n^{\alpha_{n+\frac{1}{2}}}\left\|u^0\right\| - d_1^{\alpha_{n+\frac{1}{2}}}\left\|u^n\right\| + \right.$$
$$\left. + \sum_{k=0}^{n-1}\left(d_{n-k}^{\alpha_{n+\frac{1}{2}}} - d_{n-k+1}^{\alpha_{n+\frac{1}{2}}}\right)\left\|u^k\right\|\right].$$

Lemma 2 *Let u^n be the solution to Problem 2, and u be the solution to Problem 1. Then there exists $\tau_0 > 0$ such that, under assumptions (AI) and (AII), the following inequality holds for all $\tau \leq \tau_0$:*

$$\left\|u(t_{n+1}) - u^{n+1}\right\| + \sqrt{\frac{\gamma\tau^{\alpha_{n+\frac{1}{2}}}\Gamma\left(2-\alpha_{n+\frac{1}{2}}\right)}{2d_0^{\alpha_{n+\frac{1}{2}}}}}\left\|\nabla\left(u(t_{n+1}) - u^{n+1}\right)\right\|_{\mathbf{L}^2(\Omega)} \leq C\tau^2, \quad (16)$$

where C is a constant that depends on the solution norms, but does not depend on the grid parameters.

Proof. Consider the difference between the identities (5) and (9):

$$\left(\partial_t^{\alpha(x)}u\left(t_{n+\frac{1}{2}}\right) - \Delta_{0,t}^{\alpha_{n+\frac{1}{2}}}u^{n+\frac{1}{2}}, v\right) + \gamma\left(\nabla u\left(t_{n+\frac{1}{2}}\right) - \nabla u^{n+\frac{1}{2}}, \nabla v\right) = 0. \quad (17)$$

Let us denote $\pi^n = u(t_n) - u^n$ and note that

$$\partial_t^{\alpha(x)}u\left(t_{n+\frac{1}{2}}\right) - \Delta_{0,t}^{\alpha_{n+\frac{1}{2}}}u^{n+\frac{1}{2}} = \Delta_{0,t}^{\alpha_{n+\frac{1}{2}}}\pi^{n+\frac{1}{2}} + r^{\alpha_{n+\frac{1}{2}}},$$

where $r^{\alpha_{n+\frac{1}{2}}} = \partial_t^{\alpha(x)} u\left(t_{n+\frac{1}{2}}\right) - \Delta_{0,t}^{\alpha\left(t_{n+\frac{1}{2}}\right)} u\left(t_{n+\frac{1}{2}}\right)$. Choosing $v = \pi^{n+1}$ in (17), and taking into account assumptions (AI) and (AII), we obtain:

$$\left(\Delta_{0,t}^{\alpha_{n+\frac{1}{2}}} \pi^{n+\frac{1}{2}}, \pi^{n+1}\right) + \gamma\left(\nabla\pi^{n+\frac{1}{2}}, \nabla\pi^{n+1}\right) \leq \left(r^{\alpha_{n+\frac{1}{2}}}, \pi^{n+1}\right). \tag{18}$$

Using the formula for approximating the fractional derivative, we rewrite this inequality as follows:

$$\frac{\tau^{1-\alpha_{n+\frac{1}{2}}}}{\Gamma\left(2-\alpha_{n+\frac{1}{2}}\right)} \sum_{k=0}^{n} d_{n-k}^{\alpha_{n+\frac{1}{2}}} \left(\frac{1}{\tau}\left(\pi^{k+1}-\pi^{k}\right), \pi^{n+1}\right) + \frac{\gamma}{2}\left(\nabla\left(\pi^{n+1}+\pi^{n}\right), \nabla\pi^{n+1}\right) \leq$$
$$\leq \left(r^{\alpha_{n+\frac{1}{2}}}, \pi^{n+1}\right).$$

Separating the last term from the sum, using Lemma 1 and taking into account that $\|\pi^0\| = 0$ we get:

$$\frac{d_0^{\alpha_{n+\frac{1}{2}}}}{\tau^{\alpha_{n+\frac{1}{2}}}\Gamma\left(2-\alpha_{n+\frac{1}{2}}\right)} \left\|\pi^{n+1}\right\|^2 + \frac{\gamma}{4}\left\|\nabla\pi^{n+1}\right\|_{\mathbf{L}^2(\Omega)}^2 \leq$$

$$\leq \frac{1}{\tau^{\alpha_{n+\frac{1}{2}}}\Gamma\left(2-\alpha_{n+\frac{1}{2}}\right)} \sum_{k=1}^{n} \left(d_{n-k}^{\alpha_{n+\frac{1}{2}}} - d_{n-k+1}^{\alpha_{n+\frac{1}{2}}}\right) \left\|\pi^{k}\right\| \left\|\pi^{n+1}\right\| +$$
$$+\frac{\gamma}{4}\left\|\nabla\pi^{n}\right\|_{\mathbf{L}^2(\Omega)}^2 + \left\|r^{\alpha_{n+\frac{1}{2}}}\right\| \left\|\pi^{n+1}\right\|.$$

Multiply both sides by $\dfrac{\tau^{\alpha_{n+\frac{1}{2}}}\Gamma\left(2-\alpha_{n+\frac{1}{2}}\right)}{d_0^{\alpha_{n+\frac{1}{2}}}}$ and applying the Cauchy-Schwartz inequality to the terms on the right-hand side of this inequality we obtain:

$$\left\|\pi^{n+1}\right\|^2 + \frac{\tau^{\alpha_{n+\frac{1}{2}}}\Gamma\left(2-\alpha_{n+\frac{1}{2}}\right)\gamma}{4d_0^{\alpha_{n+\frac{1}{2}}}} \left\|\nabla\pi^{n+1}\right\|_{\mathbf{L}^2(\Omega)}^2 \leq$$

$$\leq \frac{1}{4\varepsilon_1\left(d_0^{\alpha_{n+\frac{1}{2}}}\right)^2} \sum_{k=1}^{n} \left(d_{n-k}^{\alpha_{n+\frac{1}{2}}} - d_{n-k+1}^{\alpha_{n+\frac{1}{2}}}\right)^2 \left\|\pi^{k}\right\|^2 +$$
$$+\varepsilon_1\left\|\pi^{n+1}\right\|^2 + \frac{\tau^{\alpha_{n+\frac{1}{2}}}\Gamma\left(2-\alpha_{n+\frac{1}{2}}\right)\gamma}{4d_0^{\alpha_{n+\frac{1}{2}}}} \left\|\nabla\pi^{n}\right\|_{\mathbf{L}^2(\Omega)}^2 +$$
$$+\frac{\left(\tau^{\alpha_{n+\frac{1}{2}}}\Gamma\left(2-\alpha_{n+\frac{1}{2}}\right)\right)^2}{4\varepsilon_2\left(d_0^{\alpha_{n+\frac{1}{2}}}\right)^2} \left\|r^{\alpha_{n+\frac{1}{2}}}\right\|^2 + \varepsilon_2\left\|\pi^{n+1}\right\|^2.$$

By choosing $\varepsilon_1 = \dfrac{1}{2}$, $\varepsilon_2 = \left(\tau^{\alpha_{n+\frac{1}{2}}}\Gamma\left(2-\alpha_{n+\frac{1}{2}}\right)\right)^2$, for small τ we get

$$\left\|\pi^{n+1}\right\|^2 + \frac{\tau^{\alpha_{n+\frac{1}{2}}}\Gamma\left(2-\alpha_{n+\frac{1}{2}}\right)\gamma}{2d_0^{\alpha_{n+\frac{1}{2}}}}\left\|\nabla\pi^{n+1}\right\|^2_{\mathbf{L}^2(\Omega)} \leq C\sum_{k=1}^{n}\left(d_{n-k}^{\alpha_{n+\frac{1}{2}}} - d_{n-k+1}^{\alpha_{n+\frac{1}{2}}}\right)^2\left\|\pi^k\right\|^2 +$$

$$+\frac{\tau^{\alpha_{n+\frac{1}{2}}}\Gamma\left(2-\alpha_{n+\frac{1}{2}}\right)\gamma}{2d_0^{\alpha_{n+\frac{1}{2}}}}\left\|\nabla\pi^{n}\right\|^2_{\mathbf{L}^2(\Omega)} + C\left\|r^{\alpha_{n+\frac{1}{2}}}\right\|^2. \tag{19}$$

Considering the inequality (19) for $n = 0$ and taking into account $r^{\alpha_{n+\frac{1}{2}}} = O\left(\tau^2\right)$, we obtain

$$\left\|\pi^{1}\right\|^2 + \frac{\tau^{\alpha_{\frac{1}{2}}}\Gamma\left(2-\alpha_{\frac{1}{2}}\right)\gamma}{2d_0^{\alpha_{\frac{1}{2}}}}\left\|\nabla\pi^{1}\right\|^2_{\mathbf{L}^2(\Omega)} \leq C\tau^4. \tag{20}$$

Let us similarly consider the inequality (19) in the case of $n = 1$. Taking into account the inequality (20) and $r^{\alpha_{n+\frac{1}{2}}} = O\left(\tau^2\right)$, we obtain

$$\left\|\pi^{2}\right\|^2 + \frac{\tau^{\alpha_{1\frac{1}{2}}}\Gamma\left(2-\alpha_{1\frac{1}{2}}\right)\gamma}{2d_0^{\alpha_{1\frac{1}{2}}}}\left\|\nabla\pi^{2}\right\|^2_{\mathbf{L}^2(\Omega)} \leq C\tau^4. \tag{21}$$

The inequality (19) is considered in the case of $n = 2$ in a similar way, and taking into account (20) and (21), we get

$$\left\|\pi^{3}\right\|^2 + \frac{\tau^{\alpha_{2\frac{1}{2}}}\Gamma\left(2-\alpha_{2\frac{1}{2}}\right)\gamma}{2d_0^{\alpha_{2\frac{1}{2}}}}\left\|\nabla\pi^{3}\right\|^2_{\mathbf{L}^2(\Omega)} \leq C\tau^4$$

By repeating this process, we arrive at the inequality

$$\left\|\pi^{n+1}\right\|^2 + \frac{\tau^{\alpha_{n+\frac{1}{2}}}\Gamma\left(2-\alpha_{n+\frac{1}{2}}\right)\gamma}{2d_0^{\alpha_{n+\frac{1}{2}}}}\left\|\nabla\pi^{n+1}\right\|^2_{\mathbf{L}^2(\Omega)} \leq C\tau^4. \tag{22}$$

By taking the square root of both sides of the inequality and applying the elementary inequality $\sqrt{a_1^2 + a_2^2} \geq \frac{1}{\sqrt{2}}\left(|a_1| + |a_2|\right)$, we obtain the inequality (16).

2.1.2 Convergence of a fully discrete scheme

Lemma 3 *Let u_h^n be the solution to Problem 3, and u^n be the solution to Problem 2. Then, under assumptions (AI) and (AII), the following inequality holds:*

$$\left\|Q_h u^{n+1} - u_h^{n+1}\right\| + \sqrt{\frac{\gamma\tau^{\alpha_{n+\frac{1}{2}}}\Gamma\left(2-\alpha_{n+\frac{1}{2}}\right)}{2d_0^{\alpha_{n+\frac{1}{2}}}}}\left\|\nabla\left(Q_h u^{n+1} - u_h^{n+1}\right)\right\|_{\mathbf{L}^2(\Omega)} \leq Ch^{k+1}\left\|u\right\|_{H^{k+1}(\Omega)},$$

where C is a constant that depends on the norms of the solution, but does not depend on the grid parameters.

Proof. Consider the difference between the identities (9) and (15):

$$\left(\Delta_{0,t}^{\alpha_{n+\frac{1}{2}}}\left(u^{n+\frac{1}{2}}-u_h^{n+\frac{1}{2}}\right),v_h\right)+\gamma\left(\nabla\left(u^{n+\frac{1}{2}}-u_h^{n+\frac{1}{2}}\right),\nabla v_h\right)=0.$$

Using the notation (14) and choosing $v_h=\xi^{n+1}$, we arrive at the following inequality:

$$\left(\Delta_{0,t}^{\alpha_{n+\frac{1}{2}}}\xi^{n+\frac{1}{2}},\xi^{n+1}\right)+\gamma\left(\nabla\xi^{n+\frac{1}{2}},\nabla\xi^{n+1}\right)+\left(\Delta_{0,t}^{\alpha_{n+\frac{1}{2}}}\psi^{n+\frac{1}{2}},\xi^{n+1}\right)+\gamma\left(\nabla\psi^{n+\frac{1}{2}},\nabla\xi^{n+1}\right)\leq 0.$$

Using the approximation formula, property (12) and multiplying both sides by $\dfrac{\tau^{\alpha_{n+\frac{1}{2}}}\Gamma\left(2-\alpha_{n+\frac{1}{2}}\right)}{d_0^{\alpha_{n+\frac{1}{2}}}}$, we get

$$\left\|\xi^{n+1}\right\|^2+\frac{\gamma\tau^{\alpha_{n+\frac{1}{2}}}\Gamma\left(2-\alpha_{n+\frac{1}{2}}\right)}{2d_0^{\alpha_{n+\frac{1}{2}}}}\left\|\nabla\xi^{n+1}\right\|^2_{\mathbf{L}^2(\Omega)}\leq$$

$$\leq\frac{\gamma\tau^{\alpha_{n+\frac{1}{2}}}\Gamma\left(2-\alpha_{n+\frac{1}{2}}\right)}{4d_0^{\alpha_{n+\frac{1}{2}}}}\left\|\nabla\xi^{n}\right\|^2_{\mathbf{L}^2(\Omega)}+\frac{2}{\left(d_0^{\alpha_{n+\frac{1}{2}}}\right)^2}\sum_{k=1}^{n}\left(d_{n-k}^{\alpha_{n+\frac{1}{2}}}-d_{n-k+1}^{\alpha_{n+\frac{1}{2}}}\right)^2\left\|\xi^k\right\|^2+$$

$$+\frac{2}{\left(d_0^{\alpha_{n+\frac{1}{2}}}\right)^2}\left(d_n^{\alpha_{n+\frac{1}{2}}}\right)^2\left\|\xi^0\right\|^2+Ch^{2k+2}\left\|u\right\|^2_{H^{k+1}(\Omega)}. \tag{23}$$

Considering the inequality (23) for $n=0$, we obtain

$$\left\|\xi^{1}\right\|^2+\frac{\gamma\tau^{\alpha_{\frac{1}{2}}}\Gamma\left(2-\alpha_{\frac{1}{2}}\right)}{2d_0^{\alpha_{\frac{1}{2}}}}\left\|\nabla\xi^{1}\right\|^2_{\mathbf{L}^2(\Omega)}\leq Ch^{2k+2}\left\|u\right\|^2_{H^{k+1}(\Omega)}. \tag{24}$$

Then consider the inequality (23) for $n=1$ and taking into account the inequality (24), we obtain

$$\left\|\xi^{2}\right\|^2+\frac{\gamma\tau^{\alpha_{1\frac{1}{2}}}\Gamma\left(2-\alpha_{1\frac{1}{2}}\right)}{2d_0^{\alpha_{1\frac{1}{2}}}}\left\|\nabla\xi^{2}\right\|^2_{\mathbf{L}^2(\Omega)}\leq Ch^{2k+2}\left\|u\right\|^2_{H^{k+1}(\Omega)}. \tag{25}$$

Similarly, consider the inequality (23) for $n=2$, and taking into account (24) and (25), we get

$$\left\|\xi^{3}\right\|^2+\frac{\gamma\tau^{\alpha_{2\frac{1}{2}}}\Gamma\left(2-\alpha_{2\frac{1}{2}}\right)}{2d_0^{\alpha_{2\frac{1}{2}}}}\left\|\nabla\xi^{3}\right\|^2_{\mathbf{L}^2(\Omega)}\leq Ch^{2k+2}\left\|u\right\|^2_{H^{k+1}(\Omega)}.$$

By repeating this process we arrive at the following inequality:

$$\left\|\xi^{n+1}\right\|^2+\frac{\gamma\tau^{\alpha_{n+\frac{1}{2}}}\Gamma\left(2-\alpha_{n+\frac{1}{2}}\right)}{2d_0^{\alpha_{n+\frac{1}{2}}}}\left\|\nabla\xi^{n+1}\right\|^2_{\mathbf{L}^2(\Omega)}\leq Ch^{2k+2}\left\|u\right\|^2_{H^{k+1}(\Omega)}.$$

By taking the square root of both sides of the inequality and applying the elementary inequality $\sqrt{a_1^2+a_2^2}\geq\frac{1}{\sqrt{2}}\left(|a_1|+|a_2|\right)$ we obtain the statement of the lemma.

Theorem 1 *Let u_h^n be the solution to Problem 3, and u be the solution to Problem 1. Then there exists $\tau_0 > 0$ such that under assumptions (AI) and (AII), the following inequality holds for all $\tau \le \tau_0$:*

$$\left\| u\left(t_{n+1}\right) - u^{n+1} \right\| + \sqrt{\frac{\gamma \tau^{\alpha_{n+\frac{1}{2}}} \Gamma\left(2 - \alpha_{n+\frac{1}{2}}\right)}{2 d_0^{\alpha_{n+\frac{1}{2}}}}} \left\| \nabla\left(u\left(t_{n+1}\right) - u^{n+1}\right) \right\|_{\mathbf{L}^2(\Omega)} \le C\left(\tau^2 + h^{k+1}\right),$$

where C is a constant that depends on the norms of the solution, but does not depend on the grid parameters.

Proof. The proof of the theorem follows from the inequality

$$\left\| u\left(t_n\right) - u_h^n \right\| \le \left\| u\left(t_n\right) - u^n \right\| + \left\| u^n - Q_h u \right\| + \left\| Q_h u - u_h^n \right\|,$$

from Lemmas 2, 3 and inequality (13).

The theorem is proven.

3 Results

Example 1 *Consider the following equation with a variable-order fractional derivative:*

$$\partial_t^{\alpha(x)} u\left(x,t\right) - \nabla^2 u\left(x,t\right) = f\left(x,t\right), \tag{26}$$

where $\alpha\left(x\right) \in \left(0,1\right)$ subject to the following initial and boundary conditions:

$$u\left(x,0\right) = 0,\ x \in \left[0,1\right],$$

$$u\left(0,t\right) = u\left(1,t\right) = 0,\ t \in \left(0,1\right].$$

The exact solution to this problem has the form

$$u\left(x,t\right) = t^2 x\left(1-x\right).$$

Let us choose various functions for the order of the fractional derivative $\alpha\left(x\right)$ and the corresponding right-hand sides of the equation:

$$\alpha_1\left(x\right) = \frac{2}{3},\ f\left(x,t\right) = \frac{9 t^{\frac{4}{3}}\left(1-x\right) x}{2 \mathit{\Gamma}\left(\frac{1}{3}\right)} - 2t^2,$$

$$\alpha_2\left(x\right) = \frac{2}{5}\left(x+1\right),\ f\left(x,t\right) = \frac{50 t^{\frac{8-2x}{5}}\left(1-x\right) x}{\left(3-2x\right)\left(8-2x\right) \mathit{\Gamma}\left(\frac{3-2x}{5}\right)} - 2t^2,$$

$$\alpha_3\left(x\right) = 1 - x^2,\ f\left(x,t\right) = \frac{t^{x^2+1}\left(1-x\right)}{x\left(x^2+1\right) \mathit{\Gamma}\left(x^2\right)} - 2t^2,$$

$$\alpha_4\left(x\right) = e^{-x},\ f\left(x,t\right) = \frac{t^{2-e^{-x}}\left(1-x\right) x}{\left(1-e^{-x}\right)\left(2-e^{-x}\right) \mathit{\Gamma}\left(1-e^{-x}\right)} - 2t^2.$$

In the experiments, the time step was chosen equal to $\tau = 10^{-2}$ when analyzing the dependence of the error on the spatial step. The step value in the spatial variable h varied between $h = \frac{1}{10}$ and $h = \frac{1}{160}$. Similarly, the spatial step was chosen equal to $h = 10^{-3}$ when analyzing the dependence of the error order on the time step, and the time step value varied between $h = \frac{1}{10}$ and $h = \frac{1}{160}$.

Tables 1–4 provide L^2-error values for various functions representing the variable order of the fractional derivative $\alpha(x)$ as well as the parameters h and τ.

Table 1 L_2-errors and orders of convergence for Example 1 (for the cases $\alpha_1(x)$ and $\alpha_2(x)$), with $\tau = 10^{-2}$

h	$\alpha_1(x) = \frac{2}{3}$		$\alpha_2(x) = \frac{2}{5}(x+1)$	
	Error	Rate	Error	Rate
1/10	$2.5002 \cdot 10^{-4}$	–	$1.0559 \cdot 10^{-4}$	–
1/20	$6.2506 \cdot 10^{-5}$	2.00	$2.6581 \cdot 10^{-5}$	1.99
1/40	$1.5626 \cdot 10^{-5}$	2.00	$6.6453 \cdot 10^{-6}$	2.00
1/80	$3.9066 \cdot 10^{-6}$	2.00	$1.6499 \cdot 10^{-6}$	2.01
1/160	$9.6991 \cdot 10^{-7}$	2.01	$4.0961 \cdot 10^{-7}$	2.01

Table 2 L_2-errors and orders of convergence for Example 1 (for the cases $\alpha_3(x)$ and $\alpha_4(x)$), with $\tau = 10^{-2}$

h	$\alpha_3(x) = 1 - x^2$		$\alpha_4(x) = e^{-x}$	
	Error	Rate	Error	Rate
1/10	$1.6410 \cdot 10^{-3}$	–	$1.0922 \cdot 10^{-2}$	–
1/20	$4.1888 \cdot 10^{-4}$	1.97	$2.7879 \cdot 10^{-3}$	1.97
1/40	$1.0618 \cdot 10^{-4}$	1.98	$7.0670 \cdot 10^{-4}$	1.98
1/80	$1.6916 \cdot 10^{-5}$	1.98	$1.7914 \cdot 10^{-4}$	1.98
1/160	$6.7290 \cdot 10^{-6}$	2.00	$4.5097 \cdot 10^{-5}$	1.99

Table 3 L_2-errors and orders of convergence for Example 1 (for the cases $\alpha_1(x)$ and $\alpha_2(x)$), with $h = 10^{-3}$

τ	$\alpha_1(x) = \frac{2}{3}$		$\alpha_2(x) = \frac{2}{5}(x+1)$	
	Error	Rate	Error	Rate
1/10	$9.3420 \cdot 10^{-4}$	–	$1.9240 \cdot 10^{-4}$	–
1/20	$2.3681 \cdot 10^{-4}$	1.98	$4.9110 \cdot 10^{-5}$	1.97
1/40	$6.0029 \cdot 10^{-5}$	1.98	$1.2449 \cdot 10^{-5}$	1.98
1/80	$1.5112 \cdot 10^{-5}$	1.99	$3.1557 \cdot 10^{-6}$	1.98
1/160	$3.7779 \cdot 10^{-6}$	2.00	$7.9441 \cdot 10^{-7}$	1.99

Table 4 L_2-errors and orders of convergence for Example 1 (for the cases $\alpha_3(x)$ and $\alpha_4(x)$), with $h = 10^{-3}$

τ	$\alpha_3(x) = 1 - x^2$		$\alpha_4(x) = e^{-x}$	
	Error	Rate	Error	Rate
1/10	$1.2587 \cdot 10^{-3}$	–	$1.2591 \cdot 10^{-2}$	–
1/20	$3.2353 \cdot 10^{-4}$	1.96	$3.2362 \cdot 10^{-3}$	1.96
1/40	$8.2582 \cdot 10^{-5}$	1.97	$8.3181 \cdot 10^{-4}$	1.96
1/80	$2.0934 \cdot 10^{-6}$	1.98	$2.1232 \cdot 10^{-4}$	1.97
1/160	$5.2698 \cdot 10^{-6}$	1.99	$5.3821 \cdot 10^{-5}$	1.98

From the presented analysis, we can conclude that the empirical convergence order of the constructed numerical scheme is $O(\tau^2 + h^2)$. Thus, the computational experiments carried out confirmed that the proposed scheme has second-order convergence in both spatial and temporal variables.

4 Conclusion

Thus, a numerical method for a fractional differential filtration equation with a variable order of the fractional derivative is constructed in this paper. A priori estimates are obtained which yield the convergence of the proposed numerical scheme. The results of computational experiments carried out for a model problem with a known exact solution showed good agreement between the empirical order of convergence and the theoretical order. In subsequent works, the authors intend to apply the constructed numerical scheme to solving more realistic filtration problems.

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